INTEGRAL FUNCTIONS OBTAINED BY COMPOUNDING POLYNOMIALS*

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1. *Introduction*. We consider a sequence of polynomials $P_n(z)$, $(n=1, 2, \cdots)$, where the degrees of the P_n do not exceed a fixed integer m and where each P_n, ordered in ascending powers of z, starts with the term z. We shall study the sequence of polynomials $Q_n(z)$ defined by

$$(1)\qquad Q_1(z) = P_1(z);\ Q_{n+1}(z) = Q_n[P_{n+1}(z)], \qquad (n = 1, 2, \cdots),$$

and also the sequence of polynomials $R_n(z)$ defined by

$$(2)\qquad R_1(z) = P_1(z);\ R_{n+1}(z) = P_{n+1}[R_n(z)], \qquad (n = 1, 2, \cdots).$$

If the coefficients, after the first, in P_n, are sufficiently small, these sequences will converge to integral functions. For instance, sin z can be obtained, in many ways, as a limit of a sequence (1). In what follows, our chief object will be to establish conditions under which the sequences converge to integral functions.

2. *The Sequence of $Q_n(z)$*. Let

$$P_n(z) = z + a_{n2}z^2 + \cdots + a_{nm}z^m, \quad (n = 1, 2, \cdots),$$

where m is an integer independent of n.

THEOREM 1. *Let a convergent series of positive numbers,*

$$(3)\qquad c_1 + c_2 + \cdots + c_n + \cdots,$$

exist such that $|a_{ni}| < c_n$, for every n and for $i=2, \cdots, m$. Then the sequence of polynomials $Q_n(z)$ converges to an integral function, the convergence being uniform in every bounded domain.

PROOF. For every n,

$$(4)\qquad U_n(z) = z + c_n(z^2 + \cdots + z^m)$$

is a majorant of $P_n(z)$. Let

$$V_1 = U_1;\ V_{n+1} = V_n(U_{n+1}), \quad (n = 1, 2, \cdots).$$

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Then V_n is a majorant of Q_n. Also, if we let

$$\alpha_n = c_n(z^2 + \cdots + z^m),$$

we have

$$\begin{aligned} V_{n+1} - V_n &= V_n(z + \alpha_{n+1}) - V_n \\ &= \frac{dV_n}{dz}\alpha_{n+1} + \frac{1}{2!}\frac{d^2V_n}{dz^2}\alpha_{n+1}^2 + \cdots, \end{aligned}$$

from which it follows easily that $V_{n+1} - V_n$ is a majorant of $Q_{n+1} - Q_n$. For every positive z, $V_{n+1}(z) > V_n(z)$. These considerations show that our theorem will be proved if we can show that the sequence of V_n converges for every positive z.

Let b be any positive number. Let

$$h = 2b + 4b^2 + \cdots + 2^{m-1}b^{m-1}. \tag{5}$$

Then the infinite product $(1+hc_1) \cdots (1+hc_n) \cdots$ converges. Let p be a fixed integer such that

$$(1 + hc_{p+1})(1 + hc_{p+2}) \cdots < 2. \tag{6}$$

Let

$$W_1 = U_{p+1}; \; W_{n+1} = W_n(U_{p+n+1}), \quad (n = 1, 2, \cdots).$$

It will plainly suffice to show that the sequence of W_n converges for $z = b$. For any n, by (4) and (5),

$$U_{p+n}(b) < b(1 + hc_{p+n}),$$

so that, by (6), $U_{p+n}(b) < 2b$. Hence

$$\begin{aligned} U_{p+n-1}[U_{p+n}(b)] &= U_{p+n}(b)[1 + c_{p+n-1}(U_{p+n}(b) + \cdots)] \\ &< U_{p+n}(b)[1 + hc_{p+n-1}] \\ &< b(1 + hc_{p+n-1})(1 + hc_{p+n}), \end{aligned}$$

and the last quantity, by (6), is less than $2b$. Continuing in this fashion, we find that, for every n,

$$W_n < b(1 + hc_{p+1}) \cdots (1 + hc_{p+n}) < 2b.$$

This shows that the $W_n(b)$, which increase with n, approach a limit. The theorem is proved.

That the condition placed on the P_n is critical with respect to the convergence of the Q_n, is seen on taking $P_n = z + c_n z^m$ with

$c_n>0$ and (3) divergent. The coefficient of z^m in Q will be $c_1+\cdots+c_n$ and Q_n will tend towards infinity with n for every positive z.

The function $\sin z$ can be expressed as a limit of polynomials Q_n. Let

$$P_n(z) = z - \frac{4}{3^{2n+1}} z^3. \tag{7}$$

The formula

$$\sin z = 3 \sin \frac{z}{3} - 4 \sin^3 \frac{z}{3}$$

gives then

$$\sin z = Q_n(3^n \sin 3^{-n}z).$$

From (7) we see that the Q_n converge to an integral function. This integral function must be $\sin z$, since $3^n \sin 3^{-n}z$ approaches z as n increases.*

3. *The Sequence of $R_n(z)$*. We shall study the sequence of $R_n(z)$ defined by (2).

THEOREM 2. *Let the $P_n(z)$ all be of degree at most $m>1$. Let a sequence of positive numbers c_n exist such that*

$$\limsup_{n\to\infty} c_n^{1/m^n} < 1, \tag{8}$$

and such that, for every n, the moduli of the coefficients of $z^2, \cdots, z^m$ in P_n are all less than c_n. Then the $R_n(z)$ converge to an integral function, the convergence being uniform in every bounded domain.

PROOF. Let r be a number which lies between the two members of (8). Then, for n large,

$$z + r^{m^n}(z^2 + \cdots + z^m)$$

will be a majorant of $P_n(z)$. A fortiori, since $m>1$,

$$U_n(z) = z + r^{m^{n-1}}z^2 + r^{2m^{n-1}}z^3 + \cdots + r^{(m-1)m^{n-1}}z^m \tag{9}$$

* In the same way, one can express as limits of polynomials Q_n a large class of the functions with rational multiplication theorems introduced by Poincaré (Journal de Mathématiques, vol. 55 (1890)).

will be a majorant of $P_n(z)$ for n large. We see now readily that it will suffice, for the proof of our theorem, to show that the sequence of $V_n(z)$ defined by

$$(10) \qquad V_1 = U_1; \; V_{n+1} = U_{n+1}(V_n), \qquad (n \geqq 1),$$

converges for every real and positive z.†

Let p be any non-negative integer. Putting

$$(11) \qquad W_1 = U_{p+1}; \; W_{n+1} = U_{p+n+1}(W_n), \qquad (n \geqq 1),$$

we shall show that the sequence of W_n converges for $z < hr^{-mp}$, where $h = 1 - r$.

By (9),

$$S_n(z) = \frac{z}{1 - r^{mp+n-1}z}, \qquad (n = 1, 2, \cdots),$$

is a majorant of U_{p+n}. If, then,

$$T_1 = S_1; \; T_{n+1} = S_{n+1}(T_n), \qquad (n \geqq 1),$$

T_n will be a majorant of W_n. Now an easy calculation shows that

$$T_n(z) = \frac{z}{1 - (r^{mp} + \cdots + r^{mp+n-1})z}.$$

For any positive z less than the reciprocal of the infinite series

$$r^{mp} + r^{mp+1} + \cdots,$$

which reciprocal we shall denote by k, the $T_n(z)$ form a sequence of numbers which increase towards $kz/(k-z)$. Also, if $0 < z < k$, $T_n(z) > W_n(z)$, so that the $W_n(z)$ will form a bounded sequence of increasing numbers and will converge to a limit. Now as $m > 1$,

$$k \geqq \frac{r^{-mp}}{1 + r + r^2 + \cdots} = hr^{-mp},$$

and our statement with respect to (11) is proved.

Thus Theorem 2 will be established if, putting $V_0(z) = z$, we show that for every positive z there is a p such that $V_p(z) < hr^{-mp}$.

† The fact that U_n may not be a majorant of P_n for n small is of no significance. One may suppress a finite number of P_n and then add a finite number of polynomials (9) to the beginning of the resulting sequence of U_n.

Let us assume that there is a positive z for which no such p exists. In what follows, we work with a fixed z of this type. We have, by (9) and (10), for any $n \geqq 0$,

$$
\begin{aligned}
V_{n+1} &= V_n + r^{m^n} V_n^2 + \cdots + r^{(m-1)m^n} V_n^m \\
&\leqq V_n(1 + r^{m^n} V_n)^{m-1}.
\end{aligned}
\tag{12}
$$

Now, for every n,

$$V_n \geqq h r^{-m^n}, \tag{13}$$

so that

$$1 \leqq \frac{r^{m^n} V_n}{h},$$

and, if we put $a = (1+1/h)^{m-1}$, we have, by (12),

$$V_{n+1} \leqq a r^{(m-1)m^n} V_n^m \leqq a r^{m^n} V_n^m.$$

We have thus

$$V_1 \leqq a r z^m, \quad V_2 \leqq a^{m+1} r^{2m} z^{m^2}, \quad V_3 \leqq a^{m^2+m+1} r^{3m^2} z^{m^3},$$

and, in general,

$$V_{n+1} \leqq a^{m^n + \cdots + 1} r^{(n+1)m^n} z^{m^{n+1}}.$$

As $m > 1$, we have $m^n + \cdots + 1 < m^{n+1}$. Then, because $a > 1$,

$$V_{n+1} < \left[r^{(n+1)/m} a z \right]^{m^{n+1}}. \tag{14}$$

As z is fixed, $r^{(n+1)/m} a z$ is small for n large, so that, by (14), V_{n+1} approaches 0 as n increases. This contradicts (13). The theorem is proved.

The condition (8) is a critical one. That we cannot let the first member of (8) be as great as unity is seen on taking $P_n = z + z^m$. The coefficient of z^m in Q_n will be n and the Q_n will diverge for every positive z. That m in the first member of (8) cannot be replaced by any smaller positive number α, is seen, taking $m = 2$, for instance, on putting $P_n = z + 2^{-\alpha^n} z^2$. For any positive z, we have

$$P_n > 2^{-\alpha^n} z^2.$$

Then

$$R_1 > 2^{-\alpha} z^2, \quad R_2 > 2^{-(\alpha^2 + 2\alpha)} z^4,$$

and, in general,

$$R_n > 2^{-(\alpha^n+2\alpha^{n-1}+\cdots+2^{n-1}\alpha)}z^{2^n}.$$

Now

$$-(\alpha^n + 2\alpha^{n-1} + \cdots + 2^{n-1}\alpha) = \frac{\alpha^{n+1} - 2^n\alpha}{2-\alpha} > -b2^n,$$

where $b = \alpha/(2-\alpha)$. Thus

$$R_n > \left(\frac{z}{2^b}\right)^{2^n},$$

so that the R_n diverge for $z > 2^b$.

Let $f(z)$ be an integral function obtained as a limit of polynomials $R_n(z)$, the approach being uniform in every bounded domain. Unless $P_n(z) = z$ for every n, $f(z)$ will not be linear, for if some $R_n(z)$ is of degree greater than unity, $f(z)$, like that $R_n(z)$, will assume certain values at more than one place. In what follows, we shall assume that $f(z)$ is not linear.

We are going to prove that, between any two branches of the inverse of $f(z)$, there exists an algebraic relation of a simple type.

Let a and b be two distinct points such that $f(a) = f(b)$ and that the derivative of $f(z)$ does not vanish at a or at b. Let A be a circle with a as center such that, in the interior of A, $f(z)$ assumes no value twice. Let B be a similar circle with center at B. We can find a neighborhood M of $f(a) = f(b)$ such that, both in A and in B, $R_n(z)$ with n large assumes all values in M. If n is large enough, $R_n(a)$ will be in M. In what follows, we deal with a fixed $R_n(z)$ for which both conditions just described are realized.

If z_a is a point in A, very close to a, there will be a z_b in B such that $f(z_b) = f(z_a)$, and, furthermore, $R_n(z_a)$ will lie in M. We shall prove that $R_n(z_a) = R_n(z_b)$. As $R_n(z_a)$ is in M, there is a ζ in B such that $R_n(\zeta) = R_n(z_a)$. Now ζ must coincide with z_b, for $f(\zeta) = f(z_a) = f(z_b)$ and $f(z)$ assumes no value twice in B.

Thus, *if we put $w = f(z)$ and if $\alpha(w)$ and $\beta(w)$ are two branches of the inverse of $f(z)$, then, for n large, $R_n[\alpha(w)] = R_n[\beta(w)]$.*

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