The method here described is based on the suggestions made by Mr. Frederick King. These suggestions have led to the evaluation of $R(x)$ as a starting point of the subsequent discussion.

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ALL INTEGERS EXCEPT 23 AND 239 ARE SUMS OF EIGHT CUBES

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Summary. In 1770 Waring stated that every positive integer is a sum of nine integral nonnegative cubes. The first proof is due to Wieferich.* I shall prove the following new result.

THEOREM. *Every positive integer other than* 23 *and* 239 *is a sum of eight integral nonnegative cubes.*

Five lemmas are required.

LEMMA 1. *Every integer greater than or equal to* $233^6 D$ *is a sum of eight cubes if* $D = 14.0029682$, *or more generally if* $D = d$, *where*†

$$d > 14 + \left(\frac{24}{167}\right)^3, \qquad d \leqq 14.1.$$

The algebraic part of Wieferich's proof holds for all integers exceeding $2\frac{1}{4}$ billion. The fact that all smaller integers are sums of nine cubes was proved by use of Table I. To prove my theorem, I shall need also the new Tables II and III.

Table I gives, for each positive integer $N \leqq 40{,}000$, the least number m such that N is a sum of m cubes.

It was computed by R. D. von Sterneck‡ by adding all cubes to

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* His errors are avoided in the much simpler proof by the writer, Transactions of this Society, vol. 30 (1928), pp. 1–18. On page 16 is proved a generalization of Landau's result that all sufficiently large numbers are sums of eight cubes.

† The proof is essentially like that given for $d = 14.1$ by W. S. Baer, *Beiträge zum Waringschen Problem*, Dissertation, Göttingen, 1913.

‡ Sitzungsberichte der Akademie der Wissenschaften, Vienna, IIa, vol. 112 (1903), pp. 1627–1666.

each sum of j, but not fewer, cubes for $j=1, 2, \cdots$. Any error in the jth step would introduce an increasingly large number of errors in the later steps. For this reason, my assistant, Miss Evelyn Garbe, computed in four weeks the table* for $12{,}000 < N \leqq 40{,}000$ by my improved method described later. The only error found is

$$32{,}822 = 3^3 + 3^3 + 32^3,$$

whose m was given to be 4 by von Sterneck. The latter stated that his table agrees completely with the table for $N \leqq 12{,}000$ computed by Dahse and published by Jacobi.†

Table II is a continuation to 123,000 of Table I.

Tables I and II imply the following fact.

LEMMA 2. *All integers from* 8,043 *to* 123,000 *are sums of six cubes.*

Table III is a list of all integers from 123,000 to 560,000 which are sums C_4 of four or fewer positive cubes.

The entries not found twice were computed independently, and likewise for the auxiliary list of C_3. Allowing for permutations of the cubes in a sum of them, when making a table of sums of $r+1$ cubes, we need add a fixed cube c^3 only to a sum of r cubes which is not less than rc^3. When finding the $C_4 > 400{,}000$ we saved much time by adding to the C_3 no cubes greater than 64,000, but found separately the C_4 which are sums of cubes all greater than 64,000 (without new computation for necessary C_2 and C_3).

The entries in Table III which are greater than or equal to $C=400{,}000$ and less than $D=401{,}000$, for example, appear on a double sheet. To deduce rapidly a provisional list L′ of C_5 between C and D, we compiled the entries of that double sheet, the same entries increased by unity, and the entries of the double sheet for the interval from $C-(10k)^3$ to $D-(10k)^3$ for $k=1, 2, \cdots$, so that such an entry increased by $(10k)^3$ is a C_5. From each number missing from the list L′ we subtracted successive cubes to see if the difference is in Table III. In this manner we obtain the following list L of all integers between A and B which are not sums of five or fewer cubes. Note that 125,564, 131,423, and 259,682 are the only entries E such that $E-1$ is also an entry. Since $E-8$ and $N-1$, when $N \neq E$, are not in L, we conclude that every number in L is a C_6.

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* But without checking that no number marked $m=6$ is a sum of 5 cubes (it certainly is not a sum of four or fewer cubes). Such a decision is unnecessary for the applications in this paper.

† Journal für die reine und angewandte Mathematik, vol. 42 (1851), pp. 41–69; *Werke*, vol. 6, pp. 322–354.

LEMMA 3. *All integers from* 123,000 *to* 560,000 *are sums of six cubes.*

LIST* L OF ALL INTEGERS BETWEEN 123,000 AND 560,000 WHICH ARE SUMS OF SIX, BUT NOT FEWER, CUBES

123097	136012	146435	162860	188149	224537	274451
268	237	767	958	159	707	284044
305	354	147110	163300	626	225598	288005
619	137219	406	165091	708	860	291074
124232	237	865	380	977	226202	293431
249	254	956	407	189094	227371	702
294	309	148073	569	590	875	891
547	317	388	919	190274	230017	295691
591	561	892	166171	337	231844	296023
898	794	149774	190	409	232052	297860
125015	138236	150278	199	454	235058	298372
095	281	287	478	191623	237263	301190
393	415	566	541	902	388	303125
563	551	836	693	192281	239963	311467
564	740	988	167243	193172	241286	315220
717	857	151105	260	559	852	322879
126401	938	231	450	685	242167	324301
626	139342	447	168268	765	266	328172
896	477	583	359	982	492	361
127489	140054	844	467	194863	244318	330466
551	071	153193	809	196898	751	331231
949	179	212	863	198778	245137	335668
958	270	310	169493	199804	705	365153
128020	459	154886	170086	200074	849	368798
840	142070	155740	357	202801	246181	370534
975	105	156217	861	204503	249457	374306
129721	348	244	171967	205574	252895	377258
946	367	371	172373	816	253138	387356
130982	375	406	173372	916	156	393809
131044	844	443	174074	924	507	414391
063	889	623	175145	206897	254534	482
422	143113	722	604	207590	255155	432598
423	140	830	684	208598	236	434642
927	887	157649	721	751	623	445478
132242	144004	158575	177350	913	259205	453479
133061	346	159376	178204	209183	222	460031
133142	436	881	682	210326	259681	473189
348	589	934	180301	211063	682	480155
807	751	160249	364	212171	260563	487156
943	779	610	182606	306	815	500162
134041	895	771	732	215518	261203	508748
150	145156	844	858	770	725	529555
222	220	934	893	216949	262777	536081
771	436	161473	184145	219245	264659	541327
135392	607	933	721	739	267475	391
580	759	162229	186331	220127	818	542714
824	146083	419	187429	910	271894	548203
895	200	598	717	224473	273983	

* This short table is remarkable in that it gives the *essential data* about sums of cubes for a range of about 440,000 numbers. By its use alone it is shown that all integers from 123,000 to 21,000,000 are sums of six cubes.

To extend Lemma 3 beyond $B=560{,}000$, note that when 44^3 is added to the numbers from 474,816 to B, we get the numbers from B to $F=645{,}184$, which will be proved to be C_6. This is evidently true when 44^3 is added to a number not in L, since it is a C_5. Hence we need only add 44^3 to the last ten numbers N of L. Theoretically we subtract 45^3 from each sum. Actually we subtract $45^3-44^3=5{,}941$ from those ten numbers N, and note that the differences are not in L and hence are C_5. Thus $N+44^3=45^3+C_5$ is a C_6.

Next we added 69^3 to the numbers from $G=316{,}675$ to B to get the numbers from F to 888,509, which are all C_6. In fact, if we subtract $70^3-69^3=14{,}491$ from the last 32 numbers of L (that is, those greater than G), no difference is in L.

After 102 more such simple steps we find, by use of List L alone, that all integers from B to $B+273^3=20{,}906{,}417$ are sums of six cubes.

For the next step, $335{,}668+274^3-275^3=109{,}617$ is below the limit for List L, but is a C_5 by Table II.

The first case of new type is

$$278^3 + 500{,}162 = 279^3 + 267{,}475 = 280^3 + 33{,}114,$$

where 267,475 is in L; but 33,114 is a C_5. The next such case is

$$284^3 + 393{,}809 = 285^3 + 150{,}988 = 283^3 + H, \quad H = 634{,}926.$$

But $H-43^3=555{,}419$ is a C_4 by Table III. Continuing, and combining the result with Lemmas 2 and 3, we obtain the next lemma.

LEMMA 4. *All integers from* 8,043 *to* $J=41{,}623{,}625$ *are sums of six cubes.*

LEMMA 5. *Given a positive integer* S *and a number* B *satisfying* $0\leqq B\leqq S$, *we can find* an integer* $i\geqq 0$ *such that*

$$B \leqq S - i^3 < B + 3S^{2/3}.$$

Take $B=8{,}043$ and choose S so that $S\geqq 8{,}043$, $B+3S^{2/3}\leqq J$, which holds if $\log S\leqq 10.7132020$. By Lemma 4, $S-i^3$ is a C_6, whence S is a C_7.

By Table I, 454 is the last integer less than 8,043 which requires eight cubes. Hence all integers from 455 to the preceding S are C_7. In Lemma 5 with S replaced by s, take $B=455$ and $B+3s^{2/3}=S$. Then $s-i^3$ is a C_7, so that S is a C_8, if $\log S=15.3541210$, $S=22.60065\times 10^{14}$. Since $S>233^6D$, the present result together with Lemma 1 shows that every integer not less than 455 is a C_8. Below 455, Table I shows that 23 and 239 are the only integers which require nine cubes. This proves our theorem. The manuscript of Tables II and III has been given to the library of the University of Chicago.

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* Transactions of this Society, vol. 30 (1928), p. 4, case $t=1$.