FRÉCHET ON THE CALCULUS OF PROBABILITY

Méthode des Fonctions Arbitraires. Théorie des Événements en Chaîne dans le Cas d'un Nombre Fini d'États Possibles. (Traité du Calcul des Probabilités et de ses Applications, vol. 1, no. 3.) By Maurice Fréchet. Paris, Gauthier-Villars, 1938. 10+315 pp.

This is the second book of vol. 1, no. 3, of the Borel series on the calculus of probability and its applications, this division being entitled *Recherches théoriques modernes sur la théorie des probabilités.* The first book was reviewed in the September, 1937, issue of this Bulletin (pp. 602–603).

We may say that two general cases are discussed in this book. First, in a brief chapter, Poincaré's method of arbitrary functions is taken up. In this the distribution of probability depends on a parameter n, which, however arbitrarily distributed initially, has a limiting distribution as $n \to \infty$. This he illustrates with the roulette of equal numbers n of red and black divisions, red ones of length R, black ones of length N. Whereas the older hypotheses of probability *assumed* the probability of a red division stopping before the pointer to be $R/(R+N)$, under integrability assumptions on the distribution function of rotation it is proved that $R/(R+N)$ is the *limit* of that probability as the number of divisions n approaches infinity. In this chapter is also given Hostinsky's interesting illustration of the meaning of chance in connection with the geometrical interpretation of the mechanics of a die throw.

The second and major treatment consists in determining when and in what manner the ergodic principle is fulfilled in the case of events in a chain with a finite number r of possible states E_j. Define $P_{hk}^{(n)}$ to be the probability that a system in state E_h will arrive at or be in state E_k on the nth trial. Then the ergodic or "regularization" principle states in this case that $\lim_{n\to\infty} P_{hk}^{(n)} = P_k$, independent of the initial state.

In the *regular case* (Hadamard) each $P_{hk}^{(n)}$ has such a limit P_k as $n \to \infty$. Then the necessary and sufficient condition that $\sum_{n=1}^{P}[P_{hk}^{(n)} - P_k]$ be dominated by a convergent geometric progression is that for some n a row of the matrix $D^{(n)} \equiv \|P_{hk}^{(n)}\|$ be all positive. In the *positive regular case* there exists some n for which all the elements of the matrix are positive. That is, there is positive probability of passing from one arbitrary state of the system to any other arbitrary state with sufficiently large number of trials. Fréchet shows that the "*mélange des urnes*" problem comes under the positive regular case, that is, the probability of possible compositions approaches independence of the initial composition with larger number of operations. Following this are derived necessary and sufficient conditions that the *most regular case* obtain, that the limit of the $P_{hk}^{(n)}$ be a constant P, independent of both h and k. The new case thus introduced finds illustration in the cutting of a deck of cards, in which all the original "states" or "ranks" are reintroduced, but perhaps in a different order. (An example of imperfect shuffling and its effect upon the P's is given by halving the deck and shuffling each half, then rejoining; the limits of $P_{kk'}^{(n)}$ all exist but are not equal for k, k' in the same half and again in different halves.) From the single card viewpoint the probability of its going to a specified position approaches $1/r$, the reciprocal of the number of cards; when the permutation of the cards is taken as the *state*, the probability approaches $1/r!$. Under the assumption that $\sum_{k=1}^{r} P_{hk}^{(n)} = 1$ is not satisfied, the analysis is set up to prove the existence of unique solutions of the iteration equations $P_{hk} = \sum_{j=1}^{r} P_{hj} P_{jk}$ and the conditions $\sum_{j=1}^{r} P_{hj} = 1$, which must be the P_k,

($k=0, 1, \cdots, r$), above. Applying the results to the *mélange des urnes* problem the author obtains the result that the probability that at the end of n operations the first urn shall have k white balls tends, as n increases, to that of drawing from the *mixture* of the two urns' contents a number of balls equal to that of the first urn with exactly k white ones included.

If the initial state of a system is supposed to be a result of chance, then each member is subject to a law of initial probability ω_h taking values $\omega_1, \cdots, \omega_r$, say. Then $\omega_j^{(n)}=\sum_{h=1}^{r}P_{hj}^{(n)}\omega_h$ is the probability of arriving at the state E_j in n trials, the initial state subject to chance. In the regular case the $\omega_j^{(n)}$ tend toward law-limits which are the P_j limits aforementioned. They are called *stable* laws of initial probability. Then the necessary and sufficient condition that the inverse probability in chain, $q_j^{(n)}$, be independent of n is that the law of initial probability be positively *stable*. On inverse chain probabilities the author gives recent results of Mihoc, Hostinsky, Kolmogoroff, Bernstein, and others, with certain additions. In case sequences of trials are considered as unlimited in both directions, then the concept of initial probability is replaced by that of absolute probability.

In the regular case the mean value of an aleatory variable in chain tends to a limit, as n increases, independent of its initial value. Mean frequency is also discussed from the notion of the aleatory variable. The Markoff discussion of the dispersion of the arithmetic mean of an aleatory variable with some discussion of special cases, in particular the conditions in the regular case under which $\sigma=0$, is given. The choice of wording, *regular case*, is quite fitting since the usual situations associated with probability come under that category.

At this stage the author, having finished an exposition of Markoff's results and methods with certain supplements, proceeds with a method developed by Poincaré-Romanovsky, to obtain more complete results. In general it is the study of $P_{ik}^{(n)}$ expressed as a function of n, for the regular case.

The iterated probabilities in chain $P_{jk}^{(n)}$ always converge in the arithmetic mean, differing from their limits by infinitesimals of the order of $1/n$. That the $P_{jk}^{(n)}$ be convergent in the ordinary sense it is necessary and sufficient that each root of $\Delta(s)=0$ of modulus 1 be 1. Here $\Delta(s)$ is the characteristic equation of the determinant of the $P_{jk}^{(1)}=p_{jk}$. In case the roots of $\Delta(s)=0$ are all simple and not zero, an explicit expression for the $P_{jk}^{(n)}$ as a function of n can be written. Let $\prod_{ji}^{(n)}$ be the arithmetic mean of $P_{ji}^{(1)}, \cdots, P_{ji}^{(n)}$, and $\prod_{ji}=\lim_{n\to\infty}\prod_{ji}^{(n)}$. Then the νth moments of the semiregular aleatory variable x_j (when the $M_j=\sum\prod_{ji}x_i$ are all equal) are at most of order n.

The matrix $D^{(n)}$ is said to be *decomposable* when a matric array within it is zero. From examination of the corresponding conditions one of the results is, for example, *the positive regular case prevails if and only if the table $D^{(n)}$ is indecomposable for every positive integer n.* The set of possible states can be divided into *final* groupings and groupings *de passage*. Once in a state of a final grouping, there is zero probability of going from that state to one outside that grouping. These in turn can be distributed in cyclic subgroupings, to which various theorems apply. We say that E_k is a *consequent* of order n of state E_j if $P_{jk}^{(n)}\neq 0$. Thus in order that the set of possible states G be indecomposable it is necessary and sufficient that every pair of states be consequents of each other. Then is given the method of direct construction of indecomposable groupings from the complete set of possible states as indicated by Kolmogoroff and Doeblin.

The last section of the book, comprising fifty-four pages, is devoted to generalizing the basis $P_{jk}^{(n)}$ of an enumerable set of trials to $P_{jk}(s, t)$ consisting of a set of trials continuous with time. Here $P_{jk}(s, t)$ is the probability that a system in the state E_j

at the time s will be in the state E_k at time t. Equations and conditions on the new functions analogous to those of Kolmogoroff on the enumerable set are introduced. More generally, the $P_{jk}(s, t)$ are replaced by functions $\phi_{jk}(s, t)$ not necessarily interpretable as probabilities, and methods of solving and solutions of the functional equation $\phi_{jk}(s, t)=\sum_{i=1}^{r}\phi_{ji}(s, u)\phi_{ik}(u, t)$ are studied. The four methods given are by Kolmogoroff, Hostinsky, and the author. In one case the solutions are expressed as series of multiple integrals of increasing order, in finite terms in another.

Four notes appended to the text explain and amplify some of the basic algebra involved. Specifically, they concern systems of linear equations in finite differences of the first order with constant coefficients.

The number of typographical errors is not annoyingly large, though they will be encountered occasionally, usually in the form of wrong subscripts. A few interchanges of capital and small letters of defined symbols and a couple of wrong references to equations complete the observed list of errors.

In conclusion, the book is an effective summary and extension of recent contributions to the theory of probability of events in simple chain and of a finite number of possible states. The author's style of writing and presentation are very clear and interesting, but his prefatory statement that "since he wishes the book to be read also by those interested in probability but who are not professional mathematicians, attention to detail is sometimes sacrificed" is a true indication that the reader is apt to encounter difficulty as he attempts to verify many of the statements. However, the reader will find a wealth of reference material at the end of the book.

J. A. GREENWOOD