CONCERNING SIMILARITY TRANSFORMATIONS OF LINEARLY ORDERED SETS[1]

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1. **Introduction.** As is well known, two linearly ordered sets A and B are said to be similar if there exists a 1-1 correspondence between their elements which preserves order. A function f which defines such a 1-1 correspondence may be called a similarity transformation on A to B. In this note we consider two problems concerning similarity transformations which do not seem to have received attention heretofore. The first problem is the following:

(A) *Is it true that every infinite ordered set is similar to a proper subset of itself?*[2]

Before stating the second problem we recall a classical theorem concerning well-ordered sets.[3]

THEOREM. *If the set A is well-ordered, and if f is any similarity transformation on A to a subset of A, then $f(a) \geqq a$ for every a in A.*

It is natural to inquire whether this theorem characterizes well-ordered sets—and this is our second problem; more explicitly:

(B) *Let A be a linearly ordered set such that if f is any similarity transformation on A to a subset of A then $f(a) \geqq a$ for every a in A. Is it true that any such set A is well-ordered?*

We will demonstrate that if the set A is denumerable, then the answer to both questions is in the affirmative. An example will then be constructed to show that these conclusions need not hold if the set A is nondenumerable.

2. **The denumerable case.** We obtain first the following result:

THEOREM 1. *Every denumerably infinite linearly ordered set A contains a proper subset A' to which it is similar.*

PROOF. For any two elements a and b of A, we will say that a and b are *congruent* if either $a = b$ or if there is only a finite number of ele-

[1] Presented to the Society, September 8, 1939.

[2] This question is a natural one, in view of the familiar definition of an infinite set as one which is *equivalent* to a proper subset of itself.

[3] For theorems mentioned in this paper one may refer to Hausdorff's *Grundzüge der Mengenlehre*, 1st edition, 1914, or to Sierpiński's *Leçons sur les Nombres Transfinis*, 1928.

ments in A which lie between a and b; we will indicate this by writing $a \equiv b$. The set of all elements of A congruent to a given element a will be designated by $C(a)$ and will be called the *congruence set* corresponding to a. It is obvious that if $b \in C(a)$, then $C(b) = C(a)$, and hence that any two different congruence sets have no elements in common. We now distinguish two cases.

Case 1. There is an a in A for which $C(a)$ is infinite.

Since every element of $C(a)$ has either an immediate predecessor or an immediate successor in $C(a)$, it is clear that the order-type of $C(a)$ is either ω or ω^* or $\omega^* + \omega$. Suppose, for example, that the order-type of $C(a)$ is ω. Since every element of A which is not in $C(a)$ either precedes or succeeds *all* the elements of $C(a)$, we will have $A = A_1 + A_2 + A_3$, where $A_2 = C(a)$, A_1 is the set of all elements of A which precede $C(a)$, and A_3 is the set of all the elements of A which follow $C(a)$. We now define a function f on A to a subset of A as follows:

(a) If $\alpha \in A_1$, or $\alpha \in A_3$, then $f(\alpha) = \alpha$.

(b) If $\alpha \in A_2$, then $f(\alpha)$ is the successor of α in $C(a)$.

This function is clearly a similarity transformation; moreover, the set A' into which A is transformed by f does not contain the first element of $C(a)$, and A' is thus a proper subset of A. The case where the order-type of $C(a)$ is ω^* or $\omega^* + \omega$ may be handled in an analogous fashion.

Case 2. All the different congruence sets in A are finite sets.

Since A is denumerable, the set of all congruence sets in A is also denumerable. Let $\overline{A}$ be the ordered subset of A which consists of all first elements of congruence sets in A, and let α and β, $(\alpha < \beta)$, be any two elements of $\overline{A}$. There must exist a γ in $\overline{A}$ for which $\alpha < \gamma < \beta$ (for, if not, then clearly $\alpha \equiv \beta$ in A, which would contradict the fact that α and β belong to two different congruence sets). Thus $\overline{A}$ is a denumerable dense set, and any open interval of it, for example, the set (α, β) of all elements of $\overline{A}$ between α and β, would also be a denumerable dense set. It is well known[4] that such a set contains a subset similar to any given denumerable ordered set. Let A' be a subset of (α, β) which is similar to A. It is clear that A' is a proper subset of A, and Theorem 1 is thus completely demonstrated.

THEOREM 2. *If the denumerably infinite ordered set A is such that any similarity transformation f on A to a subset of A has the property $f(a) \geq a$ for every a in A, then A is a well-ordered set.*

PROOF. We notice at once that A cannot contain a subset $\overline{A}$ which

[4] See, for example, Sierpiński, loc. cit., pp. 147–148.

is dense. For, if $\overline{A}$ is such a subset, let $\overline{\overline{A}}$ be the lower segment of $\overline{A}$ which consists of all the elements of $\overline{A}$ which precede some element b in $\overline{A}$. Then $\overline{\overline{A}}$ would be a denumerable dense set, and, as was pointed out in the proof of Theorem 1, $\overline{\overline{A}}$ would contain a subset A' similar to A, that is, there would exist a similarity transformation f on A to A'. Since $f(b) \varepsilon A'$, and all elements of A' precede b, we would obtain $f(b) < b$; but this is impossible, by our hypothesis.

We now redefine "congruent" elements of A as follows: $a \equiv b$ if and only if (1) $a = b$ or (2) the closed interval of A which consists of a and b and all the elements of A between a and b is, as an ordered subset of A, a well-ordered set. In the present sense, the congruence sets have the following properties:

(1) Two different congruence sets have no elements in common.

(2) Any upper segment, or any open or closed interval of a congruence set, is a well-ordered set.

(3) If $a \not\varepsilon C(b)$, then a either precedes all, or succeeds all of the elements of $C(b)$.

It may be emphasized, however, that a lower segment of a congruence set is well-ordered only if the congruence set has a first element. We again distinguish two cases.

Case 1. Every congruence set in A has a first element, that is, every congruence set is well-ordered.

In this case, we will show that there can be only one congruence set altogether; the set A will thus be identical with this unique congruence set, and will therefore be well-ordered. To see that we cannot have more than one congruence set here, suppose the contrary, and let α and β, $(\alpha < \beta)$, be the first elements of any two different congruence sets, which sets may be designated by $C(\alpha)$ and $C(\beta)$. If there were in A no element c which separates $C(\alpha)$ and $C(\beta)$, then the set $D = C(\alpha) + C(\beta)$, considered as an ordered subset of A, would be an interval of A which is well-ordered. This would mean that $\alpha \equiv \beta$; but this cannot be, since α and β belong to different congruence sets. Hence, there must exist a congruence set $C(\gamma)$, whose first element is γ, such that $C(\alpha) < C(\gamma) < C(\beta)$, and we would thus have $\alpha < \gamma < \beta$. This means that the set A' of all the first elements of congruence sets in A will be a dense set—contrary to the fact that A cannot contain such a subset.

Case 2. There is a congruence set C without a first element.

We will show that this supposition leads to a contradiction, so that this case cannot actually arise. The proof of our theorem will then be complete.

We can find in C a sequence of elements $c_1 > c_2 > c_3 > \cdots > c_n > \cdots$

such that for every d in C there is a natural number n for which $c_n \leqq d$. The set of elements $[c_{i+1}, c_i)$ of C between c_i and c_{i+1}, inclusive of c_{i+1}, is a well-ordered set; let β_i be the order-type of this set, and consider the sequence of ordinals $\beta_1, \beta_2, \cdots, \beta_n, \cdots$ thus defined. There can be only a finite number of indices k such that for all $n > k$, $\beta_n < \beta_k$ (otherwise one would obtain an infinite sequence of decreasing ordinals, which is impossible). Hence, there will exist an index r such that for every $p \geqq r$ there will be infinitely many indices $q > p$ for which $\beta_p \leqq \beta_q$. In other words, for $i = r, r+1, r+2, \cdots$ the set $[c_{i+1}, c_i)$ will be similar to a subset of some $[c_{q_i+1}, c_{q_i})$ where $q_i > i$, and where moreover, $q_r < q_{r+1} < q_{r+2} < \cdots$. We now express A as the sum of three sets A_1, A_2 and A_3, where A_2 consists of all the elements of C preceding c_r, A_1 consists of all elements of A preceding every element of A_2, and A_3 consists of all the elements of A following every element of A_2. Define now a function f on A to a subset of A as follows: If $\alpha \varepsilon A_1$ or $\alpha \varepsilon A_3$, $f(\alpha) = \alpha$. Suppose that $\alpha \varepsilon A_2$. There will be an $i \geqq r$ for which $\alpha \varepsilon [c_{i+1}, c_i)$, and since $[c_{i+1}, c_i)$ is similar to a subset of $[c_{q_i+1}, c_{q_i})$, there will be a γ_α in this last set which corresponds to α under such a similarity. We define, for $\alpha \varepsilon A_2$, $f(\alpha) = \gamma_\alpha$.

This function is clearly a similarity transformation on A to a subset of itself, and for any α in A_2 we have $f(\alpha) = \gamma_\alpha < \alpha$; but this is impossible, by our hypothesis.

It may be observed that up to this point it has not been necessary to make use of Zermelo's axiom of choice.

3. **The nondenumerable case.** We now prove the following result:

THEOREM 3. *The linear continuum C contains a set E, of power $\mathfrak{c}$, which is not similar to any proper subset of itself.*

PROOF. A similarity transformation of C into a subset of itself is a monotonic increasing function of a real variable, and vice versa. Hence, there are exactly $\mathfrak{c}$ such transformations. Let $\Omega_\mathfrak{c}$ denote the first ordinal to correspond to the cardinal number of the continuum, and arrange all of these transformations, with the exception of the identity, in a well-ordered series of type $\Omega_\mathfrak{c}$: $T_1, T_2, \cdots, T_\alpha, \cdots$, $(\alpha < \Omega_\mathfrak{c})$.

Now, the fixed points under a monotonic transformation which is not the identity cannot be everywhere dense in C. Hence, if T is such a transformation, there are $\mathfrak{c}$ points p such that $T(p) \neq p$. We may accordingly choose a point p_1 such that $T_1(p_1) = q_1 \neq p_1$. We now assume that p_β and q_β have been defined for all $\beta < \alpha < \Omega_\mathfrak{c}$, and choose distinct points p_α and q_α in $C - \sum_{\beta<\alpha}(p_\beta + q_\beta)$ such that $T_\alpha(p_\alpha) = q_\alpha$.

This is possible for every $\alpha < \Omega_\mathfrak{c}$, since the power of $\sum_{\beta<\alpha}(p_\beta + q_\beta)$ is less than $\mathfrak{c}$.

We notice first that if C' is any interval on C, then there exists a monotonic increasing function which is the identity on $C - C'$ and is not the identity on C'. Therefore there is an α such that $p_\alpha \in C'$. Hence the set $E = \sum_{\alpha<\Omega_\mathfrak{c}} p_\alpha$ is dense on C. It now follows that there is no similarity transformation of E into a proper subset of itself. In fact, we can show that the only similarity transformation of E into a subset of itself is the identity. For assume that τ is a similarity transformation, other than the identity, of E into a subset of itself. Since τ is not the identity and since E is dense in C, there exists an α such that T_α agrees with τ on E. But this is impossible, since $T_\alpha(p_\alpha) = q_\alpha$, and q_α is not a point of E.

THEOREM 4. *The linear continuum contains a set E which is not well-ordered, but which has the property that if f is any similarity transformation of E into a subset of itself, then $f(e) \geqq e$ for every e in E.*

PROOF. The set E constructed in the proof of Theorem 3 satisfies these requirements.

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