alizes the theorem of Thompson and Tait. We can prove, in fact, that a condition for an affirmative answer to our question is that, on any tube of (S), either all or none of the transversal curves should be closed.

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ON THE CONDITION THAT TWO ZEHFUSS MATRICES BE EQUAL

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1. *Introduction.* In a recent paper* Williamson has considered matrices whose sth compounds are equal. The present paper considers the somewhat analogous problem of finding the conditions that two Zehfuss matrices be equal.

Suppose that R is a matrix of n_1 rows and m_1 columns whose ijth element is r_{ij}, and that P is another matrix of n_2 rows and m_2 columns. Now, if the matrix Q of n_1n_2 rows and m_1m_2 columns can be partitioned into submatrices each of n_2 rows and m_2 columns such that the ijth submatrix is $r_{ij}P$, then Q is a *Zehfuss matrix*† or the *direct product matrix*‡ of R and P. We shall write

$$Q = R\langle P\rangle = \langle P\rangle R.$$

In general, however, $R\langle P\rangle \neq \langle P\rangle R$.

It is the purpose of this paper to find out under what conditions the matrix equation

$$A\langle B\rangle = C\langle D\rangle$$

is true. That is, we shall find the most general form of the matrices A, B, C, D when the above equation holds.

2. *The Simplest Case.* We shall begin by considering the simplest case, where A, B, C, D are row vectors, where A and D are of order m_1, where B and C are of order m_2, and where

$$(m_1, m_2) = 1;$$

that is to say, m_1 and m_2 are prime to one another. Suppose that

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* J. Williamson, this Bulletin, vol. 39 (1933), p. 109.

† G. Zehfuss, Zeitschrift für Mathematik und Physik, vol. 3 (1858), p. 298.

‡ L. E. Dickson, *Algebras and Their Arithmetics*, p. 119.

$$A = [a_1, a_2, \cdots, a_{m_1}], \quad B = [b_1, b_2, \cdots, b_{m_2}],$$
$$C = [c_1, c_2, \cdots, c_{m_2}], \quad D = [d_1, d_2, \cdots, d_{m_1}],$$

and let $m_2 > m_1$. From definition, we have

$$A\langle B\rangle = [a_1b_1, a_1b_2, \cdots, a_1b_{m_2}, a_2b_1, \cdots, a_2b_{m_2}, \cdots, a_{m_1}b_1, \cdots, a_{m_1}b_{m_2}]$$

and

$$C\langle D\rangle = [c_1d_1, c_1d_2, \cdots, c_1d_{m_1}, c_2d_1, \cdots, c_2d_{m_1}, \cdots, c_{m_2}d_1, \cdots, c_{m_2}d_{m_1}].$$

Identifying these two row vectors element by element, we get m_1m_2 equations, determining the relations which must hold between the elements of A, B, C, D. Amongst these m_1m_2 equations, consider the following:

$$\begin{aligned} \alpha_1 b_1 &= \gamma_1 d_1, \\ \alpha_2 b_2 &= \gamma_2 d_1, \\ &\cdots\cdots\cdots \\ \alpha_{m_2-m_1+1} b_{m_2-m_1+1} &= \gamma_{m_2-m_1+1} d_1, \end{aligned} \tag{1}$$

where each α represents some one of $a_1, \cdots, a_{m_1}$, and each γ some one of $c_1, \cdots, c_{m_2}$. A little consideration will show that no two α's represent the same a and no two γ's represent the same c. It is obvious that $\alpha_1 = a_1$ and $\gamma_1 = c_1$. From equations (1) and from the construction of $A\langle B\rangle$ and $C\langle D\rangle$ it follows that

$$\begin{aligned} \alpha_1[b_1, \cdots, b_{m_1}] &= \gamma_1[d_1, \cdots, d_{m_1}], \\ \alpha_2[b_2, \cdots, b_{m_1+1}] &= \gamma_2[d_1, \cdots, d_{m_1}], \\ &\cdots\cdots\cdots \\ \alpha_{m_2-m_1+1}[b_{m_2-m_1+1}, \cdots, b_{m_2}] &= \gamma_{m_2-m_1+1}[d_1, \cdots, d_{m_1}]. \end{aligned} \tag{2}$$

From equations (2) we deduce

$$b_2 = \frac{\gamma_1}{\alpha_1}d_2 = \frac{\gamma_2}{\alpha_2}d_1, \quad \text{whence} \quad \frac{d_2}{d_1} = \frac{\gamma_2\alpha_1}{\gamma_1\alpha_2} = s, \text{ say,}$$

and

$$b_3 = \frac{\gamma_2}{\alpha_2}d_2 = \frac{\gamma_3}{\alpha_3}d_1, \quad \text{whence} \quad \frac{d_2}{d_1} = \frac{\gamma_3\alpha_2}{\gamma_2\alpha_3} = s,$$

and so on. In this way we can show that

$$s = \frac{\gamma_2\alpha_1}{\gamma_1\alpha_2} = \frac{\gamma_3\alpha_2}{\gamma_2\alpha_3} = \cdots = \frac{\gamma_{m_2-m_1+1}\alpha_{m_2-m_1}}{\gamma_{m_2-m_1}\alpha_{m_2-m_1+1}}. \tag{3}$$

Again, from equations (2), we find that

$$b_3 = \frac{\gamma_1}{\alpha_1}d_3 = \frac{\gamma_2}{\alpha_2}d_2, \quad \text{whence} \quad \frac{d_3}{d_2} = \frac{\gamma_2\alpha_1}{\gamma_1\alpha_2} = s.$$

By a repetition of such an argument, we can show that

$$s = \frac{d_2}{d_1} = \frac{d_3}{d_2} = \cdots = \frac{d_{m_1}}{d_{m_1-1}}.$$

Hence

$$D = d_1[1, s, s^2, \cdots, s^{m_1-1}].$$

Equating the last elements in each of the matrix equations (2), we find that by equations (3), since $d_{m_1}=d_1s^{m_1-1}$,

$$b_{m_1+1} = \frac{\gamma_2}{\alpha_2}d_{m_1} = \frac{\gamma_2\alpha_1}{\gamma_1\alpha_2}\frac{\gamma_1}{\alpha_1}d_1s^{m_1-1} = \frac{\gamma_1}{\alpha_1}d_1s^{m_1} = \frac{c_1}{a_1}d_1s^{m_1},$$

$$b_{m_1+2} = \frac{\gamma_3}{\alpha_3}d_{m_1} = \frac{\gamma_3\alpha_2}{\gamma_2\alpha_3}\frac{\gamma_2\alpha_1}{\gamma_1\alpha_2}\frac{\gamma_1}{\alpha_1}d_1s^{m_1-1} = \frac{\gamma_1}{\alpha_1}d_1s^{m_1+1} = \frac{c_1}{a_1}d_1s^{m_1+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{m_2} = \frac{\gamma_{m_2-m_1+1}}{\alpha_{m_2-m_1+1}}d_{m_1} = \cdots = \frac{\gamma_1}{\alpha_1}d_1s^{m_2-1} = \frac{c_1}{a_1}d_1s^{m_2-1}.$$

Hence, since

$$[b_1, \cdots, b_{m_1}] = \frac{\gamma_1}{\alpha_1}d_1[1, s, \cdots, s^{m_1-1}] = \frac{c_1}{a_1}d_1[1, s, \cdots, s^{m_1-1}],$$

we may write

$$B = [b_1, \cdots, b_{m_2}] = \frac{c_1}{a_1}d_1[1, s, \cdots, s^{m_2-1}] = b_1[1, s, \cdots, s^{m_2-1}].$$

We have now shown that with the possible exceptions of the elements $a_2b_1, a_3b_1, \cdots, a_{m_1}b_1$, every element of $A\langle B\rangle$ is s times the preceding element; that is, every element, with the possible exception of the

$$(m_2+1)\text{th}, \quad (2m_2+1)\text{th}, \cdots, \quad (m_2(m_1-1)+1)\text{th} \tag{4}$$

elements. But this is also true of $C\langle D\rangle$ with the possible exception of the

(5) $\quad (m_1+1)\text{th}, \quad (2m_1+1)\text{th}, \cdots, (m_1(m_2-1)+1)\text{th}$

elements. But, since $(m_1, m_2)=1$, no member of the set (4) is a member of the set (5); and since the elements of $A\langle B\rangle$ are identical with those of $C\langle D\rangle$, there are no exceptions and every element of $A\langle B\rangle$ or $C\langle D\rangle$ is s times the preceding element.

Hence

$$a_2b_1 = sa_1b_{m_2} = sa_1b_1s^{m_2-1},$$

so that

$$a_2 = a_1s^{m_2},$$

and similarly

$$a_3 = a_2s^{m_2} = a_1s^{2m_2},$$
$$\cdots\cdots\cdots$$
$$a_{m_1} = a_1s^{m_2(m_1-1)}.$$

It follows that

$$A = a_1[1, s^{m_2}, s^{2m_2}, \cdots, s^{m_2(m_1-1)}],$$

and similarly

$$C = c_1[1, s^{m_1}, s^{2m_1}, \cdots, s^{m_1(m_2-1)}].$$

We have not yet considered all the m_1m_2 equations connecting the elements of A, B, C, D; but, since the above values of A, B, C, D give a solution for any values of the arbitrary quantities a_1, c_1, d_1, s, they also give the most general solution.

3. *A More General Case.* We shall now consider a more general case, where A and D are rectangular matrices of n_1 rows and m_1 columns, where B and C are rectangular matrices of n_2 rows and m_2 columns, and where $(n_1, n_2)=1$, and $(m_1, m_2)=1$. Let

$$A = \begin{bmatrix} a_{11}, & a_{12}, \cdots, a_{1m_1} \\ \cdots & \cdots \\ a_{n_11}, & a_{n_12}, \cdots, a_{n_1m_1} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11}, & b_{12}, \cdots, b_{1m_2} \\ \cdots & \cdots \\ b_{n_21}, & b_{n_22}, \cdots, b_{n_2m_2} \end{bmatrix},$$

$$C = \begin{bmatrix} c_{11}, & c_{12}, \cdots, c_{1m_2} \\ \cdots & \cdots \\ c_{n_21}, & c_{n_22}, \cdots, c_{n_2m_2} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11}, & d_{12}, \cdots, d_{1m_1} \\ \cdots & \cdots \\ d_{n_11}, & d_{n_12}, \cdots, d_{n_1m_1} \end{bmatrix}.$$

By equating the first rows of $A\langle B\rangle$ and $C\langle D\rangle$ we obtain from §2 the relation

$$(6)\qquad [b_{11}, b_{12}, \cdots, b_{1m_2}] = b_{11}[1, s, \cdots, s^{m_2-1}].$$

Similarly, by equating the second rows of $A\langle B\rangle$ and $C\langle D\rangle$ we deduce that

$$(7)\qquad [b_{21}, b_{22}, \cdots, b_{2m_2}] = b_{21}[1, s, \cdots, s^{m_2-1}],$$

for the a's and c's occurring are the same in both rows and hence the s must be the same in (7) as in (6). Proceeding in this way with the rows of $A\langle B\rangle$ and $C\langle D\rangle$, we obtain eventually

$$B = \begin{bmatrix} b_{11}, & b_{12}, & \cdots, & b_{1m_2} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n_21}, & b_{n_22}, & \cdots, & b_{n_2m_2} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{n_21} \end{bmatrix} [1, s, \cdots, s^{m_2-1}].$$

We shall find it more convenient to denote the first factor on the right hand side by $\{b_{11}, b_{21}, \cdots, b_{n_21}\}$, as is frequently done, that is, the curly brackets denote a column vector.

Now, by equating the first columns in $A\langle B\rangle$ and $C\langle D\rangle$, we obtain, in the same manner as (6) was obtained, the relation

$$\{b_{11}, b_{21}, \cdots, b_{n_21}\} = b_{11}\{1, t, t^2, \cdots, t^{n_2-1}\},$$

where t is a new arbitrary quantity. Hence

$$B = b_{11}\{1, t, \cdots, t^{n_2-1}\}[1, s, \cdots, s^{m_2-1}],$$

and in the same manner

$$D = d_{11}\{1, t, \cdots, t^{n_1-1}\}[1, s, \cdots, s^{m_1-1}],$$

$$A = a_{11}\{1, t^{n_2}, t^{2n_2}, \cdots, t^{n_2(n_1-1)}\}[1, s^{m_2}, s^{2m_2}, \cdots, s^{m_2(m_1-1)}],$$

$$C = c_{11}\{1, t^{n_1}, t^{2n_1}, \cdots, t^{n_1(n_2-1)}\}[1, s^{m_1}, s^{2m_1}, \cdots, s^{m_1(m_2-1)}],$$

where $a_{11}b_{11} = c_{11}d_{11}$. It follows that since the above values of A, B, C, D give a solution of $A\langle B\rangle = C\langle D\rangle$, for any values of $a_{11}, c_{11}, d_{11}, s, t$, they give the most general solution.

4. *The Most General Case.* We shall now consider the most general case and show that its solution is dependent upon the one just obtained. Suppose that the matrices A, B, C, D have

n_1, n_2, n_3, n_4 rows and m_1, m_2, m_3, m_4 columns respectively. Since $A\langle B\rangle = C\langle D\rangle$,

$$n_1 n_2 = n_3 n_4, \text{ and } m_1 m_2 = m_3 m_4. \tag{8}$$

Let the highest common factor of n_1 and n_3 be k_1. We write this $(n_1, n_3) = k_1$. Let $n_1 = \nu_1 k_1$ and $n_3 = \nu_3 k_1$, where $(\nu_1, \nu_3) = 1$. Similarly, let $n_2 = \nu_2 k_2$ and $n_4 = \nu_4 k_2$, where $(\nu_2, \nu_4) = 1$; let $m_1 = \mu_1 h_1$ and $m_3 = \mu_3 h_1$, where $(\mu_1, \mu_3) = 1$; and let $m_2 = \mu_2 h_2$ and $m_4 = \mu_4 h_2$, where $(\mu_2, \mu_4) = 1$. From equation (8) $\nu_1\nu_2 k_1 k_2 = \nu_3\nu_4 k_1 k_2$ and therefore $\nu_1\nu_2 = \nu_3\nu_4$. Now since $(\nu_1, \nu_3) = 1$, ν_1 must be a factor of ν_4, and since $(\nu_2, \nu_4) = 1$, ν_4 must be a factor of ν_1. Hence $\nu_1 = \nu_4$ and $\nu_2 = \nu_3$, also $(\nu_1, \nu_2) = 1$. Similarly $\mu_1 = \mu_4$ and $\mu_2 = \mu_3$, also $(\mu_1, \mu_2) = 1$. The procedure is now quite simple, although it is somewhat difficult to explain in writing. Consider the very simple case

$$\begin{aligned}&[a_1, a_2, a_3, a_4, a_5, a_6]\langle [b_1, b_2, b_3, b_4]\rangle \\ &\qquad = [c_1, c_2, c_3, c_4]\langle [d_1, d_2, d_3, d_4, d_5, d_6]\rangle.\end{aligned}$$

In this example, $h_1 = 2$, and we see that the above equation can be split up into the two equations

$$\begin{aligned}{}[a_1, a_2, a_3]\langle [b_1, b_2, b_3, b_4]\rangle &= [c_1, c_2]\langle [d_1, d_2, d_3, d_4, d_5, d_6]\rangle, \\ [a_4, a_5, a_6]\langle [b_1, b_2, b_3, b_4]\rangle &= [c_3, c_4]\langle [d_1, d_2, d_3, d_4, d_5, d_6]\rangle.\end{aligned}$$

In this example $h_2 = 2$, and we can split up each of the above into two equations and so we can reduce this case to the following four examples of the case considered in §2:

$$\begin{aligned}{}[a_1, a_2, a_3]\langle [b_1, b_3]\rangle &= [c_1, c_2]\langle [d_1, d_3, d_5]\rangle, \\ [a_1, a_2, a_3]\langle [b_2, b_4]\rangle &= [c_1, c_2]\langle [d_2, d_4, d_6]\rangle, \\ [a_4, a_5, a_6]\langle [b_1, b_3]\rangle &= [c_3, c_4]\langle [d_1, d_3, d_5]\rangle, \\ [a_4, a_5, a_6]\langle [b_2, b_4]\rangle &= [c_3, c_4]\langle [d_2, d_4, d_6]\rangle.\end{aligned}$$

In the most general case, we can split up the equation

$$A\langle B\rangle = C\langle D\rangle$$

into $k_1 k_2 h_1 h_2$ equations

$$A_{xy}\langle B_{zu}\rangle = C_{xy}\langle D_{zu}\rangle,$$

where $x = 1, 2, \cdots, k_1$; $y = 1, 2, \cdots, h_1$; $z = 1, 2, \cdots, k_2$; $u = 1, 2, \cdots, h_2$, and where A_{xy} is the matrix of ν_1 rows and μ_1

columns whose ijth element is $a_{(x-1)\nu_1+i,(y-1)\mu_1+j}$. That is to say, $A_{xy}=[a_{(x-1)\nu_1+i,(y-1)\mu_1+j}]$ has ν_1 rows and μ_1 columns. Similarly, $C_{xy}=[c_{(x-1)\nu_2+i,(y-1)\mu_2+j}]$ has ν_2 rows and μ_2 columns; while $B_{zu}=[b_{(i-1)k_2+z,(j-1)h_2+u}]$ has ν_2 rows and μ_2 columns and $D_{zu}=[d_{(i-1)k_2+z,(j-1)h_2+u}]$ has ν_1 rows and μ_1 columns. But, since $(\nu_1, \nu_2)=1$ and $(\mu_1, \mu_2)=1$, the most general case is composed of $k_1k_2h_1h_2$ examples of the case treated in §3. For brevity, let us write

$$
\begin{aligned}
E &= \{1, t^{\nu_2}, \cdots, t^{\nu_2(\nu_1-1)}\}[1, s^{\mu_2}, \cdots, s^{\mu_2(\mu_1-1)}],\\
F &= \{1, t, \cdots, t^{\nu_2-1}\}[1, s, \cdots, s^{\mu_2-1}],\\
G &= \{1, t^{\nu_1}, \cdots, t^{\nu_1(\nu_2-1)}\}[1, s^{\mu_1}, \cdots, s^{\mu_1(\mu_2-1)}],\\
H &= \{1, t, \cdots, t^{\nu_1-1}\}[1, s, \cdots, s^{\mu_1-1}].
\end{aligned}
$$

Now, solving $A_{11}\langle B_{11}\rangle=C_{11}\langle D_{11}\rangle$ by the method of §3, we find

$$A_{11} = a_{11}E, \quad B_{11} = b_{11}F, \quad C_{11} = c_{11}G, \quad D_{11} = d_{11}H,$$

where $a_{11}b_{11}=c_{11}d_{11}$. Similarly solving $A_{xy}\langle B_{zu}\rangle=C_{xy}\langle D_{zu}\rangle$, we have

$$
\begin{aligned}
A_{xy} &= a_{(x-1)\nu_1+1,(y-1)\mu_1+1}E, \qquad & B_{zu} &= b_{zu}F,\\
C_{xy} &= a_{(x-1)\nu_2+1,(y-1)\mu_2+1}G, \qquad & D_{zu} &= d_{zu}H,
\end{aligned}
$$

where $a_{(x-1)\nu_1+1,(y-1)\mu_1+1}b_{zu} = c_{(x-1)\nu_2+1,(y-1)\mu_2+1}d_{zu}$ for all values of x, y, z, u. Hence

$$\frac{a_{(x-1)\nu_1+1,(y-1)\mu_1+1}}{c_{(x-1)\nu_2+1,(y-1)\mu_2+1}} = \frac{d_{zu}}{b_{zu}} = q,$$

where q is a constant for all values of x, y, z, u. We notice that E, F, G, H are the same for all values of x, y, z, u, for otherwise we would find some B_{zu} having two different values at once. It follows that

$$
\begin{aligned}
A &= \begin{bmatrix} A_{11}, & \cdots, & A_{1h_1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ A_{k_11}, & \cdots, & A_{k_1h_1} \end{bmatrix}\\
&= \begin{bmatrix} a_{11}, & \cdots, & a_{1,(h_1-1)\mu_1+1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ a_{(k_1-1)\nu_1+1,1} & \cdots & a_{(k_1-1)\nu_1+1,(h_1-1)\mu_1+1} \end{bmatrix}\langle E\rangle
\end{aligned}
$$

$$= q\begin{bmatrix} c_{11}, & \cdots, & c_{1,(h_1-1)\mu_2+1} \\ \cdot & \cdot & \cdot \\ c_{(k_1-1)\nu_2+1,1}, & \cdots, & c_{(k_1-1)\nu_2+1,(h_1-1)\mu_2+1} \end{bmatrix}\langle E\rangle.$$

Similarly

$$C = \begin{bmatrix} c_{11}, & \cdots, & c_{1,(h_1-1)\mu_2+1} \\ \cdot & \cdot & \cdot \\ c_{(k_1-1)\nu_2+1,1}, & \cdots, & c_{(k_1-1)\nu_2+1,(h_1-1)\mu_2+1} \end{bmatrix}\langle G\rangle.$$

In the same way it can be shown that

$$B = F\left\langle\begin{bmatrix} b_{11}, & \cdots, & b_{1h_2} \\ \cdot & \cdot & \cdot \\ b_{k_21}, & \cdots, & b_{k_2h_2} \end{bmatrix}\right\rangle \text{ and } D = qH\left\langle\begin{bmatrix} b_{11}, & \cdots, & b_{1h_2} \\ \cdot & \cdot & \cdot \\ b_{k_21}, & \cdots, & b_{k_2h_2} \end{bmatrix}\right\rangle.$$

The above values for A, B, C, D give the most general solution of the equation

$$A\langle B\rangle = C\langle D\rangle.$$

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