# DECOMPOSITION OF ELEMENTS IN ABELIAN GROUPS*

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1. **Introduction.** Let $G$ be an abelian group of elements $g$ with operation $+$ (sum), and unit element 0 (zero). We shall be concerned with the following property:

PROPERTY $P_n$. *There exist n distinct elements in G such that their sum vanishes.*

In the present paper we determine necessary and sufficient conditions for an abelian group to have the property $P_n$. The author first proved the validity of these conditions for fields, but, as noted by T. Nakayama, the operation $\times$ does not occur, so that the theory may be stated for a system of elements with only one operation $+$ defined for these elements. Groups with the property $P_n$ have useful algebraic applications.†

One is naturally led to consider the decomposability of any given element of a group $G$ into a sum of distinct elements of $G$. This problem is treated in §3.

It is necessary in the treatment of decompositions of elements to distinguish only between nonzero elements of order 2, elements of order different from 2, and the zero element.

2. **Groups with property** $P_n$. A pair $(h, k)$ of elements $h$, $k$ in a group $G$ satisfying

$$h + k = g$$

is called a *g-pair* in $G$. If one or both of the elements $h$, $k$ is zero, the pair $(h, k)$ is called a *null g-pair*.

An element $q$ in $G$ of order 2 $(q+q=0)$ will be said to be *singular*. The remaining elements of $G$ are said to be *nonsingular*.

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† Let the minimal number $m(F)$ of a form $F$ of degree $r$, with respect to a field $K^*$, designate the smallest value of $\sigma$ for which $F$ can be written as a sum $\lambda_1 L_1^r + \cdots + \lambda_\sigma L_\sigma^r$, where the $\lambda$'s are in $K^*$, and the $L$'s are linear forms with coefficients in $K^*$. Let $K$ denote a field whose characteristic and order are such that the symmetric $q$-way and $(q+1)$-way matrices of the forms $Q$ and $QL$ of degree $q$ and $(q+1)$, respectively, are unique, the coefficients being in $K$. We can prove that if $K$ has the property $P_{q+1}$, the following inequality is true:

$$m(QL) \leqq (q+1)m(Q),$$

where the $m$'s denote minimal numbers with respect to $K$.

We shall use the following obvious lemmas in the development of this paper:

LEMMA 1. *The nonsingular elements of an abelian group may be grouped into distinct* 0*-pairs.*

LEMMA 2. *The singular elements of an abelian group $G$ may be grouped into distinct $q$-pairs for each nonvanishing singular $q$ in $G$.*

LEMMA 3. *The four elements of two distinct* 0*-pairs of nonsingular elements in an abelian group are mutually distinct.*

LEMMA 4. *The four elements in two distinct $q$-pairs, $(q\neq 0)$, of singular elements in an abelian group are mutually distinct.*

It is to be observed that the null $q$-pair is $(0, q)$.

If there are two or more singular elements in an abelian group $G$, these elements form a subgroup of $G$, which we shall call the *singular subgroup* $G'$ of $G$. If $G=G'$, the group $G$ is said to be *singular*.

If $g=g_1+\cdots+g_n$, where $g_1, \cdots, g_n$ are in $G$, we shall say that $(g_1, \cdots, g_n)$ is an *$n$-representation* of $g$. If $g_1, \cdots, g_n$ are distinct we shall call the representation *proper*.

We shall prove the following theorem where it is naturally understood that the order of the group is not less than $n$:

THEOREM 1. *An abelian group $G$ possesses Property $P_n$ except in the following cases:*

(a) *$n=2$, and $G$ is singular;*

(b) *$G$ is singular, of finite order $m$, and $n=m-2$;*

(c) *$G$ possesses a singular subgroup of order* 2*, $G$ is of finite order, and $n$ equals the order of $G$.*

We let $G$ be singular. If for a pair of elements $a$, $b$ in $G$ we have $a+b=0$, then $a=b$; whence it follows that $G$ does not possess Property $P_2$. In case (b), $m\geqq 4$. The elements of $G$ may be paired into an even number of distinct $q$-pairs for any given nonzero element $q$ in $G$, whence the sum of all of the elements in $G$ is zero. If the sum of $m-2$ distinct elements of $G$ vanishes, $G$ possesses Property $P_2$, which has been proved to be impossible.

If part (c) of Theorem 1 is satisfied, $G$ possesses a singular subgroup $G'$ of elements $0$, $q$. By Lemma 1 the sum of elements of $G$ not contained in $G'$ is zero; whence the sum of all elements in $G$ is $q$. Thus Property $P_n$ does not hold in case (c).

It remains to show that if (a), (b), and (c) are not satisfied by $G$ and $n$, the group $G$ possesses Property $P_n$.

It is to be remarked that if $G$ is infinite and contains at least one nonsingular element, it contains an infinite number of such elements.

Let $d$ denote the number of nonsingular elements in $G$, where $\infty \geqq d \geqq 0$. Assume $d>0$, and $n \leqq d$. If $n=2k$, we obtain a proper $n$-representation of zero by taking $k$ 0-pairs of nonsingular elements; if $n=2k+1$, we adjoin 0 to the $k$ 0-pairs just mentioned.

We assume now that $d$ is finite and that $n=d+s$, where in turn $s=4k+i$, $(i=1, 2, 3, 4, k \geqq 0)$. The group $G$ now has a singular subgroup $G'$ whose order we shall denote by $m'$, where $m'$ is finite. If $i=1$, we obtain a proper $n$-representation of zero by adjoining zero to the sum of the nonsingular elements of $G$, and the elements in $2k$ distinct non-null $q$-pairs, $(q \neq 0)$, of $G'$. If $i=3$, we adjoin $q$ to the elements in $(2k+1)$ non-null $q$-pairs of singular elements, $(q \neq 0)$, and the nonsingular elements of $G$. If $i=4$, we adjoin $(2k+2)$ $q$-pairs, $(q \neq 0)$, of singular elements to the nonsingular elements of $G$.

Let $i=2$, and $k=0$. Since it is assumed that (a) is not satisfied, we have $d>0$. Since (c) is not satisfied, $m' \geqq 4$. We adjoin two distinct $q$-pairs, $(q \neq 0)$, from $G'$ to $(d-2)/2$ distinct 0-pairs of nonsingular elements. It remains to take $i=2$, $k \geqq 1$. We can now write $s=4k'+6$, where $k' \geqq 0$. Since the order $m'$ of $G'$ is finite, it is even. We assume that $s \neq m'-2$. Since $s$ is even, $s \leqq m'-4$; whence

$$m' \geqq 4k' + 10. \tag{1}$$

Actually $m' \geqq 4k'+12$, but this fact is not needed. It follows from (1) that the order $m'$ of $G'$ is at least $2^4$. Select from $G'$ three distinct non-null $q$-pairs, $(q \neq 0)$,

$$(q_1, q_2), (q_3, q_4), (q_5, q_6) \tag{2}$$

such that $q_1+q_3=q_5$, whence

$$q_1 + q_3 + q_5 = 0. \tag{3}$$

It is readily seen from $-q_i=q_i$ that $q_1+q_3$ is different from the elements in the pairs $(0, q)$, $(q_1, q_2)$, $(q_3, q_4)$; so $q_5$ actually occurs in a $q$-pair distinct from these. Let $(q_7, q_8)$ be a $q$-pair in $G'$ which is non-null and distinct from (2). Then

$$q + q_7 + q_8 = 0. \tag{4}$$

By (3) and (4) we have

$$q + q_1 + q_3 + q_5 + q_7 + q_8 = 0. \tag{5}$$

We obtain a proper $(d+4k'+6)$-representation of zero by adjoining to the sum exhibited in (5) the nonsingular elements of $G$ and the elements of $2k'$ distinct $q$-pairs in $G'$ which are non-null and different from the pairs (2) and $(q_7, q_8)$. That this is possible follows from (1) and the fact that the pairs (2), $(q_7, q_8)$, and $(0, q)$ contain exactly 10 elements.

If, finally, $i=2$ and $s=m'-2$, since the conditions of (b) are assumed to be not satisfied, there exist nonsingular elements in $G$, whence $d>0$. We have $m=4l$. We obtain a proper $n$-representation of 0 by taking all elements of $G$ except one 0-pair of nonsingular elements.

COROLLARY 1. *An abelian group $G$ of infinite order possesses the Property $P_n$, except when $G$ is singular and $n=2$.*

COROLLARY 2. *An abelian group $G$ with no singular subgroup possesses Property $P_n$ for each $n$.*

COROLLARY 3. *A field of characteristic $p$ and order $m$ (finite or infinite) possesses the Property $P_n$ if and only if $p\neq 2$, or if $p=2$, $n\neq 2$, $m-2$.*

COROLLARY 4. *The sum of the elements of a finite abelian group $G$ is zero except when $G$ contains a singular subgroup $G'$ of order 2; in the latter case it is equal to the nonzero element in $G'$.*

COROLLARY 5. *The sum of elements of a finite field $K$ of characteristic $p$ and order $m$ is zero except when $p=m=2$.*

One can prove that the sum of the elements of a field $K$ with characteristic different from 2 is zero rather simply and directly in another manner. Let $q_1, \cdots, q_m$ denote the distinct elements of $K$. Let $q_1+\cdots+q_m=q$, $q$ being in $K$. The set $-q_1, \cdots, -q_m$ is obviously the same as the set $q_1, \cdots, q_m$. Hence $q=0$.

3. **Decomposability of nonzero elements of abelian groups.** It will appear from the development of this section that the *existence of a proper n-representation of a nonsingular element of an abelian group G implies the existence of a proper n-representation of each nonsingular element* of $G$. Similarly, the existence of a proper $n$-representation of a nonzero singular element of $G$ implies the existence of a proper $n$-representation for each such element of $G$. We therefore make the following definitions:

PROPERTY $P_n'$. *There exists a proper n-representation of each nonsingular element in G.*

PROPERTY $P_n''$. *There exists a proper n-representation of each nonzero singular element in G.*

THEOREM 2. *An abelian group $G$ containing nonsingular elements possesses the property $P_n'$ unless $n$ equals the order of the group.*

By Corollary 4 of §2, if $n$ is the order of the group, there is no proper $n$-representation of a nonsingular element in $G$. In what follows we therefore assume that $n$ is less than the order of the group.

As above, we let $d$ denote the number of nonsingular elements of $G$, where now $\infty \geqq d \geqq 0$. Assume $n \leqq d$. We obtain a proper $n$-representation of a nonsingular element $g$ by taking a sufficient number of 0-pairs of nonsingular elements different from $(g, -g)$ and adjoining $g$ or $0, g$ according as $n$ is odd or even.

We assume now that $d$ is finite, and $n = d+s$, $(s>0)$. Let $m$ denote the order of the singular subgroup $G'$. Since $d$ is finite, $m$ is finite. Let the nonsingular elements of $G$ be denoted by $g_1, \cdots, g_d$. Since $s<m$, there exist at least a set of distinct elements $q_1, \cdots, q_{s+1}$ in $G'$. For some element $q_\alpha$ in $G'$ we have the equality

$$g_1 + \cdots + g_d + q_1 + \cdots + q_{s+1} = q_\alpha. \tag{6}$$

The equality

$$q_\alpha + x = g,$$

where $g$ is nonsingular, is satisfied for a nonsingular element $g_i$ in $G$. There is a value of $j$ such that $g_j = -g_i$. Adding $g_i$ to both sides of (6) we obtain the proper $(d+s)$-representation of $g$:

$$(g_1, \cdots, g_{j-1}, g_{j+1}, \cdots, g_d, q_1, \cdots, q_{s+1}).$$

THEOREM 3. *An abelian group $G$ containing non-null singular elements possesses the property $P_n''$ unless $n$ is the order of $G$ and the order of the singular subgroup $G'$ of $G$ is greater than 2.*

By Corollary 4, if $G$ is finite of order $n$, there is a proper $n$-representation of a non-null singular element of $G$ if and only if the singular subgroup $G'$ of $G$ is of order 2.

In what follows we assume that $n$ is less than the order of $G$. The group $G$ contains a singular subgroup $G'$ of order $m$, where $m \geqq 2$. Let the order of $G$ be $d+m$, where $\infty \geqq d \geqq 0$. If $n \leqq d$, we obtain a proper $n$-representation of a nonzero singular element $q$ by choosing elements from the pair $(0, q)$, and 0-pairs of nonsingular elements. If $n = d+4k+i$, where $k \geqq 0$ and $i = 1$, 2, or 3, we obtain a proper $n$-representation of $q$ by taking all nonsingular elements of $G$, a sufficient number of non-null $q$-pairs from $G'$, and elements from the null pair $(0, q)$. It remains to consider the case where $n = d+s$, $s = 4k+4$. Since $m = 4l$, and $s \leqq m-4$, we have

$$(7) \qquad 4k \leqq m - 8.$$

We choose distinct non-null $q$-pairs of elements of $G'$ given by

$$(8) \qquad (q_1, q_2), (q_3, q_4), (q_5, q_6)$$

so that $q_1+q_3=q_5$. By (7) there are at least $2k$ distinct non-null $q$-pairs $P_1, \cdots, P_{2k}$ in $G'$ different from (8). We obtain a proper $n$-representation of $q$ by adding $q$, $q_1$, $q_3$, $q_5$ to the elements in $P_1, \cdots, P_{2k}$ and the nonsingular elements of $G$.

4. **Construction of representations.** We used very special proper representations in §§2 and 3 to prove the existence of proper representations. In the present section we show how, under very general circumstances, proper representations of a given element $g$ may be obtained from proper representations of other elements in the group.

THEOREM 4. *Let $S$ be a set of $n$ distinct elements in an abelian group $G$, and let $h$ denote the sum of the elements in $S$. Let $g$ be an element of $G$ distinct from $h$, and let $p$ be the order of $k$, where $k=g-h$. The element $g$ has a proper $n$-representation $P$ containing at least $n-1$ elements of $S$ unless $p\neq 0$ and the elements of $S$ can be grouped into cosets of the cyclic subgroup $K=[0, k, 2k, \cdots, (p-1)k]$.*

Let $g_1, \cdots, g_n$ be $n$ distinct elements of $G$ whose sum is $h$. If for some $i$ we have

$$g_i + k \neq g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n,$$

the representation $P_i=(g_1, \cdots, g_{i-1}, g_i+k, g_{i+1}, \cdots, g_n)$ is a proper $n$-representation of $g$. Assume therefore that for each $i$ we have $g_i+k$ equal to one of the quantities $g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n$. Since $g_i\neq g_j$ for $i\neq j$, the set $(g_1+k, \cdots, g_n+k)$ is equal to the set $(g_1, \cdots, g_n)$. We can evidently order the subscripts on the $g$'s and choose $r$, $s$, so that

$$(9) \qquad g_r + k = g_1, \quad g_i + k = g_{i+1}, \qquad i = 1, \cdots, r-1,$$

and

$$g_{r+s} + k = g_{r+1}, \quad g_j + k = g_{j+1}, j = r+1, \cdots, r+s-1,$$

and so on. Adding the equations (9) we obtain

$$(10) \qquad \sum_{i=1}^{r} g_i + rk = \sum_{i=1}^{r} g_i.$$

If $p$ does not divide $r$, the formula (10) implies that $k=0$, contradicting $k\neq 0$. Hence $r$, $s$, and so on, are each divisible by $p$, so that $p$ is not zero. Let $r=qp$ where $q>1$. By (9) $g_{i+1}=g_1+ik$ for $i=1, 2, \cdots, qp-1$. Hence $g_{p+1}=g_1+pk=g_1$, contradicting $g_1\neq g_{p+1}$. It follows that the $g$'s may be grouped into cosets of $K$, whence the theorem is proved.

COROLLARY 6. *Let $S$ be a set of $n$ distinct elements $g_1, \cdots, g_n$ of an abelian group $G$, where the elements of $S$ are not the elements comprising cosets of a cyclic subgroup of $G$. Let the sum of the elements in $S$ be denoted by $h$. For each element $g$ in $G$ there is a proper representation $(g_1, \cdots, g_{i-1}, g_i+k, g_{i+1}, \cdots, g_n)$ of $g$ where $k=g-h$.*

THE INSTITUTE FOR ADVANCED STUDY

# THE MAXIMUM NUMBER OF DISTINCT CONTACTS OF TWO ALGEBRAIC SURFACES*

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1. **Introduction.** For more than half a century, it has been known that the maximum number of distinct contacts of two algebraic surfaces whose intersection curve is irreducible is the genus of that curve.†

The number of contacts of two surfaces whose entire intersection curve consists of straight lines has been found.‡ This is the maximum number of contacts of two surfaces of given orders.

The purpose of this paper is to obtain the maximum number of distinct contacts of two algebraic surfaces when the intersection curve consists of any given number of components of given orders and genera. Cases are treated in which the two surfaces have singular points or singular curves in common.

2. **Method.** From any point of $S_3$ not on the developable of $C$, a space curve $C$ with $h$ apparent double points projects into a plane curve $C'$ with $h$ nodes. Since $C'$ may have $p$ additional nodes, where $p$ is the genus of both $C$ and $C'$, the space curve $C$ may have $p$ nodes. The necessary and sufficient condition for the two surfaces to have a contact is for $C$ to have a node.§

If $C$ is irreducible and is the complete intersection of two surfaces $M$ and $N$, the maximum number of contacts of $M$ and $N$ is the genus

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† E. Pascal, Repertorium der höheren Mathematik, vol. 2 (1902), p. 225. In a later supplementary volume of the Repertorium (vol. $2_2$ (1922), pp. 653–656), more than two pages are devoted to the topic, "Contacts of two surfaces," but the problem of this paper is not treated.

‡ Encyklopädie der mathematischen Wissenschaften, vol. 3, C, 9, (1926), p. 1277.

§ E. Pascal, Repertorium der höheren Mathematik, vol. $2_2$ (1922), p. 654.