mediate functions and (2) the roots of $y_{n'} = 0$. In the second set the contribution due to (1) persists. By the introduction of the additional functions in the second set, the change contributed by a root of $y_{n'} = 0$ in the first set is transferred to a root of $y_{n''} = 0$ in the second set. Also no change arises for a root of $y_{n'} = 0$, $y_{n'+1} = 0$, $\cdots$, $y_{n''-1} = 0$. Since the total number of losses is the same in the two cases, the conclusion will be that the number of roots for $y_{n'} = 0$ and $y_{n''} = 0$ in the range $(-\infty, a)$ will be the same; similarly for the range (b, ∞). Also in (a, b) the numbers of roots of $y_{n'} = 0$ and $y_{n''} = 0$ are $n'-p-q$ and $n''-p-q$. Hence we conclude that the number of imaginary roots of $y_{n'} = 0$ and $y_{n''} = 0$ will be the same for $n'' > n' \geqq n_0$.

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NOTE ON THE EXISTENCE OF AN nTH DERIVATIVE DEFINED BY MEANS OF A SINGLE LIMIT

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The nth derivative of a function $f(x)$ may be defined without the use of derivatives of lower order by means of the limit of a certain quotient. Conditions necessary and sufficient for the existence and continuity of $f^{(n)}(x)$ at a point $x = a$ and also for the mere existence of $f^{(n)}(a)$ have been recently given by Franklin.* The purpose of the present note is to state necessary and sufficient conditions of a somewhat more general form with proofs which use only Rolle's theorem and elementary properties of determinants.

Let $f_i(x)$ and $\phi_i(x)$, $(i = 1, 2, \cdots, n+1)$, be functions possessing derivatives of the nth order, continuous in an interval I. Let $x_1, x_2, \cdots, x_{n+1}$ be points of I which close down in an arbitrary manner on a point a, in the sense that

(1) $$|x_j - a| < \epsilon_k, \qquad \lim_{k \to \infty} \epsilon_k = 0.$$

We shall use the notation

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* This Bulletin, vol. 41 (1935), p. 573.

$$(2) \qquad B_{n,k} = \frac{\left| f_1(x_j) \quad f_2(x_j) \quad \cdots \quad f_{n+1}(x_j) \right|}{\left| \phi_1(x_j) \quad \phi_2(x_j) \quad \cdots \quad \phi_{n+1}(x_j) \right|},$$

in which numerator and denominator are determinants of which only the jth row of each is written. The following theorem[*] requires for its proof only a repeated use of Rolle's theorem

$$(3 \qquad B_{n,k} = \frac{\begin{vmatrix} f_1(\xi_1) & \cdots & f_{n+1}(\xi_1) \\ f_1'(\xi_2) & \cdots & f_{n+1}'(\xi_2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_1^{(n)}(\xi_{n+1}) & \cdots & f_{n+1}^{(n)}(\xi_{n+1}) \end{vmatrix}}{\begin{vmatrix} \phi_1(\xi_1) & \cdots & \phi_{n+1}(\xi_1) \\ \phi_1'(\xi_2) & \cdots & \phi_{n+1}'(\xi_2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \phi_1^{(n)}(\xi_{n+1}) & \cdots & \phi_{n+1}^{(n)}(\xi_{n+1}) \end{vmatrix}},$$

where $\xi_1 = x_1$ and $\xi_2, \cdots, \xi_{n+1}$ lie between the extreme points of x_j. If the points x_j close down in any manner on a point a of I, then $\xi_j \to a$, $(j = 1, 2, \cdots, n+1)$, and on account of the continuity of the derivatives we have the following theorem.

THEOREM 1. *If the points x_j close down on a in an arbitrary manner subject to* (1) *and* $W(\phi_1, \phi_2, \cdots, \phi_{n+1}) \neq 0$ *at the point a, where W denotes the Wronskian of the functions indicated, then $B_{n,k}$ has a unique limit*

$$(4) \qquad \lim_{k \to \infty} B_{n,k} = \left[\frac{W(f_1, \cdots, f_{n+1})}{W(\phi_1, \cdots, \phi_{n+1})} \right]_{x=a}.$$

Conversely, however, it is not true that if $B_{n,k}$ has a unique limit for any approach of the points to a, then the derivatives involved in the Wronskians all exist. If, for example, $f_j \equiv \phi_j$, $(j = 1, 2, \cdots, n+1)$, then $B_{n,k} \equiv 1$ and $\lim B_{n,k} = 1$ for any ap-

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* Pólya and Szegö, *Aufgaben und Lehrsätze*, vol. 2, p. 54. The theorem in a slightly less general form was given by Schwarz, Annali di Matematica, (2), vol. 10 (1880), p. 129; Collected Works, vol. 2, p. 296. A proof based on Rolle's theorem was given by Stieltjes, Nouvelles Annales de Mathématique, (3), vol. 7 (1888), p. 26; Collected Works, p. 105. See also Miller, Transactions of the Royal Society of Canada, 1931, p. 195.

proach, even though the functions be non-differentiable. If, however, all the functions but one, say $f_1(x)$, possess nth derivatives continuous at a and are subject to certain further restrictions stated below, the existence of a unique limit for $B_{n,k}$ as the points x_j close down on a in an arbitrary manner* furnishes a sufficient condition for the existence and continuity of $f_1^{(n)}(x)$ at $x = a$. If the restriction be made that one of the points x_j remain at a while the others approach a in any manner, the existence of $\lim B_{n,k}$ in this case suffices to ensure the existence of $f_1^{(n)}(a)$ without the continuity of the derivative at this point. This latter theorem will first be proved.

THEOREM 2. *Hypotheses*: (i) *In an interval I containing the point a the n functions f_i, $(i = 2, 3, \cdots, n+1)$, are linearly independent in every subinterval and possess derivatives of the nth order,* (ii) *the functions ϕ_i, $(i = 1, 2, 3, \cdots, n+1)$, possess derivatives of the nth order,* (iii) *$B_{n,k}$ possesses a unique limit as the point x_1 remains at a while the points $x_2, \cdots, x_{n+1}$ close down on a in an arbitrary manner,* (iv) *$W(f_2, f_3, \cdots, f_{n+1}) \neq 0$ at the point a. Then f_1 also possesses a derivative of the nth order at $x = a$ and the derivatives of f_1 satisfy the relation (4).*

We prove first the existence of $f_1'(a)$. For this purpose set $x_1 = a$ and replace the elements of the second row of the numerator of (2) by $[f_j(x_2) - f_j(a)]/(x_2 - a)$, $(j = 1, 2, \cdots, n+1)$, with a similar replacement in the second row of the denominator. Now, leaving $x_3, \cdots, x_{n+1}$ fixed, let $x_2 \to a$. By hypothesis

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* Since the denominator of $B_{n,k}$ vanishes whenever two or more of the points x_j coincide, the quotient is not defined at such points. The hypothesis here stated shall be understood to require that the discontinuities at all such points be removable. That is, if in a sufficiently small neighbourhood of a some of the points x_j remain fixed while the remainder approach a point (either a or a_1 near a) in an arbitrary manner, the limit of $B_{n,k}$ must exist. The following example is instructive. On the parabola $y = x^2$ take a set of points whose abscissas are 1, 1/2, 1/4, 1/8, $\cdots$. Join successive points of this sequence by chords, forming a broken line curve, to which the limiting point $(0, 0)$ is adjoined. Call $f(x)$ the function represented by this broken line curve. It has the property that $\lim_{x_1 \to 0,\ x_2 \to 0} [f(x_2) - f(x_1)]/(x_2 - x_1) = 0$ for $x_2 \neq x_1$, also $\lim_{x_1 \to 0} \overline{\lim}_{x_2 \to x_1} [f(x_2) - f(x_1)]/(x_2 - x_1) = 0$ and $\lim_{x_1 \to 0} \underline{\lim}_{x_2 \to x_1} [f(x_2) - f(x_1)]/(x_2 - x_1) = 0$. But $\lim_{x_1 \to 0,\ x_2 \to 0} [f(x_2) - f(x_1)]/(x_2 - x_1)$ does not exist in the sense in which we are using the term, since the quotient has irremovable discontinuities at a set of points condensing on the origin.

(iii) $B_{n,k}$ must then approach a limit (if $x_3, \cdots, x_{n+1}$ are close enough to a). Also $\lim [f_j(x_2) - f_j(a)]/(x_2 - a) = f_j'(a)$ for $j \neq 1$ and $\lim [\phi_j(x_2) - \phi_j(a)]/(x_2 - a) = \phi_j'(a)$ for $j = 1, 2, \cdots, n+1$. The only element whose limit is in doubt is $[f_1(x_2) - f_1(a)]/(x_2 - a)$. It will follow that this element approaches a limit and therefore that $f_1'(a)$ exists as soon as we show that its cofactor in the numerator is not zero. This cofactor is, except for sign,

$$C_{12} = \begin{vmatrix} f_2(a) & f_3(a) & \cdots & f_{n+1}(a) \\ f_2(x_3) & f_3(x_3) & \cdots & f_{n+1}(x_3) \\ f_2(x_4) & f_3(x_4) & \cdots & f_{n+1}(x_4) \\ \cdots & \cdots & \cdots & \cdots \\ f_2(x_{n+1}) & f_3(x_{n+1}) & \cdots & f_{n+1}(x_{n+1}) \end{vmatrix}.$$

The elements of the first row are not all zero, by hypothesis (iv). In any subinterval of I a point x_3 exists such that not all the two-rowed minors of the first two rows are zero, for if not the n functions would be linearly dependent in this subinterval. Let such a point x_3 be chosen. Then in any subinterval of I, x_4 may be chosen such that not all the three-rowed minors of the first three rows are zero, and so on. Finally, $x_3, x_4, \cdots, x_{n+1}$ may be chosen in an infinite variety of ways such that $C_{12} \neq 0$. The existence of $f_1'(a)$ is then established.

Proceeding now by induction suppose that $f_1^{(r)}(a)$ exists. In order to show the existence of $f_1^{(r+1)}(a)$ we first transform $B_{n,k}$ in the manner in which (3) is obtained but without carrying the transformation so far. Consider

(5)
$$\begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n+1}(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n+1}(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_{n+1}) & f_2(x_{n+1}) & \cdots & f_{n+1}(x_{n+1}) \end{vmatrix}$$
$$- B_{n,k} \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_{n+1}(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_{n+1}(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1(x_{n+1}) & \phi_2(x_{n+1}) & \cdots & \phi_{n+1}(x_{n+1}) \end{vmatrix}.$$

Replace x_{n+1} by x. The resulting function of x vanishes at

$x = x_{n+1}$ and at* $x = x_n$. Hence its derivative vanishes at x_{n+1}, between x_n and x_{n+1}. In the last rows of (5) we may replace $f_j(x_{n+1})$ by $f_j'(x_{n+1}')$ and $\phi_j(x_{n+1})$ by $\phi_j'(x_{n+1}')$, so that the resulting expression vanishes. By a similar operation we may replace the elements of the second last rows by derivatives and x_n by x_n' which is between x_{n-1} and x_n. This may be repeated until all the elements except those of the first rows have been replaced by derivatives, the resulting expression being zero. Beginning again at the last rows replace x_{n+1}' by x. The resulting function of x vanishes at x_{n+1}' and at x_n'. An application of Rolle's theorem leads to the substitution of second derivatives in the last rows, and of x_{n+1}'' for x_{n+1}', where x_{n+1}'' is between x_n' and x_{n+1}'. Repetitions of the operation will replace by second derivatives all the elements of the rows beginning with the third. In a similar manner we replace by third derivatives all the elements of the rows beginning with the fourth. Continuing this process until we arrive at rth derivatives, we get for $B_{n,k}$ a quotient of which the numerator is

$$(6) \quad \begin{vmatrix} f_1(\eta_1) & f_2(\eta_1) & \cdots & f_{n+1}(\eta_1) \\ f_1'(\eta_2) & f_2'(\eta_2) & \cdots & f_{n+1}'(\eta_2) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_1^{(r)}(\eta_{r+1}) & f_2^{(r)}(\eta_{r+1}) & \cdots & f_{n+1}^{(r)}(\eta_{r+1}) \\ f_1^{(r)}(\eta_{r+2}) & f_2^{(r)}(\eta_{r+2}) & \cdots & f_{n+1}^{(r)}(\eta_{r+2}) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_1^{(r)}(\eta_{n+1}) & f_2^{(r)}(\eta_{n+1}) & \cdots & f_{n+1}^{(r)}(\eta_{n+1}) \end{vmatrix},$$

and the denominator is a similar expression with ϕ replacing f, where $\eta_1 = x_1$, η_2 is between x_1 and x_2, η_3 is between the extremes of x_1, x_2 and x_3, and so on.

We now make use of the resulting expression for $B_{n,k}$ as follows. Let $x_1 = x_2 = \cdots = x_{r+1} = a$. Then $\eta_1 = \eta_2 = \cdots = \eta_{r+1} = a$. In (6) replace $f_j^{(r)}(\eta_{r+2})$ by

$$\frac{f_j^{(r)}(\eta_{r+2}) - f_j^{(r)}(a)}{\eta_{r+2} - a}, \quad (j = 1, 2, \cdots, n+1),$$

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* For the deduction of the result which follows, the points x_j are taken distinct. When, in the result, some of the points are made to coincide, it is the limit of $B_{n,k}$ which is used.

with similar replacements in the $(r+2)$th row of the denominator of $B_{n,k}$. Thus (6) becomes

$$(7) \quad \begin{vmatrix} f_1(a) & f_2(a) & \cdots & f_{n+1}(a) \\ f_1'(a) & f_2'(a) & \cdots & f_{n+1}'(a) \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ f_1^{(r)}(a) & f_2^{(r)}(a) & \cdots & f_{n+1}^{(r)}(a) \\ \dfrac{f_1^{(r)}(\eta_{r+2}) - f_1^{(r)}(a)}{\eta_{r+2} - a} & & \cdots & \dfrac{f_{n+1}^{(r)}(\eta_{r+2}) - f_{n+1}^{(r)}(a)}{\eta_{r+2} - a} \\ f_1^{(r)}(\eta_{r+3}) & & \cdots & f_{n+1}^{(r)}(\eta_{r+3}) \\ \cdots\cdots & & & \cdots\cdots \\ f_1^{(r)}(\eta_{n+1}) & & \cdots & f_{n+1}^{(r)}(\eta_{n+1}) \end{vmatrix}.$$

Let $x_{r+2} \to a$. Then $\eta_{r+2} \to a$. The ratio of which (7) is the numerator approaches a limit by hypothesis (iii). The only element in numerator or denominator whose limit is in doubt is $[f_1^{(r)}(\eta_{r+2}) - f_1^{(r)}(a)]/(\eta_{r+2} - a)$. Consider the cofactor of this element in (7),

$$(8) \quad \begin{vmatrix} f_2(a) & \cdots & f_{n+1}(a) \\ f_2'(a) & \cdots & f_{n+1}'(a) \\ \cdots & \cdots & \cdots \\ f_2^{(r)}(a) & \cdots & f_{n+1}^{(r)}(a) \\ f_2^{(r)}(\eta_{r+3}) & \cdots & f_{n+1}^{(r)}(\eta_{r+3}) \\ \cdots & \cdots & \cdots \\ f_2^{(r)}(\eta_{n+1}) & \cdots & f_{n+1}^{(r)}(\eta_{n+1}) \end{vmatrix}.$$

Not all the $(r+1)$-rowed minors of the first $r+1$ rows of (8) vanish, by hypothesis (iv). If it should happen that all the $(r+2)$-rowed minors of the first $r+2$ rows are zero, then a slight change in x_{r+3} will cause a change in η_{r+3} and will lead to $(r+2)$-rowed minors not all of which are zero, on account of hypothesis (i). Similarly, $x_{r+3}, \cdots, x_{n+1}$ may be chosen in an infinite variety of ways so that (8) is not zero. Hence $\lim_{\eta_{r+2} \to a} [f_1^{(r)}(\eta_{r+2}) - f_1^{(r)}(a)]/(\eta_{r+2} - a)$ or $f_1^{(r+1)}(a)$ exists. This completes the induction and establishes Theorem 2.

In order that the derivative $f^{(n)}(x)$ be continuous at $x = a$ the hypotheses must be made slightly more stringent. The theorem

is suggested by the corresponding theorem of Franklin already referred to and the proof makes use of a device which he introduced.

THEOREM 3. *The hypotheses of Theorem 2 are postulated, with the following modifications. In* (i) *and* (ii) *the nth derivatives of* $f_2, \cdots, f_{n+1}, \phi_1, \cdots, \phi_{n+1}$ *shall be continuous.* (iii) *becomes:* $B_{n,k}$ *possesses a unique limit as the points* $x_1, x_2, \cdots, x_{n+1}$ *close down on a in an arbitrary manner. The conclusion is that* $f_1^{(n)}(x)$ *exists and is continuous at* $x = a$.

For the proof the existence of $f_1'(a)$ follows from Theorem 2. Take a_1, a point neighboring to a. Let $x_1 = a_1$ and let $x_2, \cdots, x_{n+1}$ approach a_1 and then let a_1 approach a. Since $B_{n,k}$ has the same limit for every approach of the points to a, and since the derivatives of the functions other than f_1 are continuous, it follows that $\lim_{a_1 \to a} \lim_{x_2 \to a_1} [f_1(x_2) - f_1(a_1)]/(x_2 - a_1) = f_1'(a)$, which proves the continuity of $f_1'(x)$ at a. In the same way the existence proof for $f_1^{(r+1)}(a)$ may be modified, with the hypotheses now available, to include the proof of the continuity of this derivative. Hence the induction for this case is also complete.

PARTICULAR CASES. Set $f_2(x) = x^{n-1}, \cdots, f_n(x) = x, f_{n+1}(x) = 1$; $\phi_1(x) = x^n, \phi_2(x) = x^{n-1}, \cdots, \phi_{n+1}(x) = 1$ and denote $B_{n,k}$ for this case by $A_{n,k}$. Under the hypothesis of a unique limit for $A_{n,k}$ we have,[*] dropping the subscript of $f_2(x)$, $\lim_{k \to \infty} A_{n,k} = f^{(n)}(a)/n!$ and $f^{(n)}(a) = n! \lim_{k \to \infty} A_{n,k}$. Here the mere existence of $f^{(n)}(a)$ follows or the existence and continuity of $f^{(n)}(x)$ at $x = a$ according as the method of approach of the points to a is that specified in Theorem 2 or in Theorem 3.

Again set $f_2(x) = x^{n-1}, \cdots, f_n(x) = x, f_{n+1}(x) = 1$. Under the pertinent hypotheses the expression for $f^{(n)}(a)$ is, in this case,

$$f^{(n)}(a) = (-1)^n \frac{W(\phi_1, \cdots, \phi_{n+1})}{(n-1)!(n-2)! \cdots 2!1!} \lim_{k \to \infty} B_{n,k},$$

the value of the Wronskian being taken at the point a.

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[*] Franklin, loc. cit.