ON NON-BOUNDARY SETS*

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The purpose of this note is largely methodological; namely, to complete the triology of dense, boundary, and nondense sets by adding non-boundary sets.

We adhere to the nomenclature of Kuratowski's *Topologie*† except as noted. In particular, we suppose that S is a nonvacuous space satisfying his axioms of closure, and we write $F(X)=\overline{X}\cdot\overline{CX}$ and $X^0=C\overline{CX}$ where $CX=S-X$. If the set X has the property P, we write X^P or $(X)^P$, and in the contrary case X^{CP} or $(X)^{CP}$.

A set is *dense* if its closure is the space; *boundary* if its complement is dense; *nondense* if its closure is boundary; and finally, *non-boundary* if its complement is nondense. We designate the properties by D, B, ND, and NB, respectively.

THEOREM. *The following conditions are necessary and sufficient in order that a set be*

I. *Dense: The interior of its closure is the space; the boundary of its complement is the closure of its complement; its complement is a boundary set; its closure is a non-boundary set.*

II. *A boundary set: The closure of its interior is null; its boundary is its closure; its complement is dense; its interior is nondense.*

III. *Nondense: The interior of its closure is null; the boundary of its closure is its closure; its complement is a non-boundary set; its closure is a boundary set.*

IV. *A non-boundary set: The closure of its interior is the space; the boundary of the closure of its complement is the closure of its complement; its complement is nondense; its interior is dense.*

We summarize this in the following table of equivalences. The Roman numerals correspond to the statements above, and each statement in a row is equivalent to every other statement in that row.

The proofs of these statements are as follows: Column 2 is a formulation of the definitions. In column 3 statement I 3 follows from I 2 since $S^0=S$; II 2 is equivalent to $X^0=0$, which is clearly the same as II 3; III 3 is the complement of III 2; IV 3 is IV 2.

As to column 4, we have for I 4

$$(\overline{X} = S) \rightarrow (\overline{X}\cdot\overline{CX} = \overline{CX}) \rightarrow (F(CX) = \overline{CX}).$$

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† C. Kuratowski, *Topologie* I, Warsaw, 1933.

Also $\overline{X}\cdot\overline{CX}=\overline{CX}$ is the same as saying that $\overline{CX}$ is a subset of $\overline{X}$. But $C\overline{X}\subset CX\subset\overline{CX}\subset\overline{X}$, or $C\overline{X}=0$. The remaining statements in this column follow from this one using only the definitions.

	1	2	3	4	5	6
I	X^D	$\overline{X}=S$	$\overline{X}^0=S$	$F(CX)=\overline{CX}$	$(CX)^B$	$(\overline{X})^{NB}$
II	X^B	$\overline{CX}=S$	$\overline{X^0}=0$	$F(X)=\overline{X}$	$(CX)^D$	$(X^0)^{ND}$
III	X^{ND}	$\overline{C\overline{X}}=S$	$\overline{X}^0=0$	$F(\overline{X})=\overline{X}$	$(CX)^{NB}$	$(\overline{X})^B$
IV	X^{NB}	$\overline{C\overline{C\overline{X}}}=S$	$\overline{\overline{X}^0}=S$	$F(\overline{CX})=\overline{CX}$	$(CX)^{ND}$	$(X^0)^D$

Column 5 can be deduced from column 4 and the definitions. Statement IV 6 is the same as IV 2; III 6 is II 2 with X replaced by $\overline{X}$; for II 6 we have

$$(X^0)^{ND} \equiv (\overline{C\overline{X^0}} = S) \equiv (\overline{\overline{\overline{CX}}} = S) \equiv X^B;$$

the proof of I 6 is similar.

We have the following theorem giving relations between the various properties:

THEOREM. *If a set is a non-boundary set, it is not a boundary set; if it is not a boundary set, it is not nondense; and if it is closed and not nondense, it is not a boundary set. If a set is a non-boundary set, it is dense; if it is dense, it is not nondense; if it is open and dense, it is a non-boundary set.*

These results may be seen easily from the appended diagram, the arrow indicating the direction of implication. The proofs of the state-

$$\begin{tikzcd} CB \arrow[r, shift left] & CND \arrow[l, shift left, "\text{closed}"] \\ NB \arrow[u] \arrow[r, shift right] & D \arrow[u] \arrow[l, shift right, "\text{open}"'] \end{tikzcd}$$

ments are as follows: $X^{NB}\rightarrow X^{CB}$ from IV 3 and II 3; $X^{CB}\rightarrow X^{CND}$, from II 2 and III 2; X closed and not nondense $\rightarrow X^{CB}$ in the same manner; $X^{NB}\rightarrow X^D$, from I 2 and IV 3 since $X^0\subset X$; X open and dense $\rightarrow X^{NB}$, in a similar way; $X^D\rightarrow X^{CND}$, from I 3 and III 3.

The following results are typical of non-boundary sets.

A necessary and sufficient condition that X be a non-boundary set is that X be dense and the boundary of X be nondense.

For we have

$$X^D \to (C\overline{X} = 0) \to (\overline{C\overline{X}} = 0)$$

and

$$F(X)^{ND} \to (C[\overline{C\overline{X}} + \overline{X^0}] = 0) \to (\overline{X^0} = S).$$

Conversely,

$$(\overline{X^0} = S) \to (\overline{C\overline{X}} + \overline{X^0} = S) \to (F(X)^0 = 0) \to F(X)^{ND}.$$

The product of a countable collection of non-boundary sets is a residual set; that is, the complement of a set of the first category.

For if X is such a set, then CX is the sum of a countable collection of nondense sets by IV 5.

*Every dense G_δ is residual.**

In fact, if $X = \prod X_n$, $X_n = X_n^0$, then each X_n is a non-boundary set because

$$S = \overline{X} \subset \overline{X_n} = \overline{X_n^0}.$$

In a complete metric space the product of a countable collection of non-boundary sets is not vacuous.

This is a classic theorem of R. Baire.

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* See Kuratowski, loc. cit., p. 206, Theorem V 2. I owe this reference to a referee.