CREMONA INVOLUTIONS DETERMINED BY A PENCIL OF SURFACES*

FRANK C. GENTRY

1. **Introduction.** The characteristics of the involutorial Cremona transformations determined by a pencil of surfaces of order n and containing an $(n-2)$-fold line d have been determined by Carroll [1]. The particular features of these transformations, which arise when the surfaces of the pencil are of order 3 and the curve residual to the line d in the base of the pencil is composite, have been considered in some detail by the same author [2]. Snyder [3] has suggested that a similar study of involutions defined by surfaces of higher order might be of interest.

The transformation is defined by Carroll as follows. Let

$$\lambda_1 F_n(x) - \lambda_2 F_n'(x) = 0 \tag{1}$$

be a pencil of surfaces of order n containing the line $d \equiv x_1 = 0,\ x_2 = 0$ to multiplicity $n-2$. Let $(z) = (0, 0, z_3, z_4)$ be a variable point on the line d, and let the pencil of surfaces (1) be connected with (z) by the relation

$$z_3\phi_1(\lambda_1, \lambda_2) - z_4\phi_2(\lambda_1, \lambda_2) = 0, \tag{2}$$

where ϕ_i, $(i=1, 2)$, is a binary form of order k. A point (y) of space determines a surface of the pencil (1) and hence a value of the ratio $\lambda_1:\lambda_2$, which in turn determines a point (z) of d. The line joining (y) and (z) meets the member of (1) determined by (y) in one further point (y'), the transform of (y) in the involution. The characteristics of the transformation are

$$\begin{aligned}
S_1 &\sim S_{2n(k+1)-1}: d^{2(n-2)(k+1)}\, k(n-2)\bar{d}^2 \\
&\qquad C_{4n-4}^{2k+1}\,\{(6n-8)k + 6n - 10\}g, \\
d &\sim T_{2n(k+1)-2}: d^{2(n-2)(k+1)}\, k(n-2)\bar{d} \\
&\qquad C_{4n-4}^{2k+1}\,\{(6n-8)k + 6n - 10\}g, \\
C_{4n-4} &\sim \Sigma_{4n(k+1)-4}: d^{4(n-2)(k+1)}\, k(n-2)\bar{d}^4 \\
&\qquad C_{4n-4}^{4k+1}\,\{(6n-8)k + 6n - 10\}g^2, \\
R_{nk+2n-1} &\sim R_{nk+2n-1}: d^{(n-2)(k+2)}\, k(n-2)\bar{d} \\
&\qquad C_{4n-4}^{k+1}\,\{(6n-8)k + 6n - 10\}g,
\end{aligned} \tag{3}$$

* Presented to the Society, September 6, 1938.

where C_{4n-4} is the base curve of the pencil of surfaces residual to d, T and Σ are the principal surfaces corresponding to d and C_{4n-4} respectively, R is the surface of coincident points and the $\{(6n-8)k+6n-10\}g$ are fundamental curves of the second kind which are all lines.

2. **Noether's map.** Noether [4] has shown that a surface of order n containing an $(n-2)$-fold line can be mapped on a plane by means of a web of curves of order n and genus $n-2$ whose base is made up of a fixed $(n-2)$-fold point N and $3n-4$ simple points p_i, $(i=1, \cdots, 3n-4)$. That is,

$$[N^{n-2} p_1 p_2 \cdots p_{3n-4}]^n \sim C_n,$$

the generic plane section of the surface S_n. Contained in the web of plane curves is the pencil $[N^1]^1[N^{n-3}p_1p_2 \cdots p_{3n-4}]^{n-1}$. But the pencil of lines $[N^1]^1$ corresponds to the pencil of conics C_2 cut out on the surface by a pencil of planes through the $(n-2)$-fold line d. Hence the uniquely determined curve $[N^{n-3}p_1p_2 \cdots p_{3n-4}]^{n-1} \sim d$. Another member of the pencil of surfaces meets S_n in a curve of order n^2 made of d counted $(n-2)^2$ times and the residual curve C_{4n-4} of order $4n-4$. The map of the complete intersection in the plane is of the form

$$[N^{n(n-2)} p_1^n p_2^n \cdots p_{3n-4}^n]^{n^2}.$$

Hence

$$C_{4n-4} \sim [N^{3n-6} p_1^2 p_2^2 \cdots p_{3n-4}^2]^{3n-2}.$$

It will happen $3n-4$ times that a line of the pencil on N coincides with one of the $3n-4$ lines $[Np_i]^1$. In this case the map of the plane section becomes $[Np_i]^1 p_i^* [N^{n-3}p_1p_2 \cdots p_{3n-4}]^{n-1}$, which corresponds to the $(n-2)$-fold line d and a composite conic made up of the 2 lines $u_i \sim [Np_i]^1$ and $v_i \sim p_i^*$, the directions about the base point p_i. The $6n-8$ lines u_i, v_i, $(i=1, 2, 3, \cdots, 3n-4)$, which lie by pairs in $3n-4$ planes through the $(n-2)$-fold line d, are the only lines on the surface S_n. Noether mentions that, aside from these lines and the pencil of conics C_2, to which they belong, there are no curves of order less than $n-2$ on the surface. The curve C_{4n-4} is of genus

$$(3n-3)(3n-4)/2-(3n-6)(3n-7)/2-(3n-4)=6n-11.$$

It meets d $4n-8$ times, each u_i and v_i 3 times and each C_2 4 times.

It is the purpose of this paper to examine those cases which arise when C_{4n-4} becomes composite and to determine what effect is produced on the order of the transformation and the configuration of its fundamental curves.

3. **Lines as components of base.** Let the map of C_{4n-4} be of the form

$$\overset{*}{p}_1 \overset{*}{p}_2 \overset{*}{p}_3 \cdots \overset{*}{p}_m \left[N^{3n-6} \overset{3}{p}_1 \overset{3}{p}_2 \overset{3}{p}_3 \cdots \overset{3}{p}_m \overset{2}{p}_{m+1} \cdots \overset{2}{p}_{3n-4}\right]^{3n-2}.$$

Then $C_{4n-4} \equiv v_1 v_2 \cdots v_m C_{4n-m-4} \cdot C_{4n-m-4}$ is of genus $6n-2m-11$, and meets d $4n-m-8$ times and v_i, $(i=1, 2, 3, \cdots, m)$, 3 times. Hence it meets the plane $(dv_i)^1$ in one variable point. Through any point (y) of this plane there can be drawn one and only one line meeting d, v_i and C_{4n-m-4} each once and hence lying entirely on the surface of the pencil (1) determined by (y). Now by (2), to each point of d there corresponds k surfaces of the pencil (1). From any point on d there can be drawn one line meeting v_i and C_{4n-m-4}. This determines a $(1, k)$ algebraic correspondence of valence zero which has $k+1$ coincidences. Therefore, it happens $k+1$ times that a line joining a point (y) to the point (z) of d corresponding to the surface of the pencil (1) determined by (y) lies entirely on that surface. These lines are fundamental curves of the second kind of the type (dv_iC_{4n-m-4}). There are $3n-m-4$ lines u_i and $3n-m-4$ lines v_i, $(i=m+1, m+2, \cdots, 3n-4)$, on each surface S_n meeting C_{4n-m-4} twice. From any point of d there can be drawn

$$(4n-m-5)(4n-m-6)/2-(4n-m-8)(4n-m-9)/2 \\ -(6n-2m-11) = 6n-m-10$$

lines meeting C_{4n-m-4} twice each. Hence there are $(6n-2m-8)k+6n-m-10$ F-lines of the type (dC^2_{4n-m-4}). The remaining mk F-lines of the transformation are accounted for by each v_i, $(i=1, 2, 3, \cdots, m)$, counting k times as an F-curve of the second kind, as well as an F-curve of the first kind. In order to determine the maximum possible number of lines v_i which may break away from C_{4n-4}, we note that the residual curve can exist on the surface S_n if and only if its map can exist in the plane under the prescribed conditions on its base. That is, $(3n-2)(3n+1)/2-(3n-6)(3n-5)/2-6m-3(3n-m-4) \geqq 0$ or $m \leqq 2n-2$. The same results are obtained if the map of C_{4n-4} is of the form

$$[Np_1]^1[Np_2]^1 \cdots [Np_m]^1\left[N^{3n-m-6} p_1 \cdots p_m \overset{2}{p}_{m+1} \cdots \overset{2}{p}_{3n-4}\right]^{3n-m-2}$$

except that v_i is replaced by u_i, $(i=1, 2, 3, \cdots, m)$.

4. **Base having conics as components.** Let the map of C_{4n-4} be of the form

$$\{[N]^1\}^m \cdot [N^{3n-m-6} p_1^2 p_2^2 \cdots p_{3n-4}^2]^{3n-m-2}.$$

It corresponds to $C_2' C_2'' \cdots C_2^{(m)} C_{4n-2m-4, 6n-3m-11}$, that is, to m conics and a curve of order $4n-2m-4$ and genus $6n-3m-11$. Each conic meets d twice, and $C_{4n-2m-4}$ meets d $4n-2m-8$ times and each conic 4 times. There are $(6n-8)k+6n-3m-10$ F-lines of the type $(dC_{4n-2m-4}^2)$ and no others. Hence each such conic reduces the number of F-lines of the second kind by 3. An examination of (3) shows that the plane of each such conic factors once out of each member of the homaloidal web and the P-surface $T_{2n(k+1)-2}$ of the transformation, and twice out of the P-surface $\Sigma_{4n(k+1)-4}$. Consequently, if there are m such conics, the order of the transformation is reduced to $2n(k+1)-m-1$. To determine the maximum possible number of such conics, we have

$$(3n - m - 2)(3n - m + 1)/2 - (3n - m - 6)(3n - m - 5)/2$$
$$- 3(3n - 4) = 6n - 5m - 4 \geqq 0 \text{ or } m \leqq (6n - 4)/5.$$

That is, $m=n+j$ when $n=5j+h$, $(h=4, 5, 6, 7, 8; j=0, 1, 2, 3, \cdots)$. Obviously, C_{4n-4} cannot be composed entirely of conics of this type for any value of n, since each conic must meet d twice, while C_{4n-4} meets d but $4n-8$ times. If we impose the condition that $2m=4n-8$, then $2n-4 \leqq (6n-4)/5$ and $n \leqq 4$. For $n=3, 4$, C_{4n-4} is composed of $2n-4$ conics and a $C_{4,1}$ which meets each conic 4 times but does not meet d. If $n=3$ the order of the transformation is $6k+3$ and there are $10k+2$ F-lines; if $n=4$ the order is $8k+3$ and there are $16k+2$ F-lines. The same results are obtained if some of the conics are composite, since the map of the curve residual to the conics is the same as before.

5. **One nonplanar component in base.** Let C_{4n-4} be composed of a single nondegenerate curve of order $N_1 \geqq n-2$ and conics and lines of the type mentioned above. Its map is then of the form:

$$[N^r p_1^2 p_2^2 \cdots p_s^2 p_{s+1} \cdots p_{s+j}]^m \{[N^1]^1\}^t [N p_{s+1}]^1 \cdots [N p_{s+j}]^1,$$

where

$$(4) \qquad m + t + j = 3n - 2, \quad r + t + j = 3n - 6, \quad s + j = 3n - 4,$$

and

$$(5) \qquad \begin{aligned} (m - 1)(m - 2)/2 - r(r - 1)/2 - s &\geqq 0, \\ m(m + 3)/2 - r(r + 1)/2 - 3s - j &\geqq 0. \end{aligned}$$

From these conditions it follows that

$$(6)\quad \begin{array}{lll} 4 \leqq m \leqq 3n-2, & r = m-4, & m-2 \leqq s \leqq 3m-9, \\ 0 \leqq j = 3n-s-4 \leqq 5m-3s-6, & & 0 \leqq t = s-m+2. \end{array}$$

Any set of integers which satisfies (6) will then constitute the characteristic of the map of a possible form of C_{4n-4}. The genus of the curve C_{N_1} is $p=3m-s-9$, and

$$N_1 = nm - (n-2)(m-4) - 2s - j = 4n + 2m - 2s - j - 8.$$

The curve C_{N_1} meets $d(n-1)m-(n-3)(m-4)-2s-j=N_1-4$ times. There are $2s$ of the lines u_i, v_i, $(i=1, 2, \cdots, s)$, which meet C_{N_1} twice each. From any point on d there can be drawn

$$(N_1-1)(N_1-2)/2 - (N_1-4)(N_1-5)/2 - p = 12n + 3m - 5s - 3j - 24$$

lines which meet C_{N_1} twice each. Hence, there are $2sk+12n+3m-5s-3j-24$, or $2sk+3N_1-3m+s$ F-lines of the type $(dC_{N_1}^2)$. The curve C_{N_1} meets each of the planes $(du_i)^1$, $(i=s+1, \cdots, s+j)$, in one variable point. Hence, there are $j(k+1)$ F-lines of the type $(dC_{N_1}u_i)$. Each of the lines u_i counts k times as an F-curve of the second kind.

6. **Two nonplanar components in base.** Let C_{4n-4} be composed of two curves of orders $N_1 \geqq n-2$ and $N_2 \geqq n-2$ and of genera p_1 and p_2 respectively, besides conics and lines. Let the map of C_{N_1} have the characteristic $(m_1r_1s_1j_1)$ where m_1 is the order, r_1 is the multiplicity at N, s_1 is the number of double points at base points p_i, and j_1 is the number of simple points at base points p_i. Let the map of C_{N_2} have the characteristic $(m_2r_2s_2j_2)$ with respect to the same base, and let the mapping curves have a simple points in common. Then

$$(7)\quad \begin{aligned} m_1 + m_2 + t + j_1 + j_2 - 2a &= 3n-2, \\ r_1 + r_2 + t + j_1 + j_2 - 2a &= 3n-6, \\ s_1 + s_2 + j_1 + j_2 - a &= 3n-4, \end{aligned}$$

where t is the number of conics which meet d twice. From the first two of the relations (7) $m_1-r_1+m_2-r_2=4$, whence either $r_1=m_1-1$, $r_2=m_2-3$, or $r_1=m_1-2$, $r_2=m_2-2$. In the first case

$$(8)\quad \begin{array}{ll} r_1 = m_1 - 1, & r_2 = m_2 - 3, \\ s_1 = 0, & 2m_2 - s_2 - 5 \geqq 0, \\ 2m_1 - j_1 \geqq 0, & 4m_2 - 3s_2 - j_2 - 3 \geqq 0. \end{array}$$

Hence

$$(9)\quad \begin{array}{ll} 1 \leqq m_1 \leqq 3n-5, & 3 \leqq m_2 \leqq 3n - m_1 - 2, \\ & 0 \leqq s_2 \leqq 2m_2 - 5, \\ 0 \leqq j_1 \leqq 2m_1, & 0 \leqq j_2 \leqq 4m_2 - 3s_2 - 3, \\ 0 \leqq a = j_1 + j_2 + s_2 - 3n + 4, & 0 \leqq t = a + s_2 - m_1 - m_2 + 2. \end{array}$$

Any set of integers satisfying the conditions (8) and (9) constitutes the characteristic of the map of a possible form of C_{4n-4}. Then $N_1 = n+2m_1-j_1-2, p_1=0$, $N_2=3n+2m_2-2s_2-j_2-6$, and $p_2=2m_2-s_2-5$. The curve C_{N_1} meets d N_1-1 times and C_{N_2} meets it N_2-3 times. The F-curves of the second kind are arranged as follows:

$$(10)\quad \begin{array}{r} (s+a)k + 2N_2 - 2m_2 + s_2 \text{ of the type } (dC_{N_2}^2), \\ (s+a)k + 3N_1 + N_2 - 3m_1 - m_2 + a \text{ of the type } (dC_{N_1}C_{N_2}), \\ (j_1 - a)(k+1) \text{ of the type } (du_iC_{N_1}), \\ (j_2 - a)(k+1) \text{ of the type } (du_iC_{N_2}), \\ j_1 + j_2 - 2a \text{ lines } u_i \text{ each counted } k \text{ times.} \end{array}$$

In the second case:

$$(11)\quad \begin{array}{ll} r_1 = m_1 - 2, & r_2 = m_2 - 2, \\ m_1 - s_1 - 2 \geqq 0, & m_2 - s_2 - 2 \geqq 0, \\ 3m_1 - 3s_1 - j_1 - 1 \geqq 0, & 3m_2 - 3s_2 - j_2 - 1 \geqq 0. \end{array}$$

Hence

$$(12)\quad \begin{array}{ll} 2 \leqq m_1 \leqq 3n-4, & 2 \leqq m_2 \leqq 3n - m_1 - 2, \\ 0 \leqq s_1 \leqq m_1 - 2, & 0 \leqq s_2 \leqq m_2 - 2, \\ 0 \leqq j_1 \leqq 3m_1 - 3s_1 - 1, & 0 \leqq j_2 \leqq 3m_2 - 3s_2 - 1, \\ 0 \leqq a = s_1 + s_2 + j_1 + j_2 - 3n + 4, & 0 \leqq t = a + s_1 + s_2 - m_1 - m_2 + 2. \end{array}$$

Any set of integers satisfying (11) and (12) will then constitute the characteristic of the map of a possible form of C_{4n-4}, except one containing the characteristic (3 1 0 8). These cases must be barred because this is the characteristic of the map of the multiple line d which cannot form a part of C_{4n-4}. We have $N_1 = 2n+2m_1-2s_1-j_1-4$, $p_1 = m_1 - s_1 - 2$, $N_2 = 2n+2m_2-2s_2-j_2-4$, and $p_2 = m_2 - s_2 - 2$. The curves C_{N_1} and C_{N_2} meet d N_1-2 and N_2-2 times, respectively. The F-curves of the second kind are arranged as follows:

$$
(13)\quad
\begin{aligned}
(s_1 + s_2)k + N_1 - m_1 + s_1 &\text{ of the type } (dC^2_{N_1}),\\
(s_1 + s_2)k + N_2 - m_2 + s_2 &\text{ of the type } (dC^2_{N_2}),\\
2ak + 2N_1 + 2N_2 - 2m_1 - 2m_2 + a &\text{ of the type } (dC_{N_1}C_{N_2}),\\
(j_1 - a)(k + 1) &\text{ of the type } (dC_{N_1}u_i),\\
(j_2 - a)(k + 1) &\text{ of the type } (dC_{N_2}u_i),\\
(j_1 + j_2 - 2a)\ u_i &\text{ each counted } k \text{ times.}
\end{aligned}
$$

7. **Three nonplanar components in base.** In addition to possible conics and lines, let C_{4n-4} be composed of 3 curves C_{N_1,p_1}, C_{N_2,p_2}, C_{N_3,p_3} whose maps have the characteristics $(m_1r_1s_1j_1)$, $(m_2r_2s_2j_2)$, and $(m_3r_3s_3j_3)$, respectively. Let the maps of C_{N_1} and C_{N_2} have a simple base points in common, those of C_{N_2} and C_{N_3} b such points, and those of C_{N_1} and C_{N_3} c such points. Then

$$
(14)\quad
\begin{aligned}
m_1 + m_2 + m_3 + t + j_1 + j_2 + j_3 - 2a - 2b - 2c &= 3n - 2,\\
r_1 + r_2 + r_3 + t + j_1 + j_2 + j_3 - 2a - 2b - 2c &= 3n - 6,\\
s_1 + s_2 + s_3 + j_1 + j_2 + j_3 - a - b - c &= 3n - 4.
\end{aligned}
$$

From the first 2 of the relations (14) it follows that, without loss of generality, we can set $r_1 = m_1 - 1$, $r_2 = m_2 - 1$, $r_3 = m_3 - 2$. Then

$$
(15)\quad
\begin{aligned}
s_1 &= 0, & s_2 &= 0, & m_3 - s_3 - 2 &\geqq 0,\\
2m_1 - j_1 &\geqq 0, & 2m_2 - j_2 &\geqq 0, & 3m_3 - 3s_3 - j_3 - 1 &\geqq 0.
\end{aligned}
$$

Hence

$$
(16)\quad
\begin{aligned}
&0 \leqq m_1 \leqq 3n-5, \quad 0 \leqq m_2 \leqq 3n-m_1-4, \quad 0 \leqq m_3 \leqq 3n-m_1-m_2-2,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leqq s_3 \leqq m_3-2,\\
&0 \leqq j_1 \leqq 2m_1, \qquad 0 \leqq j_2 \leqq 2m_2, \qquad 0 \leqq j_3 \leqq 3m_3-3s_3-1,\\
&0 \leqq a+c \leqq j_1, \qquad 0 \leqq a+b \leqq j_2, \qquad 0 \leqq b+c \leqq j_3,\\
&\qquad 0 \leqq a+b+c = j_1+j_2+j_3+s_3-3n+4,\\
&\qquad 0 \leqq t = a+b+c+s_3-m_1-m_2-m_3+2.
\end{aligned}
$$

Any set of integers satisfying (15) and (16) will constitute the characteristic of the map of a possible form of C_{4n-4} except one containing the characteristic (3 1 0 8). We have $N_1 = n+2m_1-j_1-2$, $p_1 = 0$, $N_2 = n+2m_2-j_2-2$, $p_2 = 0$, $N_3 = 2n+2m_3-2s_3-j_3-4$, and $p_3 = m_3 - s_3 - 2$. The curves C_{N_1}, C_{N_2}, and C_{N_3} meet d $N_1 - 1$, $N_2 - 1$, and $N_3 - 2$ times, respectively. The F-curves of the second kind are arranged thus:

$$
(17)\qquad
\begin{aligned}
&(s_3 + a)k + N_1 + N_2 - m_1 - m_2 + a \text{ of the type } (dC_{N_1}C_{N_2}),\\
&(s_3 + a)k + N_3 - m_3 + s_3 \text{ of the type } (dC_{N_3}^2),\\
&(b + c)k + 2N_1 + N_3 - 2m_1 - m_3 + c \text{ of the type } (dC_{N_1}C_{N_3}),\\
&(b + c)k + 2N_2 + N_3 - 2m_2 - m_3 + b \text{ of the type } (dC_{N_2}C_{N_3}),\\
&(j_1 - a - c)(k + 1) \text{ of the type } (dC_{N_1}u_i),\\
&(j_2 - a - b)(k + 1) \text{ of the type } (dC_{N_2}u_i),\\
&(j_3 - b - c)(k + 1) \text{ of the type } (dC_{N_3}u_i),\\
&(j_1 + j_2 + j_3 - 2a - 2b - 2c)\ u_i \text{ each counted } k \text{ times.}
\end{aligned}
$$

8. **Four nonplanar components in base.** Finally, let C_{4n-4} be made up of 4 curves C_{N_h,p_h}, $(h=1, 2, 3, 4)$, aside from conics and lines, and let the map of C_{N_h} in the plane have the characteristic $(m_h r_h s_h j_h)$. Let the maps of C_{N_1} and C_{N_2} have a simple base points in common, those of C_{N_1} and C_{N_3} b such points, those of C_{N_1} and C_{N_4} c such points, those of C_{N_2} and C_{N_3} d such points, those of C_{N_2} and C_{N_4} e such points, and those of C_{N_3} and C_{N_4} f such points.
Then

$$
(18)\qquad
\begin{aligned}
&m_1 + m_2 + m_3 + m_4 + t + j_1 + j_2 + j_3 + j_4\\
&\qquad - 2a - 2b - 2c - 2d - 2e - 2f = 3n - 2,\\
&r_1 + r_2 + r_3 + r_4 + t + j_1 + j_2 + j_3 + j_4\\
&\qquad - 2a - 2b - 2c - 2d - 2e - 2f = 3n - 6,\\
&s_1 + s_2 + s_3 + s_4 + j_1 + j_2 + j_3 + j_4\\
&\qquad - a - b - c - d - e - f = 3n - 4.
\end{aligned}
$$

From these conditions it follows that

$$
(19)\qquad r_h = m_h - 1, \qquad s_h = 0, \qquad 2m_h - j_h \geqq 0, \qquad h = 1, 2, 3, 4.
$$

Hence

$$
(20)\qquad
\begin{aligned}
&1 \leqq m_1 \leqq 3n-5, && 0 \leqq j_h \leqq 2m_h,\\
&1 \leqq m_2 \leqq 3n-m_1-4, && 0 \leqq a+b+c \leqq j_1,\\
&1 \leqq m_3 \leqq 3n-m_1-m_2-3, && 0 \leqq a+d+e \leqq j_2,\\
&1 \leqq m_4 \leqq 3n-m_1-m_2-m_3-2, && 0 \leqq b+d+f \leqq j_3,\\
& && 0 \leqq c+e+f \leqq j_4,\\
&0 \leqq t = a+b+c+d+e+f-m_1-m_2-m_3-m_4+2 \leqq 3n-6.
\end{aligned}
$$

Any set of integers satisfying conditions (19) and (20) will constitute the characteristic of a possible map of C_{4n-4}. We have $N_h=n+2m_h-j_h-2$ and $p_h=0$, $(h=1, 2, 3, 4)$. The curve C_{N_h} meets d N_h-1 times. The F-lines of the second kind may be classified as follows:

$$
\begin{aligned}
&(a+f)k+N_1+N_2-m_1-m_2+a \text{ of the type } (dC_{N_1}C_{N_2}),\\
&(b+e)k+N_1+N_3-m_1-m_3+b \text{ of the type } (dC_{N_1}C_{N_3}),\\
&(c+d)k+N_1+N_4-m_1-m_4+c \text{ of the type } (dC_{N_1}C_{N_4}),\\
&(c+d)k+N_2+N_3-m_2-m_3+d \text{ of the type } (dC_{N_2}C_{N_3}),\\
&(b+e)k+N_2+N_4-m_2-m_4+e \text{ of the type } (dC_{N_2}C_{N_4}),\\
&(a+f)k+N_3+N_4-m_3-m_4+f \text{ of the type } (dC_{N_3}C_{N_4}),\\
&(j_1-a-b-c)(k+1) \text{ of the type } (dC_{N_1}u_i),\\
&(j_2-a-d-e)(k+1) \text{ of the type } (dC_{N_2}u_i),\\
&(j_3-b-d-f)(k+1) \text{ of the type } (dC_{N_3}u_i),\\
&(j_4-c-e-f)(k+1) \text{ of the type } (dC_{N_4}u_i),\\
&(j_1+j_2+j_3+j_4-2a-2b-2c-2d-2e-2f)\ u_i \text{ each counted } k \text{ times.}
\end{aligned}
\tag{21}
$$

C_{4n-4} cannot contain more than four curves of order greater than or equal to $n-2$.

9. **Conclusion.** All the possible forms which C_{4n-4} may take have been determined. It has been shown that if C_{4n-4} contains m conics each meeting the multiple line d twice, then the order of the transformation is reduced by m, but that no other form reduces that order. The configuration of the F-curves of the second kind varies widely from case to case, but their total number is always $(6n-8)k+6n-3m-10$.

BIBLIOGRAPHY

1. E. T. Carroll, *Systems of involutorial birational transformations contained multiply in special linear complexes*, American Journal of Mathematics, vol. 54 (1932), pp. 707–717.

2. E. Carroll-Rusk, *Cremona involutions defined by a pencil of cubic surfaces*, American Journal of Mathematics, vol. 56 (1934), pp. 96–108.

3. V. Snyder, *Some recent contributions to algebraic geometry*, this Bulletin, vol. 40 (1934), pp. 673–687.

4. M. Noether, *Ueber Flachen, welche Schaaren rationaler Curven besitzen*, Mathematische Annalen, vol. 3 (1871), pp. 161–227.

LOUISIANA POLYTECHNIC INSTITUTE