ON THE SINGULAR POINT LOCUS IN THE THEORY OF FIELDS OF PARALLEL VECTORS

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The surface of an ordinary (circular) half cone is frequently employed as an illustration of a surface on which parallel displacement of a vector around a closed circuit may produce a change in the vector. It may appear at first sight that the reason for this is related to the fact that the circuit in question encircles the vertex of the cone and the vertex is in some sense a singular or exceptional point. But this cannot be correct since we can remove the vertex from the surface under consideration without effecting the parallel displacement of the vector. If it is then argued that the resulting surface is not simply connected and that the above phenomenon depends on the fact that the circuit along which the vector is displaced cannot be shrunk to a point, we can offset this by smoothing out the surface in the neighborhood of the vertex so that it becomes a continuous, differentiable, and simply connected surface (which is evidently possible). Clearly then, such considerations do not provide us with the inner reason as to why the parallel displacement of a vector around certain closed circuits on the surface yields the original vector while for other circuits a different vector is obtained as result of the parallel displacement.

A satisfactory answer to the above question is contained in a general theory of the parallel displacement of vectors in an affinely connected space by Mayer and Thomas, *Fields of parallel vectors in non-analytic manifolds in the large*, Compositio Mathematica, vol. 5 (1938), pp. 193–207. It can be shown that an affinely connected space of class C^r, where $r \geqq n+1$ and n is the dimensionality of the space, breaks up *in virtue of its intrinsic nature* into a finite or infinite number of open point sets K, called components, with each of which there is associated a definitely determined integer m having a value from zero to n inclusive. Denote by K_m any component K for which m is the associated integer. Then any open, connected, and simply connected point set $O \subset K_m$ admits exactly m independent fields of parallel vectors. The definition of the components K is as follows: Consider the set of equations

$$(E_0)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma} = 0; \; (E_1)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1} = 0; \cdots ; \; (E_n)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1,\cdots,\delta_n} = 0$$

as equations for the determination of the n quantities ξ^{μ}, the coefficients B being the components of the curvature tensor and its successive covariant derivatives. We shall say that a point P of the space is a *regular point with respect to the system* $(E_0)+\cdots+(E_t)$ if there exists a neighborhood $U(P)$ in which the rank of the matrix of the coefficients B of this system is constant. Otherwise the point P will be said to be *singular with respect to the above system.* It is easily demonstrated that the set of singular points so defined is nowhere dense in the space. Denote by R_{n-1} the set of all points regular with respect to the system $(E_0)+\cdots+(E_{n-1})$. A component $K=K(P)$ is defined as the greatest open connected point set in R_{n-1} which contains P. Evidently $K(P)=K(Q)$ if $Q\subset K(P)$. It can be shown that the matrix of the system $(E_0)+\cdots+(E_{n-1})$ has a constant rank in any component $K(P)$. If the rank of this matrix is ρ in $K(P)$, then this component $K(P)$ is one of the above components K_m where $m=n-\rho$. It is furthermore proved in the above paper that if Q is any point in the above point set $O\subset K_m$ and if ξ_Q is any solution vector of the system $(E_0)+\cdots+(E_{n-1})$ at Q, then a field of parallel vectors is generated in O by the process of parallel displacement of the vector ξ_Q to the various points of the set O. Taking m independent solution vectors $\xi_Q^1, \cdots, \xi_Q^m$ at Q, we thus obtain the existence of m independent fields of parallel vectors in O as above stated.

In the case of the cone with vertex deleted, we have a (locally) flat affine space which falls under the above general theory. This space contains only one component K_n, so that there exist n independent fields of parallel vectors in any of the open point sets O; hence if any vector is displaced by parallel displacement around a circuit lying entirely in O, it will return to the original vector. If, on the other hand, the circuit is not contained in a simply connected region O, the original vector need not be obtained as a result of the parallel displacement. If we smooth out the cone at the vertex so as to obtain a simply connected space of class C^r, this space must contain in addition to the component K_n at least one component K_m where $m\neq n$. Otherwise, we would have a contradiction with the fact that there exist circuits about which the parallel displacement of a vector does not yield the original vector.

When the affinely connected space is analytic, the components K into which the space breaks up in accordance with the above general theory lose their real significance. Indeed, the entire space in the analytic case appears to play the same role as the various components K under the non-analytic hypothesis. See Thomas, *Fields of parallel vectors in the large*, Compositio Mathematica, vol. 3 (1936), pp. 453–468. The fact that the various components K which may exist in an analytic affinely connected space are without especial significance in the theory of the parallel displacement of vectors in this space is an objection to the above definition of these components. It is the primary purpose of the following paper to modify the definition of the components K of an affinely connected space so as to eliminate all nonessential components at least as far as this is possible in view of the continuity and differentiability properties which we have at our disposal. As a consequence of this new definition, an analytic affinely connected space will be without singular points and hence (if topologically connected) will contain only one component which is in strict conformity with the above mentioned character of the analytic space.

Let S be an affinely connected space of class C^u. If u is a finite integer, this means that the components of the affine (or linear) connection L are continuous and have all continuous derivatives to the order u inclusive; if $u=\infty$, the components of the affine connection are continuous and have all continuous partial derivatives without exception; and if $u=\omega$, these components are analytic functions of the coordinates of the various admissible coordinate neighborhoods by which the space is covered.

If u is a finite integer, we put $u=r+2$ and assume $r \geqq n-1$, where n is the dimensionality of the space, so that we can construct the sets of equations (E_0), (E_1), $\cdots$, (E_r), (E_{r+1}) which correspond to and in-include those above considered. In the case of a space of class C^∞ or C^ω (analytic affinely connected space) the infinite sequence of equations (E_0), (E_1), (E_2), $\cdots$ can be constructed over the space. Denote by M the matrix of the system $(E_0)+\cdots+(E_r)$ for the space of class C^{r+2} and the (infinite) matrix of the system $(E_0)+(E_1)+(E_2)+\cdots$ in the case of an analytic space or space of class C^∞.

We shall say that a point P of S is *regular* if there exists a neighborhood $U(P)$ in which the rank of the above matrix M is constant. Otherwise P will be said to be a *singular* point. Denote by R the set of all regular points in S. By a *component* $K(P)$ we shall mean the greatest open connected point set in R which contains the point P. Then $K(P)=K(Q)$ if $Q \subset K(P)$. It can be shown that if P is any regular point of a space S of class C^u for which $u=r+2$ is finite, then the system (E_{r+1}) can be expressed linearly and homogeneously in terms of the systems (E_0), $\cdots$, (E_r) in some neighborhood $U(P)$, the coefficients in these expressions being continuous functions of the coordinates of $U(P)$. When S is an analytic space or space of class

C^∞, it is immediately evident that there exists a neighborhood $U(P)$ of any regular point P in which the system (E_t), for some value of the integer t, can be expressed linearly and homogeneously in terms of the systems $(E_0), \cdots, (E_{t-1})$ with coefficients which are analytic or functions of class C^∞, respectively, of the coordinates of $U(P)$. On the basis of the procedures which are employed in the paper by Mayer and Thomas (loc. cit.), the following two theorems can be established.

THEOREM I. *The rank of the matrix M is contant in any component of the space S.*

THEOREM II. *If ρ is the rank of the matrix M in a component K of an affinely connected space S of class C^u and if O is any open, connected, and simply connected point set in K, then there are exactly $n-\rho$ independent fields of parallel vectors in O and these fields can be generated by the parallel displacement to the various points of O of the $n-\rho$ independent solution vectors, at any point $Q\subset O$, of the system $(E_0)+\cdots+(E_r)$ if $u=r+2$ and $r\geqq n-1$ is a finite integer where n is the dimensionality of the space S, or of the infinite system $(E_0)+(E_1)+\cdots$ if the space S is analytic or of class C^∞.*

It is easily seen that any component on the basis of the previous definition is contained entirely in one of the components which enter in the above theorems. Also points which were singular, as previously defined, may become regular points as a result of the modified definition; and in consequence of this, two or more of the components (as previously defined) may merge into a single component. We shall show that in the case of an analytic space this merging is complete; that is, there exists only a single component K in any (topologically connected) analytic affinely connected space.

Let P be any point of an analytic space S at which the matrix M has rank $\rho<n$. Let ξ be any solution vector of the system $(E_0)+(E_1)+\cdots$ at P. Let $C(t)$, where $0\leqq t\leqq 1$, be an analytic curve joining the point P to any other point Q such that $P=C(0)$ and $Q=C(1)$, which is always possible if S is topologically connected. See Thomas, *Arcs in affinely connected spaces*, Annals of Mathematics, (2), vol. 38 (1937), pp. 120–130. Displace the vector ξ at P by parallel displacement along $C(t)$ to obtain an analytic vector $\xi(t)$ defined along the curve C. Along C we may then define the analytic following functions

$$F^{\alpha}_{\beta\gamma}(t) = \xi^{\mu}(t)B^{\alpha}_{\mu\beta\gamma},\quad F^{\alpha}_{\beta\gamma\delta}(t) = \xi^{\mu}(t)B^{\alpha}_{\mu\beta\gamma,\delta},\quad \cdots$$

all of which vanish at P, that is, at $t=0$, by the above hypothesis. It follows immediately that the successive *absolute derivatives* of the tensors F all vanish at P; that is,

$$\frac{D}{Dt}\cdots\frac{D}{Dt}F^{\alpha}_{\beta\gamma}(0) = 0, \quad \frac{D}{Dt}\cdots\frac{D}{Dt}F^{\alpha}_{\beta\gamma\delta}(0) = 0, \quad \cdots .$$

From the first set of these equations it can easily be shown that all the *absolute extensions* of the first tensor F vanish at P, from the second set of these equations all the absolute extensions of the second tensor F vanish at P, and so on; that is,

$$\frac{D^{m}F^{\alpha}_{\beta\gamma}(0)}{Dt^{m}} = 0, \quad \frac{D^{m}F^{\alpha}_{\beta\gamma\delta}(0)}{Dt^{m}} = 0, \quad \cdots, \qquad m = 1, 2, 3, \cdots .$$

The definition of the absolute derivative is, of course, well known, but the definition of the mth absolute extension is perhaps not well known. Let us note, therefore, that the above mth absolute extensions are defined in the following manner: Introduce normal coordinates y^{α} in a neighborhood of the point P. Denote, for example, the components of the first of the above tensors F by $f^{\alpha}_{\beta\gamma}(t)$ with respect to the normal coordinate system. Then

$$\left[\frac{D^{m}F^{\alpha}_{\beta\gamma}}{Dt^{m}}\right]_{P} = \left[\frac{d^{m}f^{\alpha}_{\beta\gamma}}{dt^{m}}\right]_{P},$$

where the right-hand member contains the ordinary mth derivative of the components $f^{\alpha}_{\beta\gamma}(t)$, and a similar definition applies in the case of any tensor. It follows, therefore, that the successive coefficients in the power series expansions of the components of each of the tensors F vanish at $t=0$; and since these components are analytic functions of t along the entire curve $C(t)$, they must vanish along this curve; that is, in the interval $0 \leqq t \leqq 1$. In particular, we have $F^{\alpha}_{\beta\gamma}(1)=0$, $F^{\alpha}_{\beta\gamma\delta}(1)=0, \cdots$; that is, by parallel displacement of the solution vector ξ of the system $(E_0)+(E_1)+\cdots$ at P along the analytic arc C joining P to Q, we obtain a vector ξ' at Q which is also a solution vector of this system. Since independent solution vectors of the above system at P (or Q) remain independent under parallel displacement along the curve C joining P and Q, it follows that the matrix M has the same rank at P as it has at the point Q. In other words, the space S contains no singular points.

THEOREM III. *An analytic affinely connected space S can contain no singular points.*

THEOREM IV. *An analytic affinely connected space S which is topologically connected contains only a single component K.*

Let M' be the matrix of the system $(E_0)+\cdots+(E_{n-1})$, and denote by $R_m=0$, where $m=1, \cdots, n$, the equations obtained by equating to zero all determinants of order $n-m+1$ in the matrix M'. Let S be an affinely connected space of class C^u, and suppose that the conditions $R_m=0$ are satisfied over S, where m has one of the above values $1, \cdots, n$. Let K be any component of S. Since the set of points in S which are singular with respect to the system $(E_0)+\cdots+(E_{n-1})$ is nowhere dense, it follows that there is a point $P\subset K$ which is regular with respect to this system. Then there exists a neighborhood $U(P)$ in which M' has a constant rank ρ, such that the (E_n) are expressible linearly and homogeneously in terms of $(E_0), \cdots, (E_{n-1})$ with coefficients which are functions of the coordinates of $U(P)$. For a finite value of $u=r+2$, these coefficients will be functions of class C^{r-n+1}. By differentiation of these relations, it is therefore possible also to express each of the sets $(E_{n+1}), \cdots, (E_{r+1})$ linearly and homogeneously in the $(E_0), \cdots, (E_{n-1})$ with coefficients which are continuous functions of the coordinates of $U(P)$. Hence the matrix M has the constant rank ρ in $U(P)$. Hence P is a regular point of the space S as defined in §2. Similarly, if S is analytic or of class C^∞, it follows that the matrix M has the rank ρ in $U(P)$ and hence that P is a regular point in accordance with the definition of §2. But $\rho\leqq n-m$ in consequence of the assumption that $R_m=0$ over S. Hence the matrix M has a constant rank less than or equal to $n-m$ in each component K. Conversely, if M has a constant rank less than or equal to $n-m$ in each component K, it follows that $R_m=0$ is satisfied over S. If Theorem II is taken into account, the following theorem can now be stated.

THEOREM V. *A necessary and sufficient condition that there exist at least m independent fields of parallel vectors where m has one of the values $1, 2, \cdots, n$ in an arbitrary open, connected, and simply connected point set O contained in an arbitrary component K of an affinely connected space S is that $R_m=0$ over S.*

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