Therefore $\sum_{i=1}^{\infty}|a_{ik}|^p$ converges for every k and (36) is a transformation on the space (x_i), where $\sum_{i=1}^{\infty}|x_i|^{p/(p-1)}$ converges and $p/(p-1)\geqq 2$. For any such (x_i), we have

$$(38)\quad \left[\sum_k |y_k|^{p/(p-1)}\right]^{(p-1)/p} \leqq \left[\sum_{i=1}^{\infty}|x_i|^{p/(p-1)}\right]^{(p-1)/p} \cdot\left[1+\left\{\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}|a_{ik}|^p\right]^{1/(p-1)}\right\}^{(p-1)/p}\right],$$

and hence the (y_k) is in the same space. On the other hand, given (y_k) in this space, we find an equivalent system (26′) as in I having an absolutely convergent determinant and obtain the solution from (26′) by Cramer's rule as in II, (6) of the lemma being used. The discussion of the inverse of the transposed system follows that of III.

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EXAMPLES OF SURFACE TRANSFORMATIONS

BY P. A. SMITH

The purpose of this note is to illustrate by examples certain points in the structure analysis of surface transformations, and at the same time to point out certain unsolved problems which appear to be fundamental in this connection.

Every (1-1) continuous transformation of a surface S into itself admits* a closed invariant set of *central points* which possesses certain properties of regional recurrence and which is determined essentially as follows: the points of S move under indefinite iteration of T and its inverse T_{-1}, toward a certain closed invariant set M^1 which in general is a proper subset of S; M^1 contains a proper subset M^2 related to M^1 as M^1 is to S. Continuing thus, we finally arrive at a first set M^r such that

* The definitions and theorems used implicitly in this note are to be found in detail in Birkhoff and Smith, *Structure analysis of surface transformations*, Journal de Mathématiques, vol. 7 (1928). With regard to regular regions, see also P. A. Smith, *Regular components of surface transformations*, to appear shortly in the American Journal of Mathematics.

$M^{r+i}=M^r$ $(i=1, 2, \cdots)$ and M^r is the set in question. The number r is a sort of measure of the structural complexity of T, and may perhaps be a transfinite ordinal. The fact is, however, that as yet no examples are known for which $r>2$. Examples with $r=1$ are simple to construct; a sphere, for example, may be rotated through 180°. Then $M^1=S$ and hence $M^i=S$, $i>1$, and $M^r=S$, $r=1$. Transformations with $r=2$ can also be effectively constructed; to show this will be the purpose of our first example. It would be of considerable value for structure analysis to know whether or not r can be greater than 2.

For the set M^1 we may take the totality of *non-wandering* points of S,—they are defined by the property that an arbitrarily small neighborhood σ of any of them is intersected by images of σ under sufficiently great powers of T and T_{-1}. Or, we may start with N^1 for our initial set, N^1 being the totality N of α- and ω-limit points of T, together with the ordinary limit points of N. Here we may ask if N is not itself closed, and our second example is constructed to show that such need not be the case. Each point of N^1 is non-wandering and hence there can exist non-wandering points which are not α- or ω-limit points. It would be of considerable interest to go a step farther and learn whether or not there can exist non-wandering points not contained in N^1. A closely related unsolved problem is the following: the sequence N^1, N^2, $\cdots$ leads eventually to a final set N^s which is identical with M^r; it is known that s is not greater than r,—must $s=r$?

The set $S-M^1$, if it is not null, may contain certain *regular regions*, that is, regions of uniform approach toward M^1 with respect to indefinite iteration of T. The question arises as to whether such regions *must* exist. We shall describe briefly a transformation in which the open set $S-M^1$ contains no regular regions.

The Transformations. Let S be a sphere and $\theta=$const., $\phi=$const. $(-\pi/2\leqq\theta\leqq\pi/2, 0\leqq\phi\leqq 2\pi)$ be its parallels and meridians respectively. We shall first describe three preliminary transformations of S.

A. Let t be a (1-1) continuous transformation of the equator $(\theta=0)$ into itself, leaving invariant the point O $(\theta=0,\ \phi=0)$ and increasing the ϕ-coordinate of every other point by an amount less than $\pi/2$. If t is represented by $\phi_1=\phi+f(\phi)$, the first transformation A of S will be defined by

$$\theta_1 = \theta, \ \phi_1 = \phi + f(\phi).$$

Thus A leaves invariant the poles $\theta=\pm\pi/2$, and the points of the meridian $\phi=0$, and increases the ϕ of all other points by amounts always less than $\pi/2$.

B. The transformation B is given by

$$\theta_1 = \theta - \lambda \sin 2\theta, \qquad \phi_1 = \phi,$$

where the positive constant λ is chosen, as it can be, so small that B shall be (1-1). The only invariant points here are the poles and the points of the equator. The general motion of all other points on iteration of B is toward the equator. In fact if $\cdots C_{-1}, C_0, C_1, \cdots$ represent the successive images of the parallel C_0 in the upper (lower) hemisphere, the sequence $C_0, C_1, \cdots$ has the equator for its only limit circle, and the sequence $C_0, C_{-1}, \cdots$ closes down on the pole $\theta=\pi/2, (\theta=-\pi/2)$.

C. This transformation is represented by

$$\theta_1 = \theta, \qquad \phi_1 = \phi + g(\theta),$$

where $g(\theta)$ is defined as follows: assume C_0 of the preceding paragraph to lie in the upper hemisphere, and let C_n be given by $\theta=\theta_n, (n=0,\pm1,\cdots)$. Then let $g(\theta_n)=\pi(|n|+1)^{-1}$, $g(0)=g(\pi/2)=0$; on each interval (θ_n, θ_{n+1}), $g(\theta)$ is to vary linearly between the values taken at the end points. This defines $g(\theta)$ for $0\leqq\theta\leqq\pi/2$ and the definition is now completed by putting $g(-\theta)=g(\theta)$.

Consider now the product transformation $T=ABC$; *we shall show that for* T, $r=2$. We note first that T may be represented in the form

$$\theta_1 = \theta - \lambda \sin 2\theta, \qquad \phi_1 = F(\theta,\phi).$$

Hence, as for B, motion in the two hemispheres is uniformly toward the equator on iteration of T, and toward their respective poles (which are invariant points) on iteration of T_{-1}. Hence the set M^1 is contained in the set P^1+P^2+equator, P^1 and P^2 being the poles. The equator transforms under T just as it does under A, since its points are invariant under B

and C. Hence its points move toward the invariant point O on iteration of T and T_{-1}. Hence M^2 contains only the points O, P^1, P^2. Thus $M^r = M^2$, and $r \leqq 2$. To show that $r = 2$, we must show that M^2 is a proper subset of M^1. We shall show in fact that all points of the equator are in M^1.

Let σ be a small neighborhood of an arbitrary point P of the equator. For m positive and sufficiently large, σ will be intersected by C_m; let Q be a point of intersection. Now the ϕ-coordinate of Q receives non-negative increments from A and B; hence its increment from T is at least as great as that due to C, namely $\pi(1+m)^{-1}$. Hence the ϕ of Q is increased by an amount $\geqq \sum_{i=0}^{k-1} \pi(m+1+i)^{-1}$ on k applications of T, and is therefore increased without bound on infinite iteration. Thus Q describes, so to speak, infinitely many circuits about the sphere in the neighborhood of the equator, whereas P converges directly to O. Hence the successive images of σ tend to wind themselves around the sphere, and are at the same time drawn constantly closer to the equator. Clearly σ_n must intersect σ for large positive values of n, and P is therefore non-wandering, hence in M^1. This completes the proof that $r = 2$.

Consider next the transformation AC; we shall show that the set N defined above is not closed. We remark first that AC transforms into itself every circle $\theta = c$ according to the relation

$$\phi_1 = \phi + f(\phi) + g(c) = \lambda_c(\phi).$$

Since $f(\phi) \geqq 0$ for all ϕ, and $g(c) > 0$ except for $c = 0$, $\pm\pi/2$, the only invariant points are the poles and O. The function $\lambda_c(\phi)$ undergoes continuous modification as c varies continuously and hence the rotation number* $\rho(c)$ associated with $\lambda_c(\phi)$ is a continuous function of c as one can readily verify. Since O is invariant, $\rho(0) = 0$ and hence an arbitrarily small non-zero value c' of c can be chosen such that $\rho(c')$ is commensurable with π. Because of a property of rotation numbers, there exist on the circle $\theta = c$ (which is arbitrarily near the equator) a number of periodic points. Let one of them be U. Since U can not be invariant, it is at least of order 2, and iterates periodically

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* For a definition of rotation numbers and an account of their properties, see Birkhoff, *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1920), p. 87.

through a finite group of two or more periodic points on repeated application of AC. Hence the ϕ-coordinate of U increases without bound. But the successive increments of ϕ are $<\pi/2$. This is due to the definition of A and the fact that near the equator the influence of C is minute. It follows that there exists at least one periodic point on every quarter-circumference of $\theta = c'$. Clearly the totality of periodic points must therefore have at least four limit points on the equator. Since each periodic point is an α- and ω-limit point, and since the only α- or ω-limit point on the equator is 0, we have at least three points which are not α- or ω-limit points, but are limit points of such points. *Hence N is not closed.*

In our final example* we shall let S be represented by the $x\,y$ plane with a single point at infinity. Each point of S not contained between the two lines $y = O$ and $y = 1$ is to be invariant. The only further invariant points are those of the set $(\pm p/q,\ 1/q)$, where p and q are positive integers and $q > 1$. All the remaining points are to move vertically downward $(x_1 = x,\ y_1 < y)$ on iteration of T, and hence vertically upward on iteration of T_{-1}. Clearly the set M^1 consists of the invariant points only.

Let σ be a small connected region in $S - M^1$. We can choose in σ a point $P(a, b)$ where $a = m/n$, $b > 1/n$, and a point P' (a', b') where a' is irrational. Under iteration of T, all points of σ move vertically downward. The sequence $P, P_1, P_2, \cdots$ converges to $V(m/n, 1/n)$, while the sequence $P', P_1', P_2', \cdots$ converges to $(a', 0)$. Now V is an isolated point of M^1 and is at a non-zero distance δ from the remaining points of M^1. Hence no image of σ under positive powers of T can lie entirely within a distance ϵ of M^1, if we take $\epsilon < \delta/2$. Thus *there are no regions of uniform approach toward* M^1 and all points of $S - M^1$ are ω-irregular.

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* Certain details of this construction will be omitted; they are not difficult, however, and have been carefully verified.