# AN EXTENSION OF A COVARIANT DIFFERENTIATION PROCESS[1]

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Craig[2] has considered tensors $T^{\alpha\cdots}_{\beta\cdots}$ whose components are functions of $n$ variables represented by $x$ and their $m$ derivatives $x', x'', \cdots, x^{(m)}$. He obtained the covariant derivative

$$T^{\alpha\cdots}_{\beta\cdots x^{(m-1)\gamma}} - mT^{\alpha\cdots}_{\beta\cdots x^{(m)\lambda}}\left\{{\lambda \atop \gamma}\right\}, \qquad m \geqq 2, \tag{1}$$

where

$$\left\{{\lambda \atop \gamma}\right\} \equiv x'^{\alpha}\Gamma^{\lambda}_{\gamma\alpha} + (1/2)x''^{\beta}f_{\gamma\delta\beta}f^{\delta\lambda}, \tag{2}$$

and partial differentiation in (1) is denoted by the added subscript. Throughout, a repeated letter in one term indicates a sum of $n$ terms. The purpose of this note is to derive another tensor from $T^{\alpha\cdots}_{\beta\cdots}$ whose covariant rank is one larger. The general process will be shown clearly by using $T^{\alpha}(x, x', x'', x''')$.

The extended point transformation

$$x^{\alpha} = x^{\alpha}(y), \qquad x'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}}\, y'^{i},$$

$$x''^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}}\, y''^{i} + \frac{\partial^{2} x^{\alpha}}{\partial y^{i}\partial y^{j}}\, y'^{i}y'^{j}, \cdots, \qquad \alpha = 1, \cdots, n,$$

gives the tensor equations of transformation of the tensor $T^{\alpha}$ as

$$\overline{T}^{i}(y, y', y'', y''') = T^{\alpha}(x, x', x'', x''')\partial y^{i}/\partial x^{\alpha}, \tag{3}$$

where $y$ stands for the $n$ variables $y^1, y^2, \cdots, y^n$ and a similar notation is used for the derivatives $y'$, $y''$, and $y'''$. On differentiating equations (3) with respect to $y'^{k}$ it is found that

$$\overline{T}^{i}_{y'^{k}} = \left(T^{\alpha}_{x'^{\beta}}\frac{\partial x^{\beta}}{\partial y^{k}} + T^{\alpha}_{x''^{\beta}}\frac{\partial x''^{\beta}}{\partial y'^{k}} + T^{\alpha}_{x'''^{\beta}}\frac{\partial x'''^{\beta}}{\partial y'^{k}}\right)\partial y^{i}/\partial x^{\alpha}. \tag{4}$$

The derivatives can be expressed by using the following general formulas:

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[1] Presented to the Society, April 15, 1939.

[2] H. V. Craig, *On a covariant differentiation process*, this Bulletin, vol. 37 (1931), pp. 731–734.

$$(5)\qquad \frac{\partial x^{(m-1)\beta}}{\partial y^{(m-2)k}} = (m-1)\frac{\partial x'^{\beta}}{\partial y^{k}}, \qquad \frac{\partial x^{(m)\beta}}{\partial y^{(m-2)k}} = \frac{m(m-1)}{2}\frac{\partial x''^{\beta}}{\partial y^{k}},$$

in which $\partial x'^{\beta}/\partial y^{k}$ are eliminated by[3]

$$(6)\qquad \overline{\left\{ {l \atop k} \right\}} \frac{\partial x^{\beta}}{\partial y^{l}} = \frac{\partial x'^{\beta}}{\partial y^{k}} + \left\{ {\beta \atop \gamma} \right\} \frac{\partial x^{\gamma}}{\partial y^{k}} \cdot$$

The derivatives $\partial x''^{\beta}/\partial y^{k}$ are simplified by first writing

$$(7)\qquad x''^{\beta} = \frac{\partial x^{\beta}}{\partial y^{j}} y''^{j} + \overline{\Gamma}^{r}_{jk} y'^{j} y'^{k} \frac{\partial x^{\beta}}{\partial y^{r}} - \Gamma^{\beta}_{\alpha\delta} x'^{\alpha} x'^{\delta},$$

with the help of (2), (6) and[4] $f_{\alpha\beta\gamma}x'^{\beta}=0$. It is necessary also to have

$$(8)\qquad \frac{\partial^{2} x^{\beta}}{\partial y^{j}\partial y^{k}} = \overline{\Lambda}^{t}_{jk} \frac{\partial x^{\beta}}{\partial y^{t}} - \Lambda^{\beta}_{\alpha\delta} \frac{\partial x^{\alpha}}{\partial y^{j}} \frac{\partial x^{\delta}}{\partial y^{k}},$$

where

$$\Lambda^{\beta}_{\alpha\delta} = \Gamma^{\beta}_{\alpha\delta} - (1/2) f^{\beta\gamma}\left(f_{\delta\gamma\tau}\left\{ {\tau \atop \alpha} \right\} + f_{\gamma\alpha\tau}\left\{ {\tau \atop \delta} \right\} - f_{\alpha\delta\tau}\left\{ {\tau \atop \gamma} \right\}\right).$$

This is obtained from Taylor's[5] formula (19) in the following way. Multiply this formula by $(\partial y^{k}/\partial x^{\epsilon})f^{\beta\epsilon} = (\partial x^{\beta}/\partial y^{l})\bar{f}^{kl}$, and sum for $k$. Use the tensor equations for $f_{\alpha\beta\gamma}$ and substitute from (6) for $\partial x'^{\gamma}/\partial y^{j}$.

By means of formulas (6) and (8) and the tensor $Q^{\beta}(x, x', x'') \equiv x''^{\beta} + \Gamma^{\beta}_{\alpha\delta}x'^{\alpha}x'^{\delta}$ the partial derivatives of (7) have the form

$$(9)\qquad \frac{\partial x''^{\beta}}{\partial y^{k}} = -\left| {\beta \atop \gamma} \right| \frac{\partial x^{\gamma}}{\partial y^{k}} + \overline{\left| {r \atop k} \right|} \frac{\partial x^{\beta}}{\partial y^{r}} - 2\left\{ {\beta \atop \alpha} \right\} \overline{\left\{ {l \atop k} \right\}} \frac{\partial x^{\alpha}}{\partial y^{l}} + 2\overline{\left\{ {r \atop i} \right\}}\,\overline{\left\{ {i \atop k} \right\}} \frac{\partial x^{\beta}}{\partial y^{r}},$$

in which we have the nontensor form

$$(10)\qquad \left| {\beta \atop \gamma} \right| = Q^{\beta}_{x^{\gamma}} - Q^{\beta}_{x'^{\alpha}}\left\{ {\alpha \atop \gamma} \right\} + Q^{\alpha}\Lambda^{\beta}_{\alpha\gamma}.$$

If formulas (6) and (9) are substituted in equations (5) and the results used in (4), we find

$$\overline{T}^{i}_{y'^{k}} = \left(T^{\alpha}_{x'^{\beta}} - 2T^{\alpha}_{x''^{\delta}}\left\{ {\delta \atop \beta} \right\} - 3T^{\alpha}_{x'''^{\delta}}\left| {\delta \atop \beta} \right|\right) \frac{\partial x^{\beta}}{\partial y^{k}} \frac{\partial y^{i}}{\partial x^{\alpha}}$$
$$- \left(-2\overline{T}^{i}_{y''^{l}}\overline{\left\{ {l \atop k} \right\}} - 3\overline{T}^{i}_{y'''^{l}}\overline{\left| {l \atop k} \right|}\right).$$

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[3] J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*' Transactions of this Society, vol. 27 (1925), p. 255.

[4] J. H. Taylor, loc. cit., p. 248.

[5] J. H. Taylor, loc. cit., p. 254.

Hence the new tensor whose covariant rank has been increased by one is

$$T^{\alpha}_{x'\beta} - 2T^{\alpha}_{x''\delta}\left\{{\delta \atop \beta}\right\} - 3T^{\alpha}_{x'''\delta}\left|{\delta \atop \beta}\right|, \tag{11}$$

where $\left\{{\delta \atop \beta}\right\}$ and $\left|{\delta \atop \beta}\right|$ are defined in (2) and (10).

Because of the general relations in (5) it is easy to verify that the tensor

$$T^{\alpha\cdots}_{\gamma\cdots x^{(m-2)}\beta} - (m-1)T^{\alpha\cdots}_{\gamma\cdots x^{(m-1)}\delta}\left\{{\delta \atop \beta}\right\} - \frac{m(m-1)}{2}T^{\alpha\cdots}_{\gamma\cdots x^{(m)}\delta}\left|{\delta \atop \beta}\right|, \quad m \geqq 3, \tag{12}$$

has a covariant rank which is one larger than that of $T^{\alpha\cdots}_{\gamma\cdots}$ whose components are functions of $(x, x', \cdots, x^{(m)})$.

If the components of the tensor $T^{\alpha}(x, x', x'', x''')$ do not contain the derivatives $x'''$, then (11) reduces to Craig's covariant derivative (1), and if there are no $x''$ or $x'''$ derivatives, then the result is a partial differentiation with respect to $x'$.

The usual rules for the derivative of a sum of tensors of the same type and rank and for the product of any tensors are preserved by this provess.

If $m=2$, a scalar $T(x, x', x'')$ will give a covariant tensor which is similar to that in (11) when the tensor equations for $\overline{T}(y, y', y'')$ are differentiated with respect to $y$ instead of $y'$. The tensor is

$$T_{x\beta} - T_{x'\delta}\left\{{\delta \atop \beta}\right\} - T_{x''\delta}\left|{\delta \atop \beta}\right|. \tag{13}$$

However, if $m=2$ and a tensor $T^{\alpha}(x, x', x'')$ is used, an extra term $T^{\delta}\Lambda^{\alpha}_{\delta\beta}$ has to be added to three terms similar to those in (13). If this process is performed on the tensor $Q^{\alpha}(x, x', x'')$, the result is the zero tensor.

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