ON RELATIONS EXISTING BETWEEN TWO KERNELS OF THE FORM $(a, b)+b$ AND $(b, a)+b$

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Let s and t be variables in the interval from 0 to 1, and let $a, b, c, \cdots,$ be functions of s and t. Putting, as is customary,

$$(a, b) = \int_0^1 a(s\lambda)b(\lambda t)d\lambda,$$

we have

$$(a \pm b, c) = (a, c) \pm (b, c),$$
$$(a, b \pm c) = (a, b) \pm (a, c),$$
$$((a, b), c) = (a, (b, c)) = (a, b, c).$$

From this follows readily the meaning of (a, b, c, d). Putting, again,

$$[a, b] = a + (a, b) + b,$$

we have

$$[0, a] = a, \qquad [a, 0] = a, \qquad [[a, b], c] = [a, [b, c]] = [a, b, c].$$

We put finally,

$$\{a, b, c\} = (a, b, c) + (a, b) + (b, c) + b.$$

The function a is said to be reciprocable if there exists a function $\bar{a}$ such that

$$(*) \qquad [a, \bar{a}] = 0 \quad \text{and} \quad [\bar{a}, a] = 0.$$

(Each of these equations, it is well known, implies the other.) We say then that $\bar{a}$ is the reciprocal of a. If a is reciprocable, then so is $\bar{a}$, and the reciprocal of $\bar{a}$ is a. In what follows we shall designate the Fredholm determinant of a function a by D_a, and the reciprocal of a by $\bar{a}$. Of the various relationships that exist among the symbols (a, b), (a, b, c), $[a, b]$, $[a, b, c]$ and $\{a, b, c\}$, we state here the following:

$$(1) \qquad [a, b, c] = \{a, b, c\} + [a, c],$$

$$(2) \qquad [a, b, \bar{a}] = \{a, b, \bar{a}\}.$$

The following relations also hold true:

(α) $\{a, b, 0\} = (a, b)+b \quad \{0, a, b\} = (a, b)+a \quad \{a, 0, b\} = 0,$

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(β) $\{a, (b, c)+c, d\} = \{[a, b], c, d\}$,
(γ) $((a, b)+a, \{\bar{b}, c, d\}) = (a, c, d)+(a, c)$,
(δ) $\{a, b \pm c, d\} = \{a, b, d\} \pm \{a, c, d\}$;
and more generally:
(ϕ) $\{a, \{b, c, d\}, e\} = \{[a, b], c, [d, e]\}$,
(ψ) $(\{a, b, c\}, \{d, e, f\}) = \{a, (b, [c, d], e)+(b, e), f\}$.

(β) can be derived from (ϕ). For, we have by (α) and (ϕ),

$$\begin{aligned}\{a, (b, c) + c, d\} &= \{a, \{b, c, 0\}, d\} = \{[a, b], c, [0, d]\} \\ &= \{[a, b], c, d\}.\end{aligned}$$

(γ) could likewise be derived from (ψ). For we have, by (α), (ψ), and (δ),

$$\begin{aligned}((a, b) + a, \{\bar{b}, c, d\}) &= (\{0, a, b\}, \{\bar{b}, c, d\}) \\ &= \{0, (a, [b, \bar{b}], c) + (a, c), d\} \\ &= \{0, (a, 0, c) + (a, c), d\} \\ &= \{0, (a, c), d\} = (a, c, d) + (a, c).\end{aligned}$$

(γ) and (δ) are thus seen to be special cases of (ϕ) and (ψ). For what follows, however, (γ) and (δ) will be amply sufficient.

Of the Fredholm determinant it is known that

$$D_{[a,b]} = D_a \cdot D_b \tag{3}$$

(v. G. Kowalewski, *Determinanten*, 1909, p. 467), from which relation follows easily:

$$D_{[a,b,c]} = D_a \cdot D_b \cdot D_c. \tag{4}$$

From (3) we derive the known fact:

$$D_a \cdot D_{\bar{a}} = D_{[a,\bar{a}]} = D_0 = 1. \tag{5}$$

Again, by (2), (4) and (5), we have

$$D_{\{a,b,\bar{a}\}} = D_{[a,b,\bar{a}]} = D_a \cdot D_b \cdot D_{\bar{a}} = D_b. \tag{6}$$

Let $D_a \neq 0$, so that $\bar{a}$ exists. We put $c = (a, b)+b$, $e = (b, a)+b$ and conclude that

$$D_c = D_e. \tag{7}$$

To prove (7), we put $w = \{a, e, \bar{a}\}$. We have, then, by (6)

$$D_e = D_w. \tag{8}$$

On the other hand we have:

$$\begin{aligned} w = \{a, e, \bar{a}\} &= (a, e, \bar{a}) + (a, e) + (e, \bar{a}) + e \\ &= (a, (b, a) + b, \bar{a}) + (a, (b, a) + b) + ((b, a) + b, \bar{a}) + (b, a) + b \\ &= (a, b) + b + ((a, b) + b, a + (a, \bar{a}) + \bar{a}) \\ &= (a, b) + b + ((a, b) + b, 0) = (a, b) + b = c; \end{aligned}$$

therefore $D_w = D_c$. From this and (8) follows (7).

Equation (7) holds true even when $D_a = 0$. For, putting

$$c' = (\lambda a, b) + b, \qquad e' = (b, \lambda a) + b,$$

we have for all λ for which $D_{\lambda a} \neq 0$,

$$D_{c'} = D_{e'}. \tag{9}$$

$D_{c'}$ and $D_{e'}$, however, can easily be shown to be entire functions of λ, and, moreover, the zero points of $D_{\lambda a}$ accumulate nowhere. It follows, therefore, that (9) holds true for all λ, particularly for $\lambda = 1$, that is, (7) is true even in the case of $D_a = 0$.

Retaining the notation $c = (a, b) + b$, $e = (b, a) + b$, we state that if $D_a \neq 0$, and $D_c \neq 0$, so that both $\bar{a}$ and $\bar{c}$ exist, then there exists also $\bar{e}$, and we have

$$\bar{e} = \{\bar{a}, \bar{c}, a\}. \tag{10}$$

PROOF. We have $c + (c, \bar{c}) + \bar{c} = 0$. From this follows (by (δ)),

$$\{\bar{a}, c, a\} + \{\bar{a}, (c, \bar{c}), a\} + \{\bar{a}, \bar{c}, a\} = 0. \tag{11}$$

But from (α) and (β), we obtain

$$\begin{aligned} \{\bar{a}, c, a\} &= \{\bar{a}, (a, b) + b, a\} = \{[\bar{a}, a], b, a\} \\ &= \{0, b, a\} = (b, a) + b = e. \end{aligned} \tag{12}$$

Again, by (β) we have,

$$\begin{aligned} \{\bar{a}, (c, \bar{c}), a\} &= \{\bar{a}, ((a, b) + b, \bar{c}), a\} = \{\bar{a}, (a, (b, \bar{c})) + (b, \bar{c}), a\} \\ &= \{[\bar{a}, a], (b, \bar{c}), a\} = \{0, (b, \bar{c}), a\} = (b, \bar{c}, a) + (b, \bar{c}). \end{aligned}$$

On the other hand we have by (γ)

$$(e, \{\bar{a}, \bar{c}, a\}) = ((b, a) + b, \{\bar{a}, \bar{c}, a\}) = (b, \bar{c}, a) + (b, \bar{c});$$

therefore $\{\bar{a}, (c, \bar{c}), a\} = (e, \{\bar{a}, \bar{c}, a\})$, from which, and (11) and (12), follows

$$e + (e, \{\bar{a}, \bar{c}, a\}) + \{\bar{a}, \bar{c}, a\} = 0,$$

and thus the statement above is proven.

In a similar way it can be shown that if $D_a \neq 0$ and $D_e \neq 0$, so that $\bar{a}$ and $\bar{e}$ exist, then $\bar{c}$ also exists and we have

$$\bar{c} = \{a, \bar{e}, \bar{a}\}.$$

The above results are summed up in the following:

THEOREM 1. *If a and b are any functions whatever of s and t, then the Fredholm determinants of $c = (a, b) + b$ and $e = (b, a) + b$ are equal.*

If $D_a \neq 0$ and $D_c \neq 0$, so that $\bar{a}$ and $\bar{c}$ exist, then $\bar{e}$ also exists, and we have $\bar{e} = (\bar{a}, \bar{c}, a) + (\bar{a}, \bar{c}) + (\bar{c}, a) + \bar{c}$; and similarly, if $D_a \neq 0$ and $D_e \neq 0$, so that $\bar{a}$ and $\bar{e}$ exist, then $\bar{c}$ also exists, and we have

$$\bar{c} = (a, \bar{e}, \bar{a}) + (a, \bar{e}) + (\bar{e}, \bar{a}) + \bar{e}.$$

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