ON SOME INEQUALITIES OF S. BERNSTEIN AND W. MARKOFF FOR DERIVATIVES OF POLYNOMIALS*

A. C. SCHAEFFER AND R. J. DUFFIN

A well known inequality on the derivatives of polynomials is that of S. Bernstein.†

BERNSTEIN'S THEOREM. *Suppose that $f(x)$ is a polynomial of degree n or less, and that in the interval $(-1, 1)$*

$$|f(x)| \leqq 1.$$

Then

$$|f'(x)|^2 \leqq \frac{n^2}{1-x^2}, \qquad -1 \leqq x \leqq 1,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$, $|\gamma| = 1$,‡ where $T_n(x)$ is the nth Tchebychef polynomial.

The extension of this theorem of Bernstein to the higher derivatives plays an important role in this paper. Thus, if $f(x)$ satisfies the conditions given in Bernstein's theorem, we obtain the inequality

$$|f^{(p)}(x)|^2 \leqq \left(\frac{d^p}{dx^p} \cos n\theta\right)^2 + \left(\frac{d^p}{dx^p} \sin n\theta\right)^2, \qquad x = \cos\theta,$$

for x in $(-1, 1)$. Using this inequality we are able to give a simple proof of W. Markoff's theorem,§ which states that under the conditions of Bernstein's theorem

$$|f^{(p)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2p-1)}, \quad -1 \leqq x \leqq 1.$$

* Presented to the Society, April 8, 1938.

† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoires de l'Académie Royale de Belgique, (2), vol. 4 (1912), pp. 1–104. M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

‡ γ stands hereafter for a constant, real or complex, of absolute magnitude 1.

§ W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–258, translated by J. Grossman. The original appeared in Russian in 1892.

G. Szegö, *Über einen Satz von A. Markoff*, Mathematische Zeitschrift, vol. 23 (1925) pp. 45–61.

Before proceeding with the proof of these inequalities we find it necessary to establish several lemmas concerning the properties of the Tchebychef polynomial $T_n(x)$, and a related function $S_n(x)$, (not a polynomial) which are defined by the relations

$$(1)\qquad \begin{aligned} T_n(x) &= \cos n\theta, \\ S_n(x) &= \sin n\theta, \qquad x = \cos\theta, \end{aligned}$$

n being any positive integer. These functions are independent solutions of the differential equation

$$(2)\qquad (1 - x^2)R''(x) - xR'(x) + n^2R(x) = 0,$$

the general real solution of which may be written

$$R(x) = aT_n(x) + bS_n(x) = c\cos(n\theta - \alpha), \qquad x = \cos\theta,$$

where a, b, c, and α are real constants. Differentiating (2) p times we obtain

$$(3)\quad (1 - x^2)R^{(p+2)}(x) - (2p + 1)xR^{(p+1)}(x) + (n^2 - p^2)R^{(p)}(x) = 0,$$

and this may be written in the equivalent form

$$(4)\ \frac{d}{dx}\{[1 - x^2][R^{(p+1)}(x)]^2 + [n^2 - p^2][R^{(p)}(x)]^2\} = 4px[R^{(p+1)}(x)]^2.$$

For $p \leqq n$ the functions $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ are particular solutions of (3) and (4).

Let $M_p(x)$ be defined by the relation

$$(5)\qquad \begin{aligned} M_p(x) &= (T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2 \\ &\equiv \left(\frac{d^p}{dx^p}\cos n\theta\right)^2 + \left(\frac{d^p}{dx^p}\sin n\theta\right)^2. \end{aligned}$$

Then it is clear that $M_p(x)$ also satisfies (4), that is

$$(6)\qquad \frac{d}{dx}\{(1 - x^2)M_{p+1}(x) + (n^2 - p^2)M_p(x)\} = 4pxM_{p+1}(x).$$

LEMMA 1. *In the open interval* $(-1, 1)$, $T_n^{(p)}(x)$ *has* $n-p$ *zeros all of which are simple,* $(p=0, 1, 2, \cdots, n)$; $S_n^{(p)}(x)$ *has* $n+1-p$ *zeros all of which are simple,* $(p=1, 2, 3, \cdots, n+1)$; *and the zeros of* $T_n^{(p)}(x)$ *and* $S_n^{(p)}(x)$ *separate one another,* $(p=1, 2, 3, \cdots, n)$.

PROOF. From the definition (1) we see that $T_n(x)$ has n simple zeros in the interior of $(-1, 1)$, and since it is a polynomial of degree n, these are its only zeros. From Rolle's theorem it follows that

$T_n^{(p)}(x)$, $(p=1, 2, \cdots, n)$, has exactly $n-p$ zeros in $(-1, 1)$, all of which are simple. From (1) we see that $S_n(x)$ may be expressed in the form

$$S_n(x) = T_n'(x)(1-x^2)^{1/2}/n,$$

from which it follows that $S_n(x)$ has $n+1$ zeros in the closed interval $(-1, 1)$, and that $S_n'(x)$ becomes infinite as $x \to \pm 1$. By successive use of Rolle's theorem it follows that $S_n^{(p)}(x)$ must have at least $n+1-p$ distinct zeros in the open interval $(-1, 1)$. But $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ are linearly independent solutions of the differential equation (3), so by Sturm's well known theorem the zeros of $T_n^{(p)}(x)$ and $S_n^{(p)}(x)$ separate one another. It follows that $S_n^{(p)}(x)$ has exactly $n+1-p$ zeros in the interval $(-1, 1)$, all of which are simple, $(p=1, 2, 3, \cdots, n)$. For the case $p=n+1$ we see from (3) that $S_n^{(n+1)}(x)$ is a solution of the differential equation

$$(1-x^2)\frac{d}{dx}S_n^{(n+1)}(x) - (2n+1)xS_n^{(n+1)}(x) = 0.$$

Suppose that $S_n^{(n+1)}(x)$ had at an interior point x_0 a zero of order k, $(k \geqq 1)$. Then, from the differential equation, $dS_n^{(n+1)}(x)/dx$ would have a zero of order k (or of order $k+1$ if $x_0=0$). This would demand that $S_n^{(n+1)}(x)$ have a zero of order $k+1$ (or of order $k+2$ if $x_0=0$), in contradiction to the assumption. This proves that $S_n^{(n+1)}(x)$ cannot have a zero.

LEMMA 2. *In the expansion*

$$M_p(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k},$$

valid for $|x|<1$, $(1 \leqq p \leqq n)$, *all the coefficients are greater than zero.*

PROOF. The statement is evidently true for $p=1$, since, by definition,

$$M_1(x) = (T_n'(x))^2 + (S_n'(x))^2 = n^2/(1-x^2).$$

We now proceed by induction. Let

$$M_p(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k},$$

$$M_{p+1}(x) = \sum_{k=0}^{\infty} b_{2k}x^{2k}.$$

On substituting these power series in the differential equation (6) we obtain between the coefficients the relation

$$kb_{2k} + k(n^2 - p^2)a_{2k} = (2p + k)b_{2k-2}. \tag{7}$$

Suppose that every coefficient a_{2k} is greater than zero. From the relation (7) we see that if, for a particular k, b_{2k} is positive, then b_{2k-2} is also positive; and by repeated use of this relation each of the coefficients $b_{2k-4}, b_{2k-6}, \cdots, b_2, b_0$ is positive. Thus if b_{2k} is greater than zero, then the same is true of each of the preceding coefficients. The functions $S_n^{(p+1)}(x)$ and $M_{p+1}(x)$ are unbounded in the interval $(-1, 1)$ since $S_n'(x)$ becomes infinite as $x \to \pm 1$. This shows, since $M_{p+1}(x)$ is non-negative, that there are arbitrarily large values of k for which b_{2k} is greater than zero. But if one coefficient is positive, we have shown that all the preceding ones are also positive, and it follows that all b_{2k} are positive. This completes the induction.

By definition (1), $T_n(1) = 1$, and by (3)

$$(2p + 1)T_n^{(p+1)}(1) = (n^2 - p^2)T_n^{(p)}(1).$$

From this we find by induction that

$$T_n^{(p)}(1) = \frac{n^2(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (p - 1)^2)}{1 \cdot 3 \cdot 5 \cdot \quad \cdots \quad \cdot (2p - 1)}. \tag{8}$$

THEOREM 1. *Let $f(x)$ be a polynomial of degree n or less with real coefficients such that*

$$|f(x)| \leqq 1, \qquad -1 \leqq x \leqq 1,$$

and suppose that $f(x) \not\equiv \gamma T_n(x)$. Then for every real α the first n derivatives of the function

$$\cos \alpha T_n(x) + \sin \alpha S_n(x) - f(x)$$

can have only simple zeros in the interval $(-1, 1)$.

PROOF. There is no loss of generality in supposing that $0 \leqq \alpha < \pi$. Let

$$\begin{cases} R(x) = \cos \alpha T_n(x) + \sin \alpha S_n(x) = \cos (n\theta - \alpha), & x = \cos \theta \\ R'(x) = n \sin (n\theta - \alpha)/\sin \theta. \end{cases} \tag{9}$$

Then at the points where $R(x)$ vanishes we have

$$(R'(x))^2 = \frac{n^2}{1 - x^2};$$

hence, by Bernstein's theorem,

$$|f'(x)| < |R'(x)| \tag{10}$$

at these points. If $\alpha \neq 0$, this inequality is also true in a neighborhood of the points -1 and $+1$, for $S_n'(x)$ becomes infinite here while $T_n'(x)$ and $f'(x)$ are bounded in the whole interval. We now distinguish two cases, $\alpha = 0$ and $\alpha > 0$.

CASE I. $\alpha = 0$.

In this case $R(x) \equiv T_n(x)$, and the function considered reduces to

$$T_n(x) - f(x)$$

which is a polynomial of degree n or less, not identically zero. Then by (10)

$$T_n'(x) - f'(x)$$

is alternately plus and minus at the n points where $R(x)$ vanishes; so it has at least $n-1$ distinct zeros. Using Rolle's theorem we see that $T_n^{(p)}(x) - f^{(p)}(x)$ has at least $n-\mathrm{p}$ distinct zeros, $(1 \leqq p \leqq n)$, and it can have no others, as it is a polynomial of degree $n-p$. Thus all its zeros are simple and Theorem 1 is true for the case $\alpha = 0$.

CASE II. $0 < \alpha < \pi$.

We are going to show first that in this case the function

$$R'(x) - f'(x)$$

has at least n distinct zeros.

If $\alpha = \pi/2$, then $R(x) \equiv S_n(x)$, and we see that at the $n-1$ zeros of $S_n(x)$ in the interior of $(-1, 1)$ and in a neighborhood of the two end points the inequality (10) is satisfied. Thus, $S_n'(x)$ is alternately plus and minus at $n+1$ successive points where (10) is true; so the function

$$R'(x) - f'(x) \equiv S_n'(x) - f'(x)$$

has at least n distinct zeros in $(-1, 1)$.

If $\alpha \neq \pi/2$, then either $0 < \alpha < \pi/2$ or $\pi/2 < \alpha < \pi$, and as the two cases are essentially the same we shall consider only the case $0 < \alpha < \pi/2$. From relation (9) we see that $R(x)$ vanishes at the n points $\theta = [(k+\frac{1}{2})\pi + \alpha]/n$, $(k = 0, 1, \cdots, n-1)$, in $(-1, 1)$. Selecting these n points and one point from a small neighborhood of $x = 1$, we have $n+1$ points in $(-1, 1)$ at which the inequality (10) is satisfied. From the relation (9) it is seen that $R'(x)$ is alternately plus and minus at these $n+1$ successive points. Then the function

$$R'(x) - f'(x)$$

is alternately plus and minus at $n+1$ points and so has at least n distinct zeros.

Thus if $0 < \alpha < \pi$, the function

$$R'(x) - f'(x)$$

has at least n distinct zeros in $(-1, 1)$. Using Rolle's theorem one shows that

$$R^{(p)}(x) - f^{(p)}(x)$$

has at least $n+1-p$ distinct zeros. If it had one multiple zero, then $R^{(p+1)}(x) - f^{(p+1)}(x)$ would have at least $n+1-p$ distinct zeros, and finally

$$R^{(n+1)} - f^{(n+1)}$$

would have at least one zero. Since $f(x)$ and $T_n(x)$ are polynomials of degree n at most this means that $S_n^{(n+1)}(x)$ has at least one zero at some interior point of $(-1, 1)$; but by Lemma 1 this is impossible. This completes the proof of Theorem 1.

THEOREM 2. *For any polynomial $f(x)$ of degree n or less, the inequality $|f(x)| \leqq 1$ in $(-1, 1)$ implies*

$$|f^{(p)}(x)|^2 \leqq M_p(x), \qquad -1 \leqq x \leqq 1;\ p = 1, 2, 3, \cdots, n,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$.

PROOF. For simplicity we shall suppose that $f(x) \not\equiv \gamma T_n(x)$. We shall consider first the case in which all the coefficients of $f(x)$ are real. Suppose that at some point x_0, $(-1 < x_0 < 1)$, we have

$$|f^{(p)}(x_0)|^2 \geqq M_p(x_0) \tag{11}$$

for some p, $(1 \leqq p \leqq n)$. Consider the function

$$R(x) + \lambda f(x),$$

where $R(x)$ is of the form

$$R(x) = \cos \alpha T_n(x) + \sin \alpha S_n(x), \tag{12}$$

and α and λ are real constants to be determined. We shall show that if (11) is true, then real α and λ exist, $(-1 \leqq \lambda \leqq 1)$, so that at the point x_0 the pth derivative of the function $R(x) + \lambda f(x)$ has a double zero.

Let α be chosen so that the relation

$$R^{(p+1)}(x_0) f^{(p)}(x_0) - R^{(p)}(x_0) f^{(p+1)}(x_0) = 0 \tag{13}$$

is satisfied. This is always possible for, on expressing $R^{(p)}(x)$ and $R^{(p+1)}(x)$ by use of equation (12), the relation (13) is equivalent to an equation of the form

$$a \cos \alpha + b \sin \alpha = 0,$$

where a and b are real, and this has a solution.

Having chosen α we see from Cauchy's inequality that

$$| R^{(p)}(x) |^2 \leqq \{\cos^2 \alpha + \sin^2 \alpha\} \{ (T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2 \} \equiv M_p(x).$$

Hence (supposing that inequality (11) is true) we choose λ, $(-1 \leqq \lambda \leqq 1)$, so that at the point x_0

$$R^{(p)}(x_0) + \lambda f^{(p)}(x_0) = 0.$$

Substituting this in equation (13), we have

$$f^{(p)}(x_0) \{ R^{(p+1)}(x_0) + \lambda f^{(p+1)}(x_0) \} = 0$$

and this means that the second factor is zero. Thus the pth derivative of the function

$$R(x) + \lambda f(x)$$

has a double zero at the point x_0, but by Theorem 1 this is impossible. The contradiction proves Theorem 2 in the case in which all the coefficients of $f(x)$ are real.

Now allow $f(x)$ to have complex coefficients,* and choose a real constant β so that, at a point x_0 arbitrarily chosen in $(-1, 1)$,

$$e^{i\beta} f^{(p)}(x_0)$$

is real. Writing

$$e^{i\beta} f(x) = f_1(x) + i f_2(x),$$

where $f_1(x)$ and $f_2(x)$ have real coefficients, we see that if $f_1(x)$ were of the form $f_1(x) \equiv \gamma\, T_n(x)$, then $f_2(x)$ would vanish at the $n+1$ points where $T_n(x) = \pm 1$ and so would vanish identically. Then $f(x)$ itself would be of the form $f(x) \equiv \gamma\, T_n(x)$; but we have supposed that this is not the case, so $f_1(x) \not\equiv \gamma\, T_n(x)$. We have already proved that Theorem 1 applies to the polynomial $f_1(x)$, and since $e^{i\beta} f^{(p)}(x_0) = f_1^{(p)}(x_0)$, it follows that

$$| f^{(p)}(x_0) |^2 = | f_1^{(p)}(x_0) |^2 < M_p(x_0).$$

* This method of extending inequalities to polynomials with complex coefficients has been used by S. Bernstein, *Leçons sur les propriétés extrémals et meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926, p. 45.

This completes the proof of Theorem 2.

LEMMA 3. *Let $f(x)$ be a polynomial of degree n or less, such that*

$$|f(x)| \leqq 1$$

in $(-1, 1)$. If x_0 is a point of $(-1, 1)$ which lies either to the left or to the right of all zeros of $S_n^{(p)}(x)$, then

$$|f^{(p)}(x_0)| \leqq |T_n^{(p)}(x_0)|,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$.

PROOF. Let a_k, $(k=1, 2, 3, \cdots, n+1-p)$, be the zeros of $S_n^{(p)}(x)$ in the interval $(-1, 1)$. Let $\phi(x)$ be a polynomial of degree $n+1-p$ which vanishes at these $n+1-p$ points,

$$\phi(x) = (x - a_1)(x - a_2) \cdots (x - a_{n+1-p}).$$

Then using the Lagrange interpolation formula we have, since $f^{(p)}(x)$ is a polynomial of degree $n-p$ or less,

$$f^{(p)}(x) = \phi(x) \sum_{k=1}^{n+1-p} \frac{f^{(p)}(a_k)}{\phi'(a_k)(x - a_k)},$$

and there is a similar expression for $T_n^{(p)}(x)$. The zeros of $S_n^{(p)}(x)$ and $T_n^{(p)}(x)$ interspace one another, so $T_n^{(p)}(x)$ is alternately plus and minus at the successive zeros of $S_n^{(p)}(x)$, and it is easily seen that $\phi'(x)$ alternates in sign at successive zeros of $\phi(x)$. Thus all the numbers

$$\frac{T_n^{(p)}(a_k)}{\phi'(a_k)}$$

are of the same sign, $(k=1, 2, 3, \cdots, n+1-p)$. Moreover, by Theorem 2, we have at the zeros of $S_n^{(p)}(x)$

$$|f^{(p)}(a_k)|^2 \leqq M_p(a_k) = (T_n^{(p)}(a_k))^2. \tag{14}$$

Now let x_0 be a point which lies to the right of all zeros of $S_n^{(p)}(x)$, so that $x_0 - a_k > 0$, $(k = 1, 2, \cdots, n+1-p)$. Then it follows immediately that

$$\begin{aligned} |f^{(p)}(x_0)| &\leqq |\phi(x_0)| \sum \left| \frac{f^{(p)}(a_k)}{\phi'(a_k)(x_0 - a_k)} \right| \\ &\leqq |\phi(x_0)| \sum \left| \frac{T_n^{(p)}(a_k)}{\phi'(a_k)(x_0 - a_k)} \right| = |T_n^{(p)}(x_0)|. \end{aligned} \tag{15}$$

The equality can occur throughout (15) only if the equality is true in (14), and by Theorem 2 this is true only if $f(x) \equiv \gamma T_n(x)$. This proves Lemma 3 in the case where x_0 lies to the right of all zeros of $S_n^{(p)}(x)$, and the same method is available if x_0 lies to the left of all zeros of $S_n^{(p)}(x)$.

We can now prove the theorem of W. Markoff.

MARKOFF'S THEOREM. *If $f(x)$ is a polynomial of degree n or less, the inequality $|f(x)| \leqq 1$, $(-1 \leqq x \leqq 1)$, will imply*

$$|f^{(p)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2p-1)}, \quad -1 \leqq x \leqq 1,$$

for $p=1, 2, 3, \cdots, n$. The equality can occur only at $x=\pm 1$ and here only if $f(x)\equiv\gamma T_n(x)$.

PROOF. By Theorem 2 we see that

$$|f^{(p)}(x)|^2 \leqq (T_n^{(p)}(x))^2 + (S_n^{(p)}(x))^2 \equiv M_p(x),$$

and in Lemma 3 we have shown that $M_p(x)$ increases monotonically in (0, 1). Let a, $(|a|<1)$, be the zero farthest to the right of $S_n^{(p)}(x)$ in the interval $(-1, 1)$; then in the interval $(-a, a)$ we have

$$|f^{(p)}(x)|^2 \leqq M_p(x) \leqq M_p(a) = (T_n^{p)}(a))^2. \tag{16}$$

We know that $T_n^{(p)}(x)$ increases monotonically in the interval $(a, 1)$, since, by Lemma 1, its zeros lie in the interval $(-a, a)$. Moreover $T_n^{(p)}(x)$ is either an odd or an even function of x, hence the maximum of its absolute magnitude in the intervals $(-1, -a)$ and $(a, 1)$ occurs at $x=\pm 1$. Then if x_1 lies in $(-a, a)$,

$$|f^{(p)}(x_1)| \leqq |T_n^{(p)}(a)| < |T_n^{(p)}(1)|.$$

If x_2 lies in either of the intervals $(-1, -a)$, $(a, 1)$, we have from Lemma 3

$$|f^{(p)}(x_2)| \leqq |T_n^{(p)}(x_2)| \leqq |T_n^{(p)}(1)|,$$

and the equality can occur only at $x_2=\pm 1$. Thus, if x lies in the interval $(-1, 1)$, we have the inequality

$$|f^{(p)}(x)| \leqq |T_n^{(p)}(1)|,$$

and the equality can occur only at $x=\pm 1$, and here only if $f(x)\equiv\gamma T_n(x)$. The explicit expression for $T_n^{(p)}(1)$ obtained in (8) furnishes the final step in the proof of Markoff's theorem.

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