the integrals taken so that the boundary C is traversed in the positive sense. Series (15) *converges uniformly for x in every closed region in* $\mathcal{L}$, *and is therefore valid for all x in* $\mathcal{L}$.

From this we obtain the following result.

THEOREM 2.* *The series*

$$y(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \tag{17}$$

converges in a circle of radius exceeding $1/2$, *and in some neighborhood of* $x = -1/2$ *the function* $y(x)$ *is a solution of the equation*

$$\Delta y(x) = F(x). \tag{1}$$

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ON THE SUMMABILITY BY POSITIVE TYPICAL MEANS OF SEQUENCES $\{f(n\theta)\}$†

BY M. S. ROBERTSON

1. *Introduction.* In a recent paper‡ the author required an inequality for the expression

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\sin k\theta| \leqq \frac{1}{\pi}\int_0^{\pi} |\sin\theta|\, d\theta = \frac{2}{\pi}, \tag{1}$$

which apparently is due to T. Gronwall.§ This inequality suggests immediately the question: For what functions $f(\theta)$, defined in the interval $(-\pi, \pi)$, are we permitted to write

$$F(\theta; f) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\theta) \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta? \tag{2}$$

More generally, we may ask: For what functions $f(\theta)$ and sequences $\{a_n\}$ of positive numbers is the following true:

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* See Transactions of this Society, loc. cit., p. 359.

† Presented to the Society, April 11, 1936.

‡ See M. S. Robertson, *On the coefficients of a typically-real function*, this Bulletin, vol. 41 (1935), p. 569.

§ See Transactions of this Society, vol. 13 (1912), pp. 445–468.

$$(3)\qquad M(\theta;f)\equiv\lim_{n\to\infty}\sum_{k=1}^{n}a_kf(k\theta)\cdot\left(\sum_{1}^{n}a_k\right)^{-1}\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)d\theta?$$

It is the purpose of this paper to answer these questions in part by giving sufficiency conditions when the function $f(\theta)$ possesses a uniformly converging Fourier series. It will be evident from the discussion to follow that the inequality (2) is not true for all integrable functions $f(\theta)$.

2. *An Expansion Formula for $M(\theta;f)$.* We shall adopt the following notation. Let $f(\theta)$ be an integrable function in the sense of Lebesgue, defined over the interval $-\pi\leqq\theta\leqq\pi$, and periodic outside of this interval, whose Fourier series

$$(4)\qquad f(\theta)\sim c_0+\sum_{1}^{\infty}(b_m\sin m\theta+c_m\cos m\theta)$$

converges uniformly in the closed interval $-\pi\leqq\theta\leqq\pi$. Let $\{a_n\}$ be any sequence of non-negative real numbers. Let

$$(5)\qquad \overline{P}(\theta)\equiv\overline{\lim_{n\to\infty}}\,P_n(\theta)\equiv\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_k\cos k\theta\cdot\left(\sum_{1}^{n}a_k\right)^{-1},$$

$$(6)\qquad \overline{Q}(\theta)\equiv\overline{\lim_{n\to\infty}}\,Q_n(\theta)=\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_k\sin k\theta\cdot\left(\sum_{1}^{n}a_k\right)^{-1}.$$

We may denote by $\underline{P}(\theta)$ and $\underline{Q}(\theta)$ the corresponding functions obtained by taking inferior limits. If $\overline{P}(\theta)\equiv\underline{P}(\theta)$, we denote each simply by $P(\theta)$. A similar remark holds for $Q(\theta)$. Let

$$(7)\qquad \overline{M}(\theta;f)\equiv\overline{\lim_{n\to\infty}}\,M_n(\theta)\equiv\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_kf(k\theta)\cdot\left(\sum_{1}^{n}a_k\right)^{-1},$$

$$(8)\qquad \underline{M}(\theta;f)=\underline{\lim}_{n\to\infty}\,M_n(\theta).$$

With this notation, we obtain an infinite series for $\overline{M}(\theta;f)$:

$$M_n(\theta)\cdot\sum_{1}^{n}a_k=c_0\cdot\sum_{1}^{n}a_k+\sum_{k=1}^{n}a_k\left(\sum_{m=1}^{\infty}b_m\sin mk\theta+c_m\cos mk\theta\right)$$

$$=c_0\cdot\sum_{1}^{n}a_k+\sum_{m=1}^{\infty}\left[b_m\left(\sum_{k=1}^{n}a_k\sin mk\theta\right)+c_m\left(\sum_{k=1}^{n}a_k\cos mk\theta\right)\right],$$

$$(9) \qquad M_n(\theta) = c_0 + \sum_{m=1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)).$$

Since the Fourier series (4) converges uniformly for $-\pi \leqq \theta \leqq \pi$, given $\epsilon > 0$, we can choose $N(\epsilon)$ sufficiently large so that, for all θ and k,

$$\left| \sum_{m=N+1}^{\infty} (b_m \sin mk\theta + c_m \cos mk\theta) \right| < \epsilon.$$

Hence

$$\sum_{k=1}^{n} a_k \cdot \left| \sum_{m=N+1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)) \right|$$

$$= \left| \sum_{m=N+1}^{\infty} \left[b_m \left(\sum_{k=1}^{n} a_k \sin mk\theta \right) + c_m \left(\sum_{k=1}^{n} a_k \cos k\theta \right) \right] \right|$$

$$= \left| \sum_{k=1}^{n} a_k \left(\sum_{N+1}^{\infty} (b_m \sin mk\theta + c_m \cos mk\theta) \right) \right| < \epsilon \cdot \sum_{1}^{n} a_k.$$

Consequently we have, for all θ and n,

$$(10) \qquad \left| \sum_{m=N+1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)) \right| < \epsilon.$$

On account of (10), we obtain from (9) in passing to the limit,

$$(11) \qquad \overline{M}(\theta; f) = c_0 + \sum_{m=1}^{\infty} (b_m \overline{Q}(m\theta) + c_m \overline{P}(m\theta)),$$

$$(12) \qquad \underline{M}(\theta; f) = c_0 + \sum_{m=1}^{\infty} (b_m \underline{Q}(m\theta) + c_m \underline{P}(m\theta)).$$

These two series are uniformly convergent for all θ since (10) holds for all n. If it should happen that

$$\overline{Q}(\theta) \equiv \underline{Q}(\theta) \equiv Q(\theta), \qquad \overline{P}(\theta) \equiv \underline{P}(\theta) \equiv P(\theta),$$

then the limit in (3) will exist.

If we make the substitution in (11) and (12) for the Fourier coefficients, $(m \geqq 1)$,

$$(13) \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sin m\phi d\phi, \qquad c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \cos m\phi d\phi,$$

and invert the order of integration and summation which is permissible since the series is uniformly convergent in θ, we find

$$\overline{M}(\theta; f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\bar{g}(\theta, \phi)d\phi, \tag{14}$$

$$\underline{M}(\theta; f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\underline{g}(\theta, \phi)d\phi, \tag{15}$$

where

$$\bar{g}(\theta, \phi) \equiv \frac{1}{2} + \sum_{m=1}^{\infty} (\overline{Q}(m\theta)\cdot \sin m\phi + \overline{P}(m\theta)\cdot \cos m\phi) \tag{16}$$

is uniformly convergent in θ and ϕ, and where $\underline{g}(\theta, \phi)$ has the corresponding definition in terms of $\underline{Q}(\theta)$ and $\underline{P}(\theta)$. If $\bar{g}(\theta, \phi) \equiv \underline{g}(\theta_1, \phi) \equiv g(\theta, \phi)$ for all θ and ϕ, we can be sure that $M(\theta; f)$ in (3) exists.

If $\overline{Q}(\theta) \equiv \underline{Q}(\theta) \equiv Q(\theta)$, $\overline{P}(\theta) \equiv \underline{P}(\theta) \equiv P(\theta)$, then it is seen from (11) and (12) that the necessary and sufficient condition for the inequality expressed in (3) is that the function

$$\phi(\theta) = \sum_{m=1}^{\infty} (b_m Q(m\theta) + c_m P(m\theta)) \tag{17}$$

should be non-positive for all values of θ.

3. *The Functions $P(\theta)$ and $Q(\theta)$ for Special Sequences $\{a_n\}$.* Let us now consider a more restricted class of sequences $\{a_n\}$ satisfying the following conditions:

(18a) The sequence $\{a_n\}$ of non-negative numbers a_n is to be composed of two subsequences $\{a_{2n+1}\}$ and $\{a_{2n}\}$ each of which is monotone (either non-increasing or non-decreasing).

$$\mu_n \equiv \sum_{1}^{n} a_k \text{ diverges to } +\infty \text{ with } \lim_{n\to\infty} \mu_n/\mu_{n-1} = 1. \tag{18b}$$

$$\lim_{n\to\infty} \mu_n^{-1}\cdot \sum_{k=1}^{n} (-1)^k a_k = a, \qquad (-1 \leqq a \leqq 1). \tag{18c}$$

With sequences $\{a_n\}$ of this latter type we can find $P(\theta)$ and $Q(\theta)$. For $\theta \neq k\pi$, (k an integer), we have the identity

$$(19)\qquad \sum_{k=1}^{n} a_k \sin k\theta = (2\sin\theta)^{-1}$$

$$\cdot\left\{\sum_{k=1}^{n}(a_{k+1}-a_{k-1})\cos k\theta + a_1 - a_{n+1}\cos n\theta - a_n\cos(n+1)\theta\right\}.$$

Hence

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\,|Q_n(\theta)|\\ &\leqq (2\sin\theta)^{-1}\left\{\overline{\lim_{n\to\infty}}\,\mu_n^{-1}\left(\sum_{k=1}^{n}|a_{k+1}-a_{k-1}| + a_1 + a_{n+1} + a_n\right)\right\}\\ &\leqq (2\sin\theta)^{-1}\left\{\overline{\lim_{n\to\infty}}\,\frac{2(a_1+a_2+a_n+a_{n+1})}{\mu_n}\right\} = 0\end{aligned}$$

by (18a) and (18b). It follows that

$$\overline{Q}(\theta)\equiv\underline{Q}(\theta)\equiv Q(\theta)\equiv 0.$$

Again, for $\theta\neq k\pi$, we have the identity

$$(20)\qquad \sum_{k=1}^{n} a_k\cos k\theta = (2\sin\theta)^{-1}$$

$$\cdot\left\{\sum_{k=1}^{n}(a_{k+1}-a_{k-1})\sin k\theta + a_1 - a_{n+1}\sin n\theta - a_n\sin(n+1)\theta\right\},$$

from which we obtain

$$\overline{P}(\theta)\equiv\underline{P}(\theta)\equiv P(\theta)\equiv 0$$

for $\theta\neq k\pi$. If $\theta=2p\pi$, (p an integer), evidently $P(\theta)=1$. If $\theta=(2p+1)\pi$, $P(\theta)=a$ on account of (18c).

Substituting these values for $P(\theta)$ and $Q(\theta)$ in (11) or (12), we have the following values for $M(\theta;f)$ of (3):

$$(21a)\qquad \frac{\theta}{\pi}\ \text{irrational},\quad M(\theta;f) = c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta.$$

(21b) $$\frac{\theta}{\pi} = \frac{2r}{s}, \text{ where } r \text{ and } s \text{ are integers, } (2r, s) = 1,$$

$$M\left(\frac{2r\pi}{s}; f\right) = c_0 + \sum_{m=1}^{\infty} c_{ms}$$
$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left\{ \frac{\sin (2n+1) \frac{s\theta}{2}}{\sin \frac{s\theta}{2}} \right\} d\theta.$$

(21c) $$\frac{\theta}{\pi} = \frac{2r+1}{s}, \quad (2r+1, s) = 1,$$

$$M\left(\frac{2r+1}{s}\pi; f\right) = c_0 + a\sum_{m=1}^{\infty} c_{(2m-1)s} + \sum_{m=1}^{\infty} c_{2ms}$$
$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left\{ \frac{\sin (2n+1)s\theta + 2a \sin ns\theta \cdot \cos ns\theta}{\sin s\theta} \right\} d\theta.$$

Hence for sequences of the type (18) the necessary and sufficient condition for the inequality (3) is that

(22)
$$\sum_{m=1}^{\infty} c_{ms} \leqq 0 \qquad \text{for all odd positive integers } s, \text{ and}$$
$$\sum_{m=1}^{\infty} c_{2ms} + a\sum_{m=1}^{\infty} c_{(2m-1)s} \leqq 0 \qquad \text{for all positive integers } s.$$

A sufficient condition for (22) is $c_m \leqq 0$ for $m = 1, 2, 3, \cdots$. In particular if $f(\theta)$ is an even function convex for $0 \leqq \theta \leqq \pi$, it is well known that its Fourier series is of the form

$$f(\theta) \sim c_0 + \sum_{1}^{\infty} c_{2m} \cos 2m\theta,$$

where $c_{2m} \leqq 0$, and where the series converges uniformly in the closed interval $0 \leqq \theta \leqq \pi$. Hence for sequences of type (18) and any even convex function $f(\theta)$, (3) and (1) are true. In particular, if $f(\theta) = |\sin \theta|$, we obtain (1).

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