ON AN INTEGRAL TEST OF R. W. BRINK FOR THE CONVERGENCE OF SERIES

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1. *Introduction.* The test in question is embodied in the following theorem due to R. W. Brink.*

Let $\sum^\infty u_n$ *be a series of positive terms. Also let* $r(x)$ *be a function such that* (i) $r(n)=r_n=u_{n+1}/u_n$, (ii) $0<\lambda\leqq r(x)\leqq\mu$, (iii) $r'(x)$ *exists and is continuous,* $\int^\infty|r'(x)|dx$ *is convergent. Then the convergence of the integral*

$$\int^\infty e^{\int^x \log r(t)dt}dx$$

is necessary and sufficient for the convergence of the series $\sum^\infty u_n$.

It is the object of this note to show that Brink's theorem can be expressed in a more general form (Theorem 3 below) which leads at once to all the ratio tests for the convergence of series associated with Kummer's test. The ratio tests are thus welded into unity from a point of view somewhat different from that adopted by Pringsheim in his classical paper *Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern.*†

2. *Connection of Brink's Theorem with the Maclaurin-Cauchy Integral Test.* The problem which confronts us in Brink's theorem is clearly that of setting up an integral $\int^x F(t)dt$ whose behaviour at infinity is reflected by a given series $\sum^\infty u_n$. When $\sum^\infty u_n$ has all but a finite number of terms positive, the method employed to establish the Maclaurin-Cauchy integral test shows that the convergence of $\int^\infty F(x)dx$ is sufficient for that of $\sum^\infty u_n$ if for $n\leqq x\leqq n+1$, $0<u_n\leqq F(x)$, $(n=m, m+1, \cdots)$. Denoting u_{n+1}/u_n by r_n, the condition assumed is that

$$r_{n-1}\cdot r_{n-2}\cdots r_m\leqq\frac{F(x)}{u_m},\qquad (n\leqq x\leqq n+1),$$

* R. W. Brink, *A new integral test for the convergence and divergence of infinite series*, Transactions of this Society, vol. 19 (1918), p. 188.

† Mathematische Annalen, vol. 35 (1890), pp. 359–372.

that is,

$$\sum_{\nu=m}^{n-1} \log r_\nu \leqq \log \frac{F(x)}{F(m)} + \log \frac{F(m)}{u_m}$$

$$= \int_m^x \frac{F'(t)}{F(t)} dt + \log \frac{F(m)}{u_m},$$

or,

$$\sum_{\nu=m}^{n-1} \left[\log r_\nu - \int_\nu^{\nu+1} \frac{F'(t)}{F(t)} dt \right] - \int_n^x \frac{F'(t)}{F(t)} dt \leqq \log \frac{F(m)}{u_m},$$

$$(n \leqq x \leqq n+1).$$

The right-hand member of the above inequality may be altered to any arbitrary constant; for this would merely imply the multiplication of $F(x)$ by a positive constant in our initial hypothesis. Also, for the truth of the altered inequality the following conditions are sufficient:

(i) $\dfrac{F'(x)}{F(x)}$ is bounded and integrable for $x \geqq m$,

(ii) $\log r_\nu - \displaystyle\int_\nu^{\nu+1} \frac{F'(x)}{F(x)} dx \leqq \delta_\nu$,

where $\sum^n \delta_\nu$ is bounded above as $n \to \infty$, which is a consequence of

$$\log r_\nu - \frac{F'(x)}{F(x)} \leqq \delta_\nu, \qquad (\nu \leqq x \leqq \nu + 1).$$

If we put $F'(x)/F(x) = f(x)$, the integral whose convergence is sufficient for that of $\sum^\infty u_n$ assumes the form $\int^\infty e^{\int^x f(t)dt} dx$. Further, the divergence of this integral is sufficient for the divergence of $\sum^\infty u_n$ provided that in (ii) above the inequality sign is reversed and $\sum^n \delta_\nu$ is bounded below. Hence we are led to formulate the test as follows.

THEOREM 1. *Let $\sum^\infty u_n$ be a series of positive terms and $r_n = u_{n+1}/u_n$. If*

(i) *$f(x)$ is bounded and integrable for $x \geqq m$, and*

$(\mathcal{C})$: $\quad \int^{\infty} e^{\int^x f(t)dt} dx$ *is convergent,*

$\{$*or* $(\mathcal{D})$: $\quad \int^{\infty} e^{\int^x f(t)dt} dx$ *is divergent*$\}$;

(ii) *for* $n \leqq x \leqq n+1$,

$(\mathcal{C})$: $\quad \log r_n \leqq f(x) + \delta_n, \ \sum^n \delta_\nu$ *being bounded above,*

$\{$*or* $(\mathcal{D})$: $\quad \log r_n \geqq f(x) + \delta_n', \ \sum^n \delta_\nu'$ *being bounded below*$\}$;

then $\sum^\infty u_n$ *is convergent* $\{$*or divergent*$\}$.

The direct proof of the theorem is exactly on the lines of that of Theorem 2 given below.

Brink's Integral Test. This is an immediate deduction from Theorem 1. For if $r(x)$ is defined as in Brink's theorem, then

$$\log r(x) - \log r_n = \int_n^x \frac{r'(t)}{r(t)} dt,$$

and

$$\left| \log r(x) - \log r_n \right| \leqq \frac{1}{\lambda} \int_n^{n+1} \left| r'(t) \right| dt, \qquad (n \leqq x \leqq n+1).$$

Hence replacing $f(x)$ by $\log r(x)$ and taking

$$\delta_n = \frac{1}{\lambda} \int_n^{n+1} \left| r'(t) \right| dt, \qquad \delta_n' = -\frac{1}{\lambda} \int_n^{n+1} \left| r'(t) \right| dt,$$

we see that $\sum^\infty u_n$ converges or diverges with

$$\int^\infty e^{\int^x \log r(t) dt} dx.$$

Thus Theorem 1 includes Brink's integral test, as one of his own theorems in the Annals of Mathematics* includes Hardy's generalization of the Maclaurin-Cauchy integral test†

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* R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series*, Annals of Mathematics, (2), vol. 21 (1919–20), p. 41.

† G. H. Hardy, *Theorems connected with Maclaurin's test for the convergence of series*, Proceedings of the London Mathematical Society, (2), vol. 9 (1911).

3. *Preliminary Theorems and Deductions.* Theorem 1 admits of the following generalization.

THEOREM 2. *Let* $\sum^{\infty} u_n$ *be a series of positive terms and* $r_n = u_{n+1}/u_n$. *If*

(i) (D_n) *is a strictly increasing sequence tending to infinity*;

(ii) $d_n \equiv D_n - D_{n-1} = O(1)$;

(iii) $f(x)$ *is bounded and integrable for* $x \geqq D_m$, *and*

$$(\mathcal{C}): \qquad \int^{\infty} e^{\int^x f(t)dt} dx \text{ is convergent,}$$

$$\left\{ or \quad (\mathcal{D}): \qquad \int^{\infty} e^{\int^x f(t)dt} dx \text{ is divergent} \right\};$$

(iv) *for* $D_{n-1} \leqq x \leqq D_n$,

$$(\mathcal{C}): \qquad \frac{1}{d_n} \log r_n \leqq f(x) + \delta_n, \ \sum^{n} \delta_\nu d_\nu \text{ being bounded above,}$$

$$\left\{ or \ (\mathcal{D}): \qquad \frac{1}{d_n} \log r_n \geqq f(x) + \delta_n', \ \sum^{n} \delta_\nu' d_\nu \text{ being bounded below} \right\};$$

then $\sum^{\infty} u_n d_n$ *is convergent* {*or divergent*}.

PROOF OF CASE $(\mathcal{C})$. Since

$$\frac{1}{d_\nu} \log r_\nu \leqq f(t) + \delta_\nu, \qquad (D_{\nu-1} \leqq t \leqq D_\nu),$$

by integration,

$$\log r_\nu \leqq \int_{D_{\nu-1}}^{D_\nu} f(t)dt + \delta_\nu d_\nu.$$

Sum for $\nu = m+1, m+2, \cdots, n-1$; then

$$\log \frac{u_n}{u_{m+1}} \leqq \int_{D_m}^{D_{n-1}} f(t)dt + \sum_{\nu=m+1}^{n-1} \delta_\nu d_\nu$$

$$< \int_{D_m}^{D_{n-1}} f(t)dt + K_1, \qquad (K_1 \text{ fixed}).$$

Also, for $D_{n-1} \leqq x \leqq D_n$, since $|f(t)| < M$ (fixed), $d_n < K$ (fixed), we have

$$-KM < \int_{D_{n-1}}^{x} f(t)dt.$$

Add the last two inequalities; then

$$\log \frac{u_n}{u_{m+1}} < \int_{D_m}^{x} f(t)dt + K_1 + KM, \quad (D_{n-1} \leqq x \leqq D_n),$$

and

$$u_n < u_{m+1} e^{K_1+KM} e^{\int_{D_m}^{x} f(t)dt}, \qquad (D_{n-1} \leqq x \leqq D_n).$$

Hence, integrating from D_{n-1} to D_n, we have

$$u_n d_n < u_{m+1} e^{K_1+KM} \int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt} dx.$$

Compare the series $\sum^{\infty} u_n d_n$ with the series of positive terms $\sum^{\infty} \int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt} dx$, and the test for convergence ($\mathcal{C}$) follows at once. The test for divergence ($\mathcal{D}$) is similarly proved.

The following is an adjunct to Theorem 2.

THEOREM 2a. *In Theorem 2, suppose the condition* $d_n = O(1)$ *is dropped and* $f(x) < 0$ *(that is, the integrand in the test integral is a strictly decreasing function). If other conditions remain the same* $\sum^{\infty} u_{n+1} d_n$ *is convergent in case* ($\mathcal{C}$) *and* $\sum^{\infty} u_n d_n$ *is divergent in case* ($\mathcal{D}$).

A slight modification is required in our former proof of ($\mathcal{C}$):

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{D_n} f(t)dt + K_1.$$

Also, for $D_{n-1} \leqq x \leqq D_n$,

$$0 \leqq -\int_{x}^{D_n} f(t)dt.$$

We add the last two inequalities, and obtain

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{x} f(t)dt + K_1, \qquad (D_{n-1} \leqq x \leqq D_n);$$

whence, as before,

$$u_{n+1}d_n < u_{m+1}e^{K_1}\int_{D_{n-1}}^{D_n} e^{\int_{D_m}^x f(t)dt}dx,$$

and the desired result follows by comparison.

DEDUCTIONS FROM THEOREM 2a. (i) Taking $f(x) = -\rho < 0$, $\delta_n = 0$, and putting $u_{n+1}d_n = a_n$, we see that the condition

$$\frac{1}{d_n}\log\frac{a_n\cdot d_{n-1}}{a_{n-1}\cdot d_n} \leqq -\rho < 0$$

is sufficient* for the convergence of $\sum^\infty a_n$.

(ii) The condition

$$\frac{1}{d_n}\log\frac{a_n\cdot d_{n-1}}{a_{n-1}\cdot d_n} \leqq -\frac{1}{D_{n-1}} - \frac{1}{D_{n-1}\cdot l_1D_{n-1}} - \cdots - \frac{\alpha}{D_{n-1}\cdot l_1D_{n-1}\cdots l_pD_{n-1}}, \qquad (\alpha > 1),$$

where $l_1D_{n-1} = \log D_{n-1}$, $l_2D_{n-1} = \log\log D_{n-1}, \cdots$ (and $n \geqq m+1$ which is such that $l_pD_m > 0$), is sufficient for the convergence of $\sum^\infty a_n$. For this implies that

$$\frac{1}{d_n}\log\frac{u_{n+1}}{u_n} \leqq -\frac{1}{x} - \frac{1}{x\cdot l_1x} - \cdots - \frac{\alpha}{x\cdot l_1x\cdots l_px},$$
$$(\alpha > 1;\ D_{n-1} \leqq x \leqq D_n).$$

Hence taking

$$f(x) = -\frac{1}{x} - \frac{1}{x\cdot l_1x} - \cdots - \frac{\alpha}{x\cdot l_1x\cdots l_px}, \ (\alpha > 1);\ \delta_n = 0,$$

we deduce the convergence of $\sum^\infty a_n$.

(iii) Similarly the condition

$$\frac{1}{d_n}\log\frac{a_{n+1}\cdot d_n}{a_n\cdot d_{n+1}} \geqq -\frac{1}{D_n} - \frac{1}{D_n\cdot l_1D_n} - \cdots - \frac{\alpha}{D_n\cdot l_1D_n\cdots l_pD_n},$$
$$(\alpha \leqq 1),$$

is sufficient for the divergence of $\sum^\infty a_n$.

Setting $D_n = n$ in (ii) and (iii), we obtain Bertrand's logarithmic criteria for convergence and divergence.

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* A. Pringsheim, loc. cit., p. 370.

4. *Generalization of Brink's Theorem.*

THEOREM 3. *Let* $\sum^\infty a_n$ *be a series of positive terms. If*

(i) (D_n) *is a strictly increasing sequence tending to infinity*;

(ii) $d_n \equiv D_n - D_{n-1} = O(1)$;

(iii) $f(x)$ *has a continuous derivative* $f'(x)$ *and* $\int^\infty |f'(x)| dx$ *is convergent*;

$$\text{(iv)} \qquad (\mathcal{C}): \qquad \int^\infty e^{\int^x f(t)dt} dx \text{ is convergent,}$$

$$\{or, \quad (\mathcal{D}): \qquad \int^\infty e^{\int^x f(t)dt} dx \text{ is divergent}\};$$

$$\text{(v)} \qquad (\mathcal{C}): \qquad \frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \leqq f(D_n),$$

$$\{or, \quad (\mathcal{D}): \qquad \frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \geqq f(D_n)\};$$

then $\sum^\infty a_n$ *is convergent* {*or divergent*}.

PROOF OF $(\mathcal{C})$. Denoting a_n/d_n by u_n, we have in the notation of Theorem 2,

$$\frac{1}{d_n} \log r_n \leqq \int_x^{D_n} f'(t)dt + f(x)$$

$$\leqq \int_{D_{n-1}}^{D_n} | f'(t) | \, dt + f(x), \qquad (D_{n-1} \leqq x \leqq D_n).$$

Whence, choosing $\delta_n = \int_{D_{n-1}}^{D_n} |f'(t)| dt$ in Theorem 2, we deduce the convergence of $\sum^\infty u_n d_n \equiv \sum^\infty a_n$.

Proof of $(\mathcal{D})$ is similar.

DEDUCTIONS FROM THEOREM 3. (i) If

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} = f(D_n),$$

then, under the conditions assumed, the convergence of

$$\int^\infty e^{\int^x f(t)dt} dx$$

is necessary and sufficient for the convergence of $\sum^{\infty} a_n$. When $D_n = n$, we have Brink's theorem.

(ii) Taking $f(x) = -\rho < 0$, we see that the condition

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \leqq -\rho < 0$$

is sufficient for the convergence of $\sum^{\infty} a_n$.* Since $\log \gamma \leqq \gamma - 1$, $(\gamma > 0)$, it follows that the above condition can also be expressed in Kummer's form:†

$$\frac{1}{d_n}\left(\frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1\right) \leqq -\rho < 0.$$

(iii) Taking

$$f(x) = -\frac{1}{x} - \frac{1}{x \cdot l_1 x} - \cdots - \frac{\alpha}{x \cdot l_1 x \cdots l_p x}, \quad (\alpha > 1),$$

we observe that the condition

$$\left.\begin{array}{l} \dfrac{1}{d_n} \log \dfrac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \\ \text{or, } \dfrac{1}{d_n}\left(\dfrac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1\right) \end{array}\right\} \leqq -\frac{1}{D_n} - \frac{1}{D_n \cdot l_1 D_n} - \cdots - \frac{\alpha}{D_n \cdot l_1 D_n \cdots l_p D_n}, \quad (\alpha > 1),$$

is sufficient for the convergence of $\sum^{\infty} a_n$.

The corresponding divergence criterion has already been given.

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* A. Pringsheim, loc. cit., p. 371.

† A. Pringsheim, loc. cit., p. 361, footnote.