POLYGONAL VARIATIONS*

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So-called direct methods in the calculus of variations, such as those of Tonelli involving lower semi-continuity, sometimes insure the existence of an arc minimizing an integral $J=\int f dx$ in cases where many of the partial derivatives of the integrand function f which occur in the usual theory do not exist. Examples are integrals of the form $\int g(x, y)(1+y'^2)^{1/2}dx$, or in the parametric case $\int g(x, y)(x'^2+y'^2)^{1/2}dt$, where $g(x, y)$ is merely continuous and positive. When a minimizing arc is known to exist, necessary conditions assume greater importance. It seems desirable, therefore, to have methods of deriving the familiar necessary conditions of Weierstrass, Euler, and Legendre, while making as few assumptions as possible concerning the existence of partial derivatives of the integrand f. In this paper it is shown that by using the method of polygonal variations the Weierstrass necessary condition and a generalization of the Euler equation can be derived under the assumption of the existence and continuity of the partial derivative $f_{y'}$ only. A slightly generalized form of the Legendre condition can be proved, with the assumption only of the existence of the generalized second partial derivative $f_{y'y'}$. The method involves giving the dependent variable or variables variations whose graphs are polygonal lines of proper shape, depending on a parameter ϵ, and then evaluating the derivative $J'(\epsilon)$ when $\epsilon=0$. Since the method generalizes easily in the usual way to the case of more than one dependent variable and also to the parametric problem, the discussion will be given here only for the simplest case of a non-parametric problem with one dependent variable.

1. **The Weierstrass necessary condition.** Some of the methods commonly used to derive the Weierstrass necessary condition make use of such complicated notions as fields of extremals and Hilbert's invariant integral, and most of them use the Euler equation, which involves the partial derivative f_y. L. M. Graves however, has given a proof (American Mathematical Monthly, vol. 41 (1934), pp. 502–504) which is independent of the Euler equation and requires only the existence and continuity of $f_{y'}$. The following proof by means of polygonal variations is made under the same weak assumptions.

In all that follows the integrand $f(x, y, y')$ is assumed to be con-

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tinuous for x, y in a region R and for all y', and admissible arcs are of class D', lie in R, and pass through the fixed end points (a, c) and (b, d). Since polygonal variations have corners, their use requires the assumption that the integral is minimized with respect to a set of admissible arcs of class D'. In the classical proofs of the necessary conditions, the weaker assumption that comparison arcs are of class C' is all that is needed. This loss of generality could have been avoided by "rounding off the corners" of the polygonal variations. Since this procedure would make the proofs more complicated, and the variations would no longer be polygonal, it is not used here.

THEOREM 1. *If $f_{y'}(x, y, y')$ exists and is continuous for x, y in R and all y', and if $y=y(x)$ makes the integral $J=\int_a^b f(x, y, y')dx$ a strong relative minimum in the class of admissible arcs, then at each element x, y, y' of the minimizing arc and for all k,*

$$E(x, y, y', k) = f(x, y, k) - f(x, y, y') - (k - y')f_{y'}(x, y, y') \geqq 0.$$

PROOF. The proof will be made first for the left-hand end point (a, c). For a given value of k, let $m=k-y'(a)$. The polygonal variation $\eta(x)$ to be used is linear from a to $a+\epsilon$ with slope m, then linear from $a+\epsilon$ to $a+\epsilon+\delta$ with slope $-m\epsilon/\delta$, and identically zero from $a+\epsilon+\delta$ to b, (where δ is to approach zero with ϵ in the manner described below) that is,

$$\begin{aligned} &\eta(x) = m(x - a), && a \leqq x \leqq a + \epsilon, \\ (1)\quad &\eta(x) = m\epsilon(a + \epsilon + \delta - x)/\delta, && a + \epsilon \leqq x \leqq a + \epsilon + \delta, \\ &\eta(x) = 0, && a + \epsilon + \delta \leqq x \leqq b. \end{aligned}$$

Since f is continuous, there exists an $\omega(\epsilon)>\epsilon>0$ such that if $|y_1-y_2|<|m|\epsilon$, then

$$|f(x, y_1, y') - f(x, y_2, y')| < \omega(\epsilon) \tag{2}$$

for all x, y' on $y=y(x)$, where $\omega(\epsilon)$ approaches zero with ϵ. Now let $\delta=\epsilon/(\omega(\epsilon))^{1/2}$. Since $\omega(\epsilon)>\epsilon$ and $\omega(\epsilon)\rightarrow 0$, it is seen that $\delta\rightarrow 0$ and $\epsilon/\delta\rightarrow 0$. This completes the definition of $\eta(x)$. Since its slope does not approach zero with ϵ, $\eta(x)$ is a strong variation.

Now replace y and y' in J by $y(x)+\eta(x)$ and $y'(x)+\eta'(x)$, and call the result $J(\epsilon)$. Since $y=y(x)$ makes J a strong relative minimum, and ϵ is positive,

$$J'(0) = \lim_{\epsilon\rightarrow 0} [J(\epsilon) - J(0)]/\epsilon \geqq 0 \tag{3}$$

provided the limit exists. Adding and subtracting an integral and integrating over subintervals, we have

$$
(4)\qquad
\begin{aligned}
J'(0) = \lim_{\epsilon\to 0} & \left\{ \frac{1}{\epsilon}\int_a^{a+\epsilon} [f(x, y+\eta, y'+\eta') - f(x, y, y')]dx \right. \\
& + \frac{1}{\epsilon}\int_{a+\epsilon}^{a+\epsilon+\delta} [f(x, y+\eta, y'+\eta') - f(x, y+\eta, y')]dx \\
& \left. + \frac{1}{\epsilon}\int_{a+\epsilon}^{a+\epsilon+\delta} [f(x, y+\eta, y') - f(x, y, y')]dx \right\} \\
= \lim_{\epsilon\to 0} & (I_1 + I_2 + I_3) \geqq 0.
\end{aligned}
$$

The first law of the mean for integrals and the continuity of f give

$$
(5)\qquad \lim_{\epsilon\to 0} I_1 = f[a, y(a), k] - f[a, y(a), y'(a)].
$$

Likewise the law of the mean for integrals and the mean value theorem for derivatives give

$$
(6)\qquad
\begin{aligned}
I_2 &= \frac{\delta}{\epsilon}[f(\alpha, y+\eta, y'+\eta') - f(\alpha, y+\eta, y')] \\
&= -\, m f_{y'}(\alpha, y+\eta, Y'),
\end{aligned}
$$

where y, y', η, η' are evaluated at $x=\alpha$, which is some point of the interval from $a+\epsilon$ to $a+\epsilon+\delta$, and where $y'(\alpha)\leqq Y'\leqq y'(\alpha)+\eta'(\alpha)$. Since $Y'\to y'(a)$,

$$
(7)\qquad \lim_{\epsilon\to 0} I_2 = -\, m f_{y'}[a, y(a), y'(a)].
$$

The law of the mean for integrals gives

$$
(8)\qquad I_3 = \frac{\delta}{\epsilon}\{f[\beta, y(\beta)+\eta(\beta), y'(\beta)] - f[\beta, y(\beta), y'(\beta)]\},
$$

where $a+\epsilon\leqq\beta\leqq a+\epsilon+\delta$. Then from (2) and the definition of δ,

$$
(9)\qquad |I_3| \leqq \frac{\delta}{\epsilon}\omega(\epsilon) = (\omega(\epsilon))^{1/2}.
$$

Hence $I_3\to 0$; and combining (4), (5), and (7) we obtain

$$
(10)\qquad
\begin{aligned}
\lim_{\epsilon\to 0} J'(0) = f[a, y(a), k] - f[a, y(a), y'(a)] \\
- \, m f_{y'}[a, y(a), y'(a)] \geqq 0.
\end{aligned}
$$

But since $m=k-y'(a)$, this is the Weierstrass E function, and the Weierstrass necessary condition has been shown to hold at the left-hand end point. Since $y=y(x)$ also minimizes J on any subinterval, (10) also holds if a is replaced by any value of x between a and b, which proves Theorem 1.

2. **The Euler equation.** The method of polygonal variations can be used to derive a generalization of the Euler equation which does not involve the partial derivative f_y explicitly. However, the method will first be used to derive the ordinary form of the Euler equation, since the proof is very simple and avoids any use of either integration by parts, the fundamental lemma, or the du Bois-Reymond lemma.

It is first assumed that f_y and $f_{y'}$ exist and are continuous, and that $y=y(x)$ makes $J=\int_a^b f(x, y, y')dx$ a weak relative minimum in the class of admissible arcs joining (a, c) and (b, d). For a fixed $\delta>0$, define $\eta(x, \delta)$ as follows:

$$\begin{aligned} &\eta(x, \delta) = (x-a)/\delta, && a \leqq x \leqq a+\delta, \\ (11)\qquad &\eta(x, \delta) = 1, && a+\delta \leqq x \leqq b-\delta, \\ &\eta(x, \delta) = (b-x)/\delta, && b-\delta \leqq x \leqq b. \end{aligned}$$

Replace y and y' in J by $y(x)+\epsilon\eta(x)$ and $y'(x)+\epsilon\eta'(x)$, and call the result $I(\epsilon)$. The polygonal variation $\eta(x)$ is linear on the interval $[a, a+\delta]$ with slope $1/\delta$, constant on the interval $[a+\delta, b-\delta]$, and linear with slope $-1/\delta$ on the interval $[b-\delta, b]$. Since $y=y(x)$ makes J a weak relative minimum, it follows in the usual way that

$$(12)\qquad I'(0) = \int_a^b (\eta f_y + \eta' f_{y'})dx = 0$$

for all $\delta>0$. Hence, by integrating over subintervals and taking the limit as $\delta\to 0$,

$$(13)\qquad \begin{aligned} \lim_{\delta\to 0}\int_a^b (\eta f_y + \eta' f_{y'})dx &= \lim_{\delta\to 0}\left\{\int_a^{a+\delta}\frac{x-a}{\delta}f_y dx + \int_{a+\delta}^{b-\delta} 1\cdot f_y dx\right. \\ &\left. + \int_{b-\delta}^{b}\frac{b-x}{\delta}f_y dx + \int_a^{a+\delta}\frac{1}{\delta}f_{y'}dx + 0 + \int_{b-\delta}^{b} -\frac{1}{\delta}f_{y'}dx\right\} = 0. \end{aligned}$$

The first and third of these integrals approach the limit zero, as may be seen by using the law of the mean for integrals. Likewise the last two integrals approach the limits $f_{y'}(a)$ and $-f_{y'}(b)$, respectively. The limit of the second integral is seen to be $\int_a^b f_y dx$. Hence

$$(14)\qquad \int_a^b f_y dx = f_{y'}(b) - f_{y'}(a).$$

Since $y=y(x)$ also minimizes J on any subinterval, (14) holds with b

replaced by any value of x from a to b, so that

$$\int_a^x f_y dx = f_{y'}(x) - f_{y'}(a). \tag{15}$$

But this is the Euler equation in integrated form.

3. **Partial variations and the generalized Euler equation.** The partial derivative f_y occurs in the Euler equation (14) in the integral $\int_a^b f_y dx$. This integral can also be obtained in the following way. Give the dependent variable y the (non-admissible) polygonal variation $\eta(x)=\epsilon$, a constant. Then $\eta'(x)=0$, and the integral J becomes

$$J(\epsilon) = \int_a^b f[x, y(x) + \epsilon, y'(x)]dx.$$

Then, if f_y exists and is continuous, $J'(0)=\int_a^b f_y dx$. Examples show that $J'(0)$ may exist even if f_y does not. This suggests the definition

$$\partial J = J'(0) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_a^b \{f[x, y(x) + \epsilon, y'(x)] - f[x, y(x), y'(x)]\}dx.$$

We shall call ∂J the *partial variation* of J since only y and not y' is varied. The partial variation ∂J is seen to be a generalization of the integral $\int_a^b f_y dx$ which occurs in the Euler equation. The variation $\eta(x)=\epsilon$ is not admissible since it does not vanish at the end points; hence ∂J cannot be expected to vanish. We have rather the following theorem:

THEOREM 2. *If f and $f_{y'}$ are continuous for x, y in R and all y', and if $y=y(x)$ makes $J=\int_a^b f(x, y, y')dx$ a weak relative minimum in the class of admissible functions, and ∂J exists, then $\partial J=f_{y'}(b)-f_{y'}(a)$.*

PROOF. Given ϵ, define the weak polygonal variation $\eta(x)$ as follows:

$$\begin{aligned} \eta(x) &= \epsilon(x-a)/\delta, && a \leqq x \leqq a+\delta, \\ \eta(x) &= \epsilon, && a+\delta \leqq x \leqq b-\delta, \\ \eta(x) &= \epsilon(b-x)/\delta, && b-\delta \leqq x \leqq b. \end{aligned} \tag{16}$$

This variation $\eta(x)$ is linear from a to $a+\delta$ with slope ϵ/δ which is to approach zero with ϵ. From $a+\delta$ to $b-\delta$, $\eta(x)$ is constant, and from $b-\delta$ to b, $\eta(x)$ is linear with a slope which also approaches zero with ϵ. As in the proof of Theorem 1, δ is defined in terms of ϵ as follows. From the continuity of f, there exists $\omega(\epsilon)>|\epsilon|$ such that, if

$|y_1-y_2| \leqq |\epsilon|,$

$$|f(x, y_1, y') - f(x, y_2, y')| < \omega(\epsilon) \tag{17}$$

for x, y' on $y=y(x)$, where $\omega(\epsilon)\rightarrow 0$ as $\epsilon\rightarrow 0$. Now let $\delta=|\epsilon|/(\omega(\epsilon))^{1/2}$. Note that as $\epsilon\rightarrow 0$, $\delta\rightarrow 0$, and $\epsilon/\delta\rightarrow 0$, also that δ is positive, while ϵ may be positive or negative.

Replace y and y' in J by $y(x)+\eta(x)$ and $y'(x)+\eta'(x)$, and call the result $J(\epsilon)$. Since $y=y(x)$ makes J a weak relative minimum, it follows that

$$V = \lim_{\epsilon\rightarrow 0} \frac{1}{\epsilon}\int_a^b [f(x, y+\eta, y'+\eta') - f(x, y, y')]dx = 0$$

provided the limit V exists, since the integral is non-negative for ϵ sufficiently small, while ϵ may be positive or negative. After adding and subtracting two quantities this becomes

$$\begin{aligned} V = \lim_{\epsilon\rightarrow 0} \frac{1}{\epsilon}\bigg\{ &\int_a^b [f(x, y+\eta, y'+\eta') - f(x, y+\eta, y')]dx \\ &+ \int_a^b [f(x, y+\eta, y') - f(x, y+\epsilon, y')]dx \\ &+ \int_a^b [f(x, y+\epsilon, y') - f(x, y, y')]dx\bigg\} \\ = \lim_{\epsilon\rightarrow 0} \frac{1}{\epsilon}\bigg\{ &\int_a^b F_1 dx + \int_a^b F_2 dx + \int_a^b F_3 dx\bigg\} = 0. \end{aligned} \tag{18}$$

Here F_1, F_2, F_3 have been introduced for convenience. By the law of the mean for integrals and the mean value theorem for derivatives,

$$\int_a^{a+\delta} F_1 dx = \epsilon f_{y'}[\alpha, y(\alpha)+\eta(\alpha), Y'], \tag{19}$$

where $a\leqq\alpha\leqq a+\delta$ and $y'(\alpha)\leqq Y'\leqq y'(\alpha)+\eta'(\alpha)$. Since $Y'\rightarrow y'(a)$, and $f_{y'}$ is continuous,

$$\lim_{\epsilon\rightarrow 0} \frac{1}{\epsilon}\int_a^{a+\delta} F_1 dx = f_{y'}[a, y(a), y'(a)]. \tag{20}$$

Similarly,

$$\lim_{\epsilon\rightarrow 0} \frac{1}{\epsilon}\int_{b-\delta}^{b} F_1 dx = -f_{y'}[b, y(b), y'(b)]. \tag{21}$$

Since F_1 vanishes from $a+\delta$ to $b-\delta$,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^b F_1dx = f_{y'}[a, y(a), y'(a)] - f_{y'}[b, y(b), y'(b)]. \tag{22}$$

Consider now $\int_a^b F_2dx$. Since $|\eta(x)-\epsilon| \leqq |\epsilon|$, by the definition of $\omega(\epsilon)$,

$$|F_2| = |f(x, y+\eta, y') - f(x, y+\epsilon, y')| < \omega(\epsilon). \tag{23}$$

Hence, by the law of the mean,

$$\left|\frac{1}{\epsilon}\int_a^{a+\delta} F_2dx\right| \leqq \frac{\delta}{|\epsilon|}\omega(\epsilon) = (\omega(\epsilon))^{1/2}. \tag{24}$$

Therefore, since $\omega(\epsilon)\to 0$,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^{a+\delta} F_2dx = 0. \tag{25}$$

Similarly,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{b-\delta}^{b} F_2dx = 0. \tag{26}$$

Since $F_2=0$ for $a+\delta\leqq x\leqq b-\delta$, (25) and (26) give

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^b F_2dx = 0. \tag{27}$$

From the definition of ∂J,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^b F_3dx = \partial J. \tag{28}$$

Combining (22), (27), and (28) we have

$$V = f_{y'}[a, y(a), y'(a)] - f_{y'}[b, y(b), y'(b)] + \partial J = 0, \tag{29}$$

which proves Theorem 2. Note that since $y=y(x)$ also minimizes J over any subinterval, b in (29) can be replaced by any value of x between a and b, provided that ∂J is interpreted to mean the partial variation of J over the interval from a to x.

4. **The Legendre condition.** The Legendre condition is usually derived either as a consequence of the Weierstrass condition or by use of the second variation. The objection to deriving it as a consequence of the Weierstrass condition is that the Legendre condition is necessary for a weak relative minimum while the Weierstrass condition is not. To derive the Legendre condition from the second variation re-

quires the existence and continuity of numerous partial derivatives of f. Here a generalization of the Legendre condition is derived by the use of a weak polygonal variation, the proof requiring the existence only of the single generalized partial derivative $f_{y'y'}$.

THEOREM 3. *If $f(x, y, y')$ is continuous, and if the arc $y=y(x)$ makes $J=\int_a^b f dx$ a weak relative minimum in the set of arcs of class D' joining (a, c) and (b, d), and if*

$$L = \lim_{m\to 0} \frac{1}{m^2} \{f[x, y(x), y'(x) + m] - 2f[x, y(x), y'(x)] + f[x, y(x), y'(x) - m]\}$$

exists, where x is any point of the interval $[a, b]$, then $L \geqq 0$.

The proof will be given first for the case where x is the left-hand end point a. The weak polygonal variation $\eta(x)$ to be used is linear from a to $a+\epsilon$ with slope m which is to approach zero. It is linear from $a+\epsilon$ to $a+2\epsilon$ with slope $-m$, and identically zero from $a+2\epsilon$ to b; that is,

$$\begin{aligned} &\eta(x) = m(x - a), \qquad a \leqq x \leqq a + \epsilon, \\ (30) \qquad &\eta(x) = m(a + 2\epsilon - x), \qquad a + \epsilon \leqq x \leqq a + 2\epsilon, \\ &\eta(x) = 0, \qquad a + 2\epsilon \leqq x \leqq b, \end{aligned}$$

where ϵ is to be defined in terms of m as follows: From the continuity of f, $y(x)$, and $y'(x)$, for every value of m there exists an $\epsilon(m)>0$ such that $\epsilon(m)<|m|$ and such that if $|x-a|<2\epsilon$ and $|\delta|<\epsilon$, the following three inequalities hold:

$$\begin{aligned} &|f[x, y(x) + \delta, y'(x) + m] - f[a, y(a), y'(a) + m]| < m^4, \\ (31) \qquad &|f[x, y(x) + \delta, y'(x) - m] - f[a, y(a), y'(a) - m]| < m^4, \\ &|f[a, y(a), y'(a)] - f[x, y(x), y'(x)]| < m^4. \end{aligned}$$

This completes the definition of $\eta(x)$. Since $y=y(x)$ makes J a weak relative minimum, it follows that

$$(32) \qquad I = \lim_{m\to 0} \frac{1}{\epsilon m^2} \int_a^b \{f[x, y(x) + \eta(x), y'(x) + \eta'(x)] - f[x, y(x), y'(x)]\} dx \geqq 0,$$

provided the limit exists, since the integral is non-negative for m sufficiently small, and $\epsilon>0$. On the interval from $a+2\epsilon$ to b the integrand in (32) is zero; hence by integrating over subintervals and applying the law of the mean, we can write (32) as

$$
\begin{aligned}
(33)\quad I = \lim_{m\to 0}\frac{1}{m^2}\{&f[\alpha, y(\alpha)+\eta(\alpha), y'(\alpha)+m] - f[\alpha, y(\alpha), y'(\alpha)] \\
&+ f[\beta, y(\beta)+\eta(\beta), y'(\beta)-m] - f[\beta, y(\beta), y'(\beta)]\},
\end{aligned}
$$

where $a \leqq \alpha \leqq a+\epsilon$, and $a+\epsilon \leqq \beta \leqq a+2\epsilon$, and use has been made of the fact that $\eta'(\alpha)=m$ and $\eta'(\beta)=-m$ from (30).

After adding and subtracting a quantity, (33) becomes

$$
\begin{aligned}
I = \lim_{m\to 0}\frac{1}{m^2}\{&f[a, y(a), y'(a)+m] - 2f[a, y(a), y'(a)] \\
&+ f[a, y(a), y'(a)-m]\} + \lim_{m\to 0}\frac{1}{m^2}\{f[\alpha, y(\alpha)+\eta(\alpha), y'(\alpha)+m] \\
(34)\quad &- f[a, y(a), y'(a)+m] + f[a, y(a), y'(a)] - f[\alpha, y(\alpha), y'(\alpha)] \\
&+ f[\beta, y(\beta)+\eta(\beta), y'(\beta)-m] - f[a, y(a), y'(a)-m] \\
&+ f[a, y(a), y'(a)] - f[\beta, y(\beta), y'(\beta)]\} \\
= \lim_{m\to 0}\frac{1}{m^2}&B_1 + \lim_{m\to 0}\frac{1}{m^2}B_2 \geqq 0.
\end{aligned}
$$

The two expressions in braces in (34) are here designated B_1, B_2. Now apply (31) to B_2, replacing δ by $\eta(\alpha)$ and x by α in the first line of (31), replacing δ by $\eta(\beta)$ and x by β in the second line of (31), and then replacing x first by α and then by β in the third line of (31). Then

$$(35)\qquad |B_2| < 4m^4,$$

since $|\eta(x)| < \epsilon$ for $|m| < 1$ from (30). Hence

$$(36)\qquad \lim_{m\to 0}\frac{1}{m_2}B_2 = 0.$$

Therefore, from (34) and the definition of L,

$$(37)\qquad L = \lim_{m\to 0}\frac{1}{m_2}B_1 \geqq 0,$$

which proves Theorem 3 when x is the left-hand end point a. Since $y=y(x)$ minimizes J on any subinterval, the same conclusion holds when a is replaced by any value of x between a and b. It is well known that if the ordinary second partial derivative $f_{y'y'}$ exists, it is equal to L; but L may exist even when the first partial derivative $f_{y'}$ does not. Hence Theorem 3 is a generalization of Legendre's condition.

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