THE TOPOLOGY OF TRANSFORMATION GROUPS*

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1. Stated in general terms, the problem which I wish to consider is the following: A group G of transformations operates in a space S; what relations must exist between the topology of this situation on the one hand, and its group theoretical properties, on the other? This is of course a rather vague question and I shall not attempt to describe all the recent results which could be considered as being relevant. I hope rather to illustrate certain special phases of the problem by means of examples. I shall then consider in some detail one special case where certain conclusions can be drawn which may possibly be of interest in algebraic geometry. I refer particularly to a generalization to higher dimensions of Harnack's theorem concerning the number of real branches which a real algebraic curve may possess.

CONTINUOUS GROUPS

2. Let us begin by supposing that G is a continuous r-parameter group. The elements of G may then be thought of as points of a space which has locally the character of a euclidean r-space. Products and quotients of two elements are to vary continuously with those elements. What restrictions does the fact that G is a continuous group place upon the *space* G? It is very easy to see that G cannot be of arbitrary topological structure. For suppose that a is a fixed element different from the identity, and that x is an arbitrary element; then the transformation T_a: $x \rightarrow ax$, is a homeomorphism of G with itself, and no x remains fixed. Now if we assume that G is connected, a can be made to describe a continuous path toward the identity element, and T_a then undergoes a continuous modification into the identical transformation. That is, T_a belongs to the "class of the identity." But in many spaces (a sphere for example) every transformation which belongs to the class of the identity must leave at least one point fixed. Thus no two-parameter group can be topologically equivalent to a sphere.† It is just as easy to show that G must be

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† A necessary condition that a compact manifold admit transformations of the class of the identity without fixed points is that its Euler-Poincaré characteristic vanish. See Lefschetz, *Topology*, American Mathematical Society Colloquium Publications, vol. 12, New York, 1930, pp. 272, 359.

orientable (see for example [31]). It follows readily that the only surfaces which could be group-spaces are the torus, the infinite cylinder, and the euclidean plane; continuous groups of each of these types do exist.

Let us return to the situation mentioned in the beginning in which G is realized by means of continuous transformations in S. If we assume that G operates in S transitively, then S, as well as G, is restricted topologically. For example, if S is a surface, it must belong to one of six distinct topological types, at least if G is a Lie group, that is, analytic (Cartan [10]). In this special case of an analytic G, the topology of (G, S) is intimately related to the group theoretic infinitesimal structure of G. When G is compact this relationship reveals itself through a consideration of the invariant integrals on S (Cartan [8]). In particular, one may assume that S is identical with G, the transformations of the realization then being the T_a's defined above. The theory of invariant integrals then leads to an explicit enumeration of the Betti numbers of compact continuous groups. Cartan [9] has pointed out the suggestive fact that on the basis of these and related results, one may conclude that so far as the ordinary invariants of homology theory are concerned, every simply connected simple compact Lie group G is equivalent to the topological product of a finite number of spheres of varying dimensions. Whether or not G is actually homeomorphic to such a product remains to be determined.*

3. Let us now consider the problem stated at the beginning, from a somewhat different point of view. Given certain topological condi-

* Our present knowledge of the topology of Lie groups is due largely to the analytic and algebraic investigations of Cartan and Weyl. An excellent review of the situation will be found in Cartan's expository article [9]. Suffice it to mention here that the Betti numbers of compact groups were first obtained by Pontrjagin [22] by direct and elementary methods and subsequently by Brauer [5] by a method based on the theory of invariant integrals. The connecting link between the integrals and the homology groups was furnished by de Rham [24]. With regard to the realizations of G, it was pointed out by Cartan [10] that the study of the topological and group-theoretical properties of a Lie group G contains within itself the study of the properties of its transitive realizations. For suppose $\mathfrak{g}$ is a closed subgroup of G; then the cosets $x\mathfrak{g}$ (x in G) constitute, upon the introduction of natural topology, a space S such that the transformations $x\mathfrak{g}\rightarrow a(x\mathfrak{g})$ (a in G) define a transitive realization (G, S). Every transitive (G, S) is in fact equivalent to one defined in this manner by means of a suitably chosen $\mathfrak{g}$. The details justifying these remarks when the spaces G and S are quite general and analyticity is not assumed were first worked out by Freudenthal [12] and subsequently by the author in his study [33] of partially defined groups and realizations (the sort actually encountered in the classical theory of continuous groups). As concerns the homotopy groups of a continuous (non-analytic) G see Hurewicz [13]; for the special case of the Poincaré group, see also [33], [34], [35].

tions on (G, S), to what extent is the group-theoretic structure determined? Suppose, for example, one considers only the analytic case and asks: Given the dimension of S (a topological property), what are the continuous groups which can operate transitively in S? The answer constitutes, as we know, one of the important chapters in the classical works of Lie. Suppose however that one drops the assumption of analyticity and assumes merely that G and S are locally euclidean and that G, as well as the transformations by which G is realized, is continuous. Does one then have a situation which is essentially more general than that studied by Lie, or is every (G, S) of this type equivalent to a classical (analytic) one? In two rather difficult papers, Brouwer [**6**] showed that the continuous case is not more general than the analytic one when S is one-dimensional, and went far toward establishing a similar result when S is two-dimensional. With regard to G itself, we have now the theorem of von Neumann [**18**], somewhat sharpened by Pontrjagin [**21**], that every r-parameter G which, considered as a space, is compact, is topologically equivalent to an analytic G. Whether or not this holds for non-compact G's appears to be a difficult question.

These results have a special significance in connection with certain problems in the foundations of geometry which were occupying mathematicians of another generation. Helmholtz and Lie attempted to found euclidean n-dimensional geometry by assuming that space was an n-fold number manifold and then introducing certain postulates which were to define the group of rigid motions. It is well known that Hilbert criticized this procedure on the grounds that analyticity, postulated for the transformations which were to be rigid motions, did not properly belong to "foundations." Hilbert himself then proposed a set of axioms for two-dimensional geometry in which analyticity is not assumed, the decisive axiom being a topological closure requirement for the system of transformations (see Hilbert's *Grundlagen der Geometrie*, seventh edition). In this manner, the number plane received a truly topological conversion into the place of euclidean geometry. So far as I know, there has been little attempt to carry this development out in spaces of higher dimensions. The discovery that compact r-parameter groups are analytic is of course closely connected with this question; but there still remains the problem of characterizing, among the realizations (G, S) operating in a given number space S, those whose transformations, in their totality, are topologically equivalent to the rigid motions. A recent work of Montgomery and Zippin [**16**] which contains a topological-group-theoretical characterization of the group of rotations about a fixed axis can, I believe, be considered as marking a certain progress toward a solution of the problem for three dimensions.*

Finite groups

4. I shall pass over the case in which G is an infinite discrete group and assume that G is finite. One may ask, what are the finite groups of auto-homeomorphisms of a space S with given topological properties? If S is homeomorphic to a sphere, the answer is known. The groups are essentially the same as the finite groups of orthogonal transformations of a sphere into itself; for example, if it is required that orientation be preserved, the possible groups are cyclic groups of rotations about a fixed axis and the rotation groups of the regular solids. This is a topological theorem and can be proved by topological methods (Kerékjártó [**14**]). If one asks the more special question, what are the finite groups of linear fractional substitutions in one complex variable, one may arrive at the answer (essentially the same as that of the preceding questions) by function-theoretic methods (Klein, *Lectures on the Icosahedron*). These methods, in their use of Riemann surfaces, have a certain topological element, and it is perhaps this fact that led Poincaré to express the belief that the determination of the finite groups of linear transformations in n variables must depend on the solution of topological problems.

5. I wish now to consider in more detail the special case in which G is finite and cyclic; that is, the case in which we are dealing with a single transformation t of finite period. Such transformations arise in a number of ways. Suppose, for example, that t is a birational transformation of an algebraic curve C into itself. Then it is known that if the genus of C is greater than one, t must be of finite period (the Schwarz-Klein theorem†). Again, in the theory of algebraic varieties, a particular type of involution, so-called, is that generated by a periodic transformation.‡ In certain problems in the calculus of variations one encounters cyclic and symmetric products. The m-fold topological product of a space K is the space S whose "points" are the ordered sets $(p_1, \cdots, p_m)$, (p_i in K). The m-fold symmetric product

* I am informed by one of these authors that they have recently established the still more significant theorem that every (G, S) in which G is compact and connected and S is a three-dimensional number space, and which possesses no identical transformations save that which corresponds to the identity element of G, is equivalent either to the group of rotations about a fixed axis or about a fixed point. Concerning characterizations of translation groups, see [**17**].

† See Severi-Loeffler, *Vorlesungen über Algebraische Geometrie*, p. 143.

‡ See, for example, Lucien Godeaux, *Les Involutions Cycliques Appartenant à une Surface Algébrique*, Actualités Scientifiques et Industrielles, no. 270, Paris, 1935.

$K^{(m)}$ of K is obtained from S by regarding two sets as identical if one is a cyclic permutation of the other. In determining the properties of $K^{(m)}$ it is of advantage to study the periodic transformation $t:(p_1, p_2, \cdots, p_m)\to(p_2, \cdots, p_m, p_1)$ of S into itself.*

The only invariant of G is now its order m, and our general problem therefore reduces to this: Does the topological nature of S impose restrictions on m? Suppose that S is an orientable surface of genus $p+1$ and that t preserves orientation. Then we have the interesting fact that m can not exceed $4p+2$.† I wish now to establish a similar result for higher dimensions. I shall suppose that S is an analytic orientable manifold of an even number of dimensions; S could, for example, be an algebraic surface without singularities; it would then have four (real) dimensions. I shall suppose also that t is analytic and preserves orientation. We shall see later that the invariant points of t, if any exist, constitute a finite number of submanifolds, each of an even number of dimensions. In a sense, we may regard as the general case that in which the invariant set is of dimension zero.‡ In any case, we shall assume that each of the transformations $t, t^2, \cdots, t^{m-1}$ leaves fixed only a finite number of points; when m is prime this condition holds automatically for the powers of t if it holds for t.

THEOREM. *If the conditions stated above hold, and if the Euler-Poincaré characteristic κ of S is negative, then there is an upper bound for the period m, which depends only on the Betti numbers of S.*

PROOF. To every isolated fixed point of a given transformation there can be associated a topologically significant numerical "index." Let O be a fixed point of t. In terms of suitable parameters at O, t can be represented by

$$u_i' = tu_i = \phi_i(u_1, \cdots, u_{2n}), \qquad i = 1, \cdots, 2n,$$

where the ϕ's are analytic and $\phi_i(0)=0$. If $J(u)$ is the Jacobian ma-

* Cyclic products $K^{(m)}$ were defined by Walker who determined their Betti numbers in terms of those of K. The case $m=2$ (symmetric products) had already been studied by the author [**32**] and by Richardson [**26**]. For applications in the calculus of variations, it seems to be necessary to know the Betti numbers of K modulo L, where L is a certain subcomplex, homeomorphic to K. These have been obtained by Richardson [**27**] for $m=2$ and by Richardson and Smith [**29**] for arbitrary prime m.

† First proved by Wiman [**40**] for birational transformations of algebraic curves; the first topological proof is by Steiger [**37**]. Nielsen [**19**] subsequently showed that the theorem holds if it is assumed that t^m is not necessarily the identity but merely belongs to the class of the identity.

‡ The fixed points of a projective transformation, for example, are isolated unless the characteristic equation has multiple roots.

trix of this transformation and I is the identity matrix, then it can be shown that the index at O is $+1$ if the determinant of $J(0)-I$ is greater than zero (Alexandroff and Hopf, *Topologie*, p. 537).* But in the present case this condition is bound to be satisfied, as the following considerations will show.

The space of differentials at O is transformed into itself linearly; and since this transformation is periodic, it leaves invariant a positive definite invariant form; in fact if we write $du_i=\xi_i$, such a form is

$$\sum_{i=1}^{2n} (\xi_i^2 + (t\xi_i)^2 + \cdots + (t^{m-1}\xi_i)^2).$$

On making a suitable linear transformation of parameters at O, the invariant form will become euclidean. Therefore, in the power series expansions of the functions ϕ_i, the coefficients of the linear terms may be assumed to form a proper† orthogonal matrix; hence, after a further linear change of parameters, t has, in the neighborhood of O, the form

$$\begin{aligned} tx_i &= x_i \cos\theta_i - x_{i+1}\sin\theta_i + X_i, \\ tx_{i+1} &= x_i \sin\theta_i + x_{i+1}\cos\theta_i + X_{i+1}, \qquad i = 1, 3, \cdots, 2n-1, \end{aligned} \tag{1}$$

where the X's are power series beginning with terms of at least the second degree, and the θ_i are multiples of $2\pi/m$. I assert that each θ_i is different from zero (mod 2π). Suppose for example that $\theta_1=0$. Then

$$\begin{aligned} tx_1 &= x_1 + X_1(x) = \psi_1(x), \\ tx_2 &= x_2 + X_2(x) = \psi_2(x), \\ &\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

Let

$$\Omega_i(x) = \psi_i(x) + \psi_i(tx) + \cdots + \psi_i(t^{m-1}x), \qquad i = 1, 2,$$

where, of course, tx means $(tx_1, \cdots, tx_{2n})$. Obviously $\Omega_i(x)=\Omega_i(tx)$. Moreover, it is easy to see that

$$\begin{aligned} \Omega_1(x) &= mx_1 + Y_1(x), \\ \Omega_2(x) &= mx_2 + Y_2(x), \end{aligned}$$

where Y_1, Y_2 begin with terms of at least second degree. Hence the transformation

$$\bar{x}_1 = \Omega_1(x),\ \bar{x}_2 = \Omega_2(x),\ \bar{x}_3 = x_3, \cdots, \bar{x}_{2n} = x_{2n}$$

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* The condition is obviously independent of the choice of parameters at O.

† Proper, because orientation is preserved.

is (1, 1) in the neighborhood of O. We now examine the form which t takes in the variables $\bar{x}$. We have, for $i=1, 2$,

$$t\bar{x}_i = \Omega_i(tx) = \Omega_i(x) = \bar{x}_i.$$

Therefore t leaves fixed the points of the $(\bar{x}_1, \bar{x}_2)$-plane. This contradicts the assumption that the fixed points are finite in number and proves our assertion that no θ_i vanishes. Hence if we form the matrix $J(x)-I$ from equations (1), it follows immediately that its determinant is greater than zero for $(x)=(0)$. Hence O is of index $+1$.

Consider now the h-dimensional homology group H_h. It possesses a decomposition of the form

$$H_h = [\gamma_h^1] + \cdots + [\gamma_h^{R_h}] + Z_h,$$

where $[\gamma_h^i]$ is an infinite cyclic group generated by the cycle γ_h^i and Z_h is a finite group; R_h is the Betti number. The effect of t on the γ's is given by

$$(2) \qquad t\gamma_h^i \sim \sum_j a_h^{ij}\gamma_h^j \pmod{Z_h}.$$

Let $\theta=\sum(-1)^h$ trace a_h, $a_h=\|a_h^{ij}\|$. In the Lefschetz theory of transformations the number θ is an important character of t, and when the fixed points are isolated, θ equals the algebraic sum of their indices (Lefschetz, *Topology*, p. 272). In the case under consideration it follows that $\theta \geqq 0$.

Now each matrix a_h is of period m; that is, its mth power is the identity. Consequently, the characteristic roots of a_h are mth roots of unity. It cannot happen that all these roots are $+1$, for then the trace of each a_h would equal R_h and θ would equal κ, which is impossible since $\kappa<0$. In exactly the same way we can say that not all the roots of all the matrices for t^d, $(1\leqq d\leqq m-1)$, can equal $+1$.

The completion of the proof of the theorem now depends on two elementary theorems in number theory. Let $m=p_1^{\alpha_1}, \cdots, p_s^{\alpha_s}$ be the factorization of m into powers of distinct primes. Let $\Lambda(m)=p_1^{\alpha_1}+\cdots+p_s^{\alpha_s}$. Then first, if $\mu_1, \cdots, \mu_s$ is any set of integers whose l.c.m. is m, we have $\sum\mu_i\geqq\Lambda(m)$. Secondly, if R is any positive integer, the number of integers m for which $\Lambda(m)\leqq R$ is finite; we shall denote the largest one by $\chi(R)$.

Consider the "roots of t"; that is, the totality of characteristic roots of $a_0, a_1, \cdots, a_n$. Let the orders of these roots be $\lambda_1, \lambda_2, \cdots, \lambda_s, \cdots$, where $\lambda_1, \cdots, \lambda_s$ are a maximal set of *distinct* orders. Let l be the l.c.m. of $\lambda_1, \cdots, \lambda_s$. Then $l=m$. For, each λ_i is a divisor of m, therefore so is l. Suppose $l<m$; then the roots of t^l, being lth powers of

the corresponding roots of t, are all 1, which is impossible. From the preceding paragraph it follows that $\sum_1^s \lambda_i \geqq \Lambda(m)$. Let ϵ_i be a root of order λ_i, $(i=1, \cdots, s)$; it satisfies an irreducible equation $P_i(\epsilon)=0$ of degree λ_i with integer coefficients. P_i must be a factor of one of the characteristic equations. Since $\lambda_1, \cdots, \lambda_s$ are distinct, so are $P_1, \cdots, P_s$; therefore the sum of their degrees is less than or equal to the sum of the degrees of the characteristic equations; that is, $\sum_1^s \lambda_i \leqq \sum_1^s R_i = R$, say. Hence $\Lambda(m) \leqq R$ and $m \leqq \chi(R)$, which proves the theorem.

Whether or not the theorem remains true, if we drop analyticity and allow continua of fixed points, remains to be determined. It seems very unlikely that analyticity is really essential in the argument.

TYPE INVARIANTS OF PERIODIC TRANSFORMATIONS

5. Let us now adopt a slightly different point of view. Two homeomorphisms t and t_0 of S into itself are topologically equivalent, or belong to the same topological type if there exists a third homeomorphism τ such that $t_0=\tau^{-1}t\tau$. For arbitrary homeomorphisms, the problem of classifying types is too general to be of great interest, although a number of characteristic properties of types can be described in terms of the recurrence phenomena of repeated iteration (see for example [**4**]). But if t is periodic, one has invariants of a mere arithmetic nature, the period m for example. Suppose S is a circle. Then every periodic t of S into itself is equivalent to an orthogonal transformation.* In particular, if t preserves orientation it is equivalent to a rotation through an angle $2k\pi/m$, $(0<k\leqq m-1)$. If t_0 (of S into itself) is of the same type as t, then it is readily seen that m_0 must equal m, and k_0 must equal $\pm k$. Thus rotations of a circle through $2\pi/5$ and $4\pi/5$ do not belong to the same topological type. But suppose S is a torus with angular coordinates θ, ϕ and that t, t_0 are the rotations $\theta'=\theta+2\pi/5$, $\theta'=\theta+4\pi/5$, respectively. Then t and t_0 do belong to the same type (cf. Nielsen [**20**]). One can in fact see readily that if τ is the homeomorphism

$$\theta_0 = 2\theta + \phi,$$
$$\phi_0 = 5\theta + 3\phi$$

of S into itself, then $t_0=\tau t\tau^{-1}$. This suggests that the characteristic type invariants of periodic surface transformations will come to light only through a fairly penetrating analysis of surface topology. Such an analysis has recently been carried out by Nielsen [**20**] for orienta-

* First proved by J. F. Ritt [**30**].

ble surfaces and transformations which preserve orientation; the invariants obtained are sufficient to settle the question of whether or not two given periodic transformations are of the same type.

In Nielsen's work, the fundamental (Poincaré) group plays an important part. I shall now consider certain invariants which can be defined purely in terms of boundary relations. These invariants are less sharp than those of Nielsen, but they have proved to be very useful in a number of connections and can be defined for spaces of any number of dimensions.

Let $\mathfrak{g}$ be a coefficient group for a homology theory in S, and let $\rho(t)$ be a polynomial in t with coefficients in $\mathfrak{g}$; ρ is to be thought of as an operator on the chains and cycles of S. Consider an h-dimensional cycle Γ_h such that $\rho(t)\Gamma_h=0$. Such cycles will be called ρ-cycles; in their totality they form an additive group $\mathfrak{H}_h{}^\rho$. A ρ-cycle Γ_h which is the boundary of a ρ-chain is "ρ-homologous to zero," $\Gamma_h \underset{\rho}{\sim} 0$; denote the totality of these by $\mathfrak{H}_h^{0\,\rho}$. The group $H_h{}^\rho=\mathfrak{H}_h{}^\rho-\mathfrak{H}_h^{0\,\rho}$ is a "special homology group" characteristic of (S, t) and is, in fact, a type invariant of t. Of particular interest are the groups H^σ, $(\sigma=1+t+\cdots+t^{m-1})$, and H^δ, $(\delta=1-t)$, since they can be explicitly determined in a variety of cases and bear certain useful reciprocal relations to each other (see [**29**]).*

I have recently examined at some length [**36**] the case in which m is a prime and $S_n=S$ is assumed to be sphere-like in the sense that its dimension is n and that it has the same homology groups as an n-sphere. It can be shown that if $H_h{}^\rho$ is modified by taking for $\mathfrak{H}_h^{0\,\rho}$ the ρ-cycles which are ρ-homologous to zero modulo L, where L is the totality of fixed points, then if any group in the sequence

$$\overset{\delta}{H_n},\ \overset{\sigma}{H}_{n-1},\ \overset{\sigma}{H}_{n-2},\ \cdots,\ \overset{\rho}{H_0}, \qquad \rho=\sigma \text{ or } \delta,$$

vanishes, all those which follow it also vanish. Let $r=r(\mathfrak{g})$ be the dimension of the first group to vanish; I have shown that if $\mathfrak{g}=\mathfrak{m}$, the integers reduced modulo m, the modulo m dimension† of L is r; and if $r>\theta$, the Betti numbers (mod m) of L are the same, both locally and in the large, as those of an r-sphere; if $r=0$, L consists of two points. If S_3 is an actual three-sphere, then L is homeomorphic to an

* Strictly speaking, the relation $\rho\Gamma=0$ is not a topological one and in fact has little meaning unless S is a complex and t merely permutes the cells of S among themselves. For situations of a much more general nature, the theory of the σ- and δ-cycles and of the corresponding special homology groups has been worked out in my paper [**36**]. See also [**28**] of Richardson, who first introduced the special groups relative to an arbitrary (but fixed) polynomial ρ.

† Alexandroff [**2**].

actual r-sphere, ($r=2$ or 1), or consists of two points;* if t preserves orientation, $r=1$ unless L is null; if t reverses orientation, $r=0$ or 2, and m can only equal 2.

The methods which lead to these results will apply with slight modifications to the case in which m is composite, although the details are as yet unpublished. I must leave open, however, the problem of determining, in the general case, the dimensional and homological properties of L relative to coefficient groups other than m.† In case S_n is assumed to be a simplicial complex and t is also simplicial (carrying simplexes into simplexes), the results of the preceding paragraph do hold equally well when $\mathfrak{g}$ is say the group of integers. In addition, if t preserves orientation, it can now be shown that the dimension of L is of the same parity as n. Whether or not this is true for the non-simplicial case, even if "dimension" is taken to mean "dimension modulo m," I have as yet not determined.‡

If we no longer assume that S_n is sphere-like, but that it is still fairly regular locally, then although it is no longer true that L must have the homology groups of an r-sphere, locally the situation remains the same as before, and one can show that L possesses to the same degree as S_n whatever regularity can be described by modulo m homology theory. If S_n is an orientable simplicial manifold modulo m, and if t is simplicial and L a subcomplex of S_n, then L consists of a finite number of non-intersecting orientable manifolds (mod m), and the dimension of each is of the same parity as n.§

6. There is, as we shall see, a certain amount of interest in knowing the maximum number of components which L may have under certain conditions. Let us take m to be a prime and assume as above that S_n is an orientable simplicial manifold modulo m and that L is a subcomplex, t simplicial. We may write $L=L_p^1+L_p^2+\cdots+L_p^{k_p}+M$, where the L's are non-intersecting manifolds of dimension p and M is the sum of manifolds of dimension not p. I shall establish a certain

* The proof of the statement about S_3 depends on certain topological characterizations of one-dimensional manifolds by Alexandroff [3] and of higher dimensional manifolds by Wilder [39]. Only part of the theorem is stated and proved in [36]; the complete proof will be given elsewhere.

† Partial results are obtained in [36] when $\mathfrak{g}$ is the additive group of rational numbers.

‡ Except for the theorem [36] that if m is prime, the dimension of L can be zero only if n is even.

§ The dimension of L need not be of the same parity as n if S_n is non-orientable. Moreover, when $m=2$, simple examples show that L can be non-orientable when S_n is orientable; this however does not contradict our theorem, since orientability cannot be determined by modulo two topology.

relation between k_p and the group $\mathfrak{H}_{p+1}^{\sigma}$ of §5 (coefficients in $\mathfrak{m}$). Since m is a prime every homology group H is the direct sum of a number τ of cyclic groups of order m; τ will be called the rank of $\mathfrak{H}$. Let $\tau_{p+1}^{\sigma} = \text{rank } \mathfrak{H}_{p+1}^{\sigma}$, and let $\tilde{R}_p$ be the maximum number of non-intersecting p-cycles in S linearly independent with respect to homologies. Then the relation in question is

$$(1) \qquad k_p \leqq \tau_{p+1}^{\sigma} + \tilde{R}_p.$$

To prove this, choose in each L^i a p-cycle Γ^i which is not homologous to zero in L. It may happen that some or all of the Γ's are independent with respect to homologies in S_n. Suppose the Γ's so named that the first ν of them constitute a maximal set independent in S_n. Let the remaining Γ's now be denoted by $\Delta^1, \cdots, \Delta^{\mu}$, $(\nu+\mu=k_p)$. There is a homology relation between each Δ^i and the cycles $\Gamma^1, \cdots, \Gamma^{\mu}$, say

$$(2) \qquad F(A_{p+1}^i) = b^i\Delta^i + \sum_{j=1}^{\nu} a_j^i\Gamma^j, \qquad b^i \neq 0;\ i = 1, \cdots, \mu,$$

where F is the boundary operator. Now δA_{p+1}^i (see §5) is a cycle since $F\delta A = \delta FA = FA - tFA = 0$. Moreover

$$\sigma(\delta A) = (1 + t + \cdots + t^{m-1})(1-t)A = (1-t^m)A = 0.$$

Therefore δA may be considered as an element in $\mathfrak{H}_{p+1}^{\sigma}$. Let $C^1, \cdots, C^{\tau}$, $(\tau = \tau_{p+1}^{\sigma})$, be a basis for H_{p+1}^{σ}. Then we may write

$$(3) \qquad \delta A_{p+1}^i \underset{\sigma}{\sim} \sum_{j=1}^{\tau} c_j^i C^j, \qquad i = 1, \cdots, \mu.$$

I assert that $\mu \leqq \tau$. For suppose $\mu > \tau$. Then there exists a linear relation among the forms in the right of (3) with coefficients $g_1, \cdots, g_{\mu}$ not all zero. Hence from (3) we have

$$\sum_i g_i \delta A_{p+1}^i \underset{\sigma}{\sim} 0.$$

This implies the existence of a chain B_{p+2} such that $\sigma B = 0$ and $FB = \sum g_i \delta A_{p+1}^i$. The relation $\sigma B = 0$ implies the existence* of chains X_{p+2} and X_{p+2}^L, the latter in L, such that $B = \delta X + X^L$. Thus $F(\delta X) + FX^L = \delta(\sum g_i A_{p+1}^i)$. Now if $\sum g_i A_{p+1}^i$ contains a cell in L, that cell will appear in $\delta(\sum g_i A_{p+1}^i)$ with coefficient zero. Similarly, $F(\delta X) = \delta FX$ contains no cell of L. Hence $FX^L = 0$, and the bounding relation just written takes the form $F(\delta X_{p+2}) = \sum g_i \delta A_{p+1}^i$. Let

* [**36**], p. 141.

$Z_{p+1} = FX_{p+2} - \sum g_i A^i$. Then $\delta Z = 0$, so that we may write $Z_{p+1} = \sigma U_{p+1} + U^L_{p+1}$, $(U^L \subset L)$.* Now

$$
\begin{aligned}
0 = FFX &= F(\textstyle\sum g_i A^i + Z) \\
&= F(\textstyle\sum g_i A^i) + F\sigma U + FU^L \\
&= \sum_i g_i\left(b^i\Delta^i + \sum_j a_j^i \Gamma^j\right) + F\sigma U + FU^L.
\end{aligned} \tag{4}
$$

Since the Δ's, the Γ's, and $F(U^L)$ are cycles in L, $F\sigma U$ is also such a cycle. Let E_p be a simplex occurring in $F\sigma U$ with a non-zero coefficient; E_p must occur in the boundary of at least one of the chains $U, tU, \cdots, t^{m-1}U$, say

$$Ft^jU = yE_p + \cdots, \qquad y \neq 0,$$

where the dots denote a chain not containing E_p. We then have

$$\sigma FU = t^j(\sigma FU) = \sigma(Ft^jU) = y\sigma E_p + \cdots .$$

Since E_p is invariant under t, it follows that $\sigma E_p = mE_p = 0 \pmod m$, and we conclude that $\sigma FU = 0$. Thus (4) becomes

$$\sum_i g_i\left(b^i\Delta^i + \sum_j a_j^i \Gamma^j\right) = -FU^L;$$

hence the expression on the left is homologous to zero in L. Therefore the coefficients of the Δ's and Γ's must be zero. Since $b^i \neq 0$ we conclude that $g_i = 0$, $(i = 1, \cdots, \mu)$, which is impossible. This completes the proof of the inequality $\mu \leqq \tau$; and since $\nu \leqq \tilde{R}_p$, (1) is now established.

Consider the matrix a_j of (2), §5, defined now relative to the coefficient group $\mathfrak{m}$. Let q_j^δ be the rank of $e - a_j$, and q_j^σ the rank of $e + a_j + a_j^2 + \cdots + a_j^{m-1}$ (e is the identity matrix). It has been shown elsewhere [29] that

$$\tau^\sigma_{p+1} \leqq R_{p+1} + R_{p+2} + \cdots + R_n - q^\sigma_{p+1} - q^\delta_{p+2} - \cdots - q^\rho_n, \qquad \rho = \delta \text{ or } \sigma, \tag{5}$$

where the R's are the modulo m Betti numbers of S_n. If we combine (5) with (1), omitting the q's, we obtain

$$k_p \leqq \tilde{R}_p + R_{p+1} + \cdots + R_n. \tag{6}$$

Thus we have the result that *there exists, at least if m is prime, an upper bound for the number of p-dimensional components of L which is*

* [36], p. 141.

independent of the period m and, in fact, depends only on the topological structure of S. Notice that if $m=2$, orientation can play no part in the argument, and (6) then holds equally well for non-orientable manifolds.

Since $\tilde{R}_0=R_0=1$, we have the result that there can be at most $\sum_0^n R_i$ isolated fixed points. Suppose $S=S_2$ is an orientable closed surface of genus p. Then

$$\tilde{R}_1 = p, \; R_0 = R_2 = 1, \; R_1 = 2p.$$

Hence there can be at most $p+1$ pointwise invariant curves and $2p+2$ isolated invariant points.

Let $x_0, \cdots, x_s$ be homogeneous coordinates for a complex projective space P_s, and let V be an algebraic variety defined by equating to zero a finite number of polynomials in $x_0, \cdots, x_s$ with real coefficients. Let us assume that, considered as a real locus in a space of $2s$ real dimensions, V is a manifold modulo 2. The transformation t: $(x_0, \cdots, x_s)\rightarrow(\bar{x}_0, \cdots, \bar{x}_s)$, where $\bar{x}$ is the complex conjugate of x, is of period two and induces a homeomorphism of V into itself, the fixed points of which constitute the *real* folds or branches of V. Formula (6) then gives an upper bound for the number of connected real branches which V may have. In particular, suppose V is a real algebraic surface. If V has no algebraic singularities, the manifold condition is satisfied (cf. Zariski [**41**], p. 102), and we can say that there are at most $\tilde{R}_2+R_3+1$ connected real branches (the R's refer of course to V considered as a manifold of four real dimensions). If V is a real algebraic curve, it need not be restricted with regard to algebraic singularities, since V may now be replaced by its Riemann surface which is in any case a manifold. We obtain then the theorem of Harnack that a real algebraic curve of genus p can have at most $p+1$ real branches.*

* We have here taken it for granted that V can be subdivided into simplexes such that the conditions stated at the beginning of §6 are satisfied. There is no difficulty in showing that such is the case by means of the methods used by Brown and Koopman [**7**] in connection with the triangulation of analytic loci. As a matter of fact the results stated hold equally well for non-simplicial spaces and triangulations, and the assumptions used were only a convenience for exposition. It seems likely also that the condition that V be a manifold could also be dispensed with.

It should be pointed out that little is known about the modulo two Betti numbers of algebraic varieties whereas the non-modular Betti numbers are related to the birational invariants of V. It would therefore be of interest to find upper bounds for k_p in terms of Betti numbers of the latter type rather than the former.

The transformation $x_i\rightarrow\bar{x}_i$ was used by Lefschetz [**15**] in determining the number of real folds of a real abelian variety.

CYCLIC MANIFOLDS

7. We have seen that when S is a simple closed curve, the types of periodic transformations can be explicitly enumerated (§5). Let us consider the problem of enumeration when S is an odd dimensional sphere S_{2n-1}, and t is without fixed points. To make matters as simple as possible, let us suppose that t is a rotation. We may assume that S lies in a euclidean space E_{2n}. If a euclidean coordinate system in E_{2n} is properly chosen, t will be represented by the equations (1) (with the X's identically zero). The θ's are multiples of $2\pi/m$, say

$$\theta_{2i-1} = a_i \cdot 2\pi/m, \qquad 1 \leqq a_i \leqq m-1;\ i = 1, \cdots, n.$$

The a's will be called indices of t; *we shall assume that they are prime to m.* If t' is a second rotation of S with period m, and if t' has a set of indices $b_1, \cdots, b_2$ which, except for order and sign, are the same as $a_1, \cdots, a_n$, then t and t' are topologically equivalent.* Whether or not the converse is true remains, I believe, an open question of considerable interest. If has recently been shown† that the converse is true in a *combinatorial* sense. That is, if for S there exist cellular subdivisions Σ and Σ' identical in structure, with Σ invariant under t, Σ' under t', and if there is a homeomorphism τ of S into itself such that $\tau\Sigma = \Sigma'$ and such that $t' = \tau t \tau^{-1}$, then the a's and b's are the same modulo m except for order and sign. But if one assumes only topological equivalence, then the most that is known at present is that

$$a_1 a_2 \cdots a_n \equiv \pm\, b_1 b_2 \cdots b_n \pmod{m}.$$

This was first proved by de Rham [24] by means of the theory of looping coefficients. One can also establish this relation in a very elementary manner by means of our special homologies. For this purpose three simple lemmas are sufficient. Let us assume for the moment that S possesses a simplicial subdivision which is invariant under t. By a natural sort of extension, t may be thought of as carrying chains into chains in such a way as to preserve boundary relations. Taking as coefficient group either the integers or the integers modulo m, we have the following lemmas:

LEMMA 1. *If δX is a σ-cycle, then FX is a δ-cycle; if $\delta X \underset{\sigma}{\sim} 0$ then $FX \underset{\delta}{\sim} 0$. The same holds with σ, δ interchanged.*‡

LEMMA 2. *If X is a chain, then δX and $(1-t^a)X$ are σ-chains; if they are cycles, then $(1-t^a)X \underset{\sigma}{\sim} a(\delta X)$.*

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* In fact they are equivalent under an orthogonal transformation.

† de Rham [25]; see also Reidemeister [23], Franz [11].

‡ See [36], p. 143.

PROOF. Since $\sigma=\sigma t=\sigma t^a$, we have $\sigma(\delta X)=\sigma(1-t^a)X=0$, which proves the first assertion. To prove the second, let Y be a chain bounded by δX. Then $\delta FY=\delta\delta X$. Since $\sigma\delta=0$, we may write $\delta\delta X\underset{\sigma}{\sim}0$, or $(1-t)\delta X\underset{\sigma}{\sim}0$, or $t\delta X\underset{\sigma}{\sim}\delta X$, and on repeating the argument we have $t^i\delta X\underset{\sigma}{\sim}\delta X$. Hence

$$(1-t^a)X=(1+t+\cdots+t^{a-1})\delta X\underset{\sigma}{\sim}a\delta X.$$

LEMMA 3. *If X_0 is a 0-chain consisting of a single 0-simplex with coefficient ± 1, and if $p\delta X_0\underset{\sigma}{\sim}0$, then $p\equiv 0 \pmod m$.*

PROOF. Every σ-chain is of the form δV ([**36**], p. 141); hence there exists a one-chain Y_1 such that $F\delta Y_1=p\delta X_0$. Let $Z_0=FY_1-pX_0$. Then $\delta Z_0=0$; hence $Z_0=\sigma U_0$ ([**36**], p. 141). Now the sum of the coefficients of FY_1 is zero (this being the case for the boundary of each individual one-simplex). Moreover the sum of the coefficients of $Z_0=\sigma U_0$ is* zero modulo m, whereas the sum of the coefficients of pX_0 is $\pm p$. Hence $p\equiv 0 \pmod m$.

Suppose now that t is a rotation with indices $a_1,\cdots,a_n$ prime to m. It is easy to show (see [**25**], p. 741, footnote) that there exist chains $X_0,\cdots,X_{2n-1}$ such that (i) $\sigma X_{2n-1}=\pm S_{2n-1}$; (ii) X_0 is a single vertex with coefficient ± 1; and (iii)

$$\begin{aligned} FX_{2h-1}&=(1-t^{\alpha_h})X_{2h-2}, &\quad h&=1,\cdots,n,\\ FX_{2h}&=\sigma X_{2h-1}, &\quad h&=1,\cdots,n, \end{aligned}\tag{1}$$

where $\alpha_h\equiv a_h^{-1} \pmod m$. Let us call any set of chains satisfying these three conditions a normal set with indices a_i. Suppose there is a second normal set Y with indices b_i. Then

$$\prod a_i\equiv\pm\prod b_i \pmod m.\tag{2}$$

For we have $S_{2n-1}=\pm\sigma X_{2n-1}=\pm\sigma Y_{2n-1}$. Hence $\pm\sigma X_{2n-1}\underset{\delta}{\sim}\sigma Y_{2n-1}$; and from Lemma 1 and (1) we have

$$\pm(1-t^{\alpha_n})X_{2n-2}\underset{\sigma}{\sim}(1-t^{\beta_n})Y_{2n-2},\qquad \beta_n=b_n^{-1}.$$

Hence from Lemma 2, $\pm(1-t)\alpha_nX_{2n-2}\underset{\sigma}{\sim}(1-t)\beta_nY_{2n-2}$, and from Lemma 1, $\pm\sigma\alpha_nX_{2n-3}\underset{\delta}{\sim}\sigma\beta_nY_{2n-3}$. Again, from Lemma 1 then Lemma 2, we obtain $\pm\alpha_n\alpha_{n-1}\delta X_{2n-4}\sim\beta_n\beta_{n-1}\delta Y_{2n-4}$ and finally

$$\pm\alpha_n\cdots\alpha_0\delta X_0\underset{\sigma}{\sim}\beta_n\cdots\beta_0\delta Y_0.$$

* To each vertex E_0^i there are associated two 0-chains $+E_0^i$ and $-E_0^i$. If $tE_0^i=E_0^j$, it follows, from the condition that t preserve bounding relations, that $t(+E_0^i)=+E_0^j$. Thus $(1+t+\cdots+t^{m-1})(+E_0^i)=+E_0^j+E_0^h+\cdots$; hence the sum of the coefficients of σE_0^i is m.

From Lemma 2 we conclude that $\pm\prod\alpha_i\equiv\prod\beta_i$ (mod m), and from this (2) follows.

Suppose now that t' is a rotation of S_{2n-1} with indices $b_1, \cdots, b_n$. Suppose τ is a homeomorphism of S with itself such that $t=\tau^{-1}t'\tau$. Imagine first that τ carries simplexes into simplexes. If $Y_0, \cdots, Y_{2n-1}$ is a normal set relative to t', with indices b_i, then $\tau^{-1}Y_0, \cdots, \tau^{-1}Y_{2n-1}$ is a normal set relative to t with indices still b_i. Hence $\prod b_i\equiv\pm\prod a_i$ (mod m). In case τ is not simplicial, the images of the Y's under τ are no longer chains of the complex S but will become so after a suitable deformation. The deformation can in fact be so chosen that the new Y's again constitute a normal set for t with indices b_i.

The larger questions connected with the equivalence problem for rotations are of fundamental interest in topology. It has been known for some time that two manifolds, which are identical so far as the homology invariants and the fundamental group are concerned, need not be homeomorphic (J. W. Alexander [**1**]). Consider a space S and a periodic t of S into itself. The so-called modular space $S^{(t)}$ is constructed by identifying points which are images of each other under powers of t. If S is a sphere of an odd number of dimensions and t a rotation of period m and with indices prime to m, then $S^{(t)}$ is called a cyclic manifold of order m. It can be shown (de Rham [**24**]) that two cyclic manifolds $S^{(t)}$ and $S^{(t')}$ of order m have the same Betti numbers, the same torsion coefficients, and the same fundamental group. Moreover, a necessary and sufficient condition that $S^{(t)}$ and $S^{(t')}$ be homeomorphic is that there exist an h prime to m such that t^h and t' are topologically equivalent.* Suppose the indices of t and t' are a_i and b_i. Those of t^h are then ha_i. Hence by $S^{(t)}$ and $S^{(t')}$ they can be homeomorphic only if the relation $\pm\prod a_i\equiv h^n b_i$ (mod m) has a solution in h (cf. de Rham [**24**]). Suppose for example that S is a three-sphere, that $m=5$, and that the indices are (1, 1) for t and (1, 2) for t'. Then $S_3{}^{(t)}$ and $S_3{}^{(t')}$ are not homeomorphic, since the equation $\pm 1\equiv 2h^2$ (mod 5) has no solutions.†

The problem of finding new topological invariants of manifolds is of course a fundamental one. In focusing attention on cyclic manifolds we have shown the existence of invariants which are independent of the ordinary ones; they are implicit in the relation (2). The existence of a solution for (2) is, however, not sufficient to ensure

* For a given S, the modular space $S^{(t)}$ is a type invariant of t and the special groups $\mathfrak{H}^\sigma, \mathfrak{H}^\delta$ (§5) are closely related to the ordinary homology groups of $S^{(t)}$. These relations have proved to be particularly useful for determining certain relative homology characters for cyclic and symmetric products (see [**29**]).

† This is essentially the example of Alexander [**1**].

homeomorphism. If one is content to ask, when are two cyclic manifolds equivalent in a strictly combinatorial sense, one finds a complete answer in recent noteworthy papers of Reidemeister [23], de Rham [25], and Franz [11]. Whether or not these combinatorial results are really topological remains to be determined.

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