NOTE ON SEPARABILITY*

BY R. G. PUTNAM

The following theorems have been shown by R. L. Moore† to hold in a class D of Fréchet.‡

THEOREM 1. *In order that every subclass of a given class D of Fréchet should be separable, it is necessary and sufficient that every uncountable subclass of that class D should have a limit point.*

THEOREM 2. *If D_s is a separable class D, then every uncountable subclass of D_s contains a point of condensation.*

THEOREM 3. *Every subclass of a separable class D is itself separable.*

THEOREM 4. *In order that every uncountable subclass of a given class D should contain a point of condensation of itself, it is necessary and sufficient that every uncountable subclass of D should have a limit point.*

THEOREM 5. *In order that every ascending sequence of distinct closed subsets of a given class D should be countable, it is necessary and sufficient that every descending one should be.*

Theorems 3 and 4 follow from Theorems 1 and 2, and 5 is obtained with the aid of Theorems 1 and 4.

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† Fundamenta Mathematicae, vol. 8, p. 189. Theorems 1, 2, and 3 have been previously considered by W. Gross in *Zur Theorie der Mengen, in denen ein Distanzbegriff definiert ist,* Sitzungsberichte, Wien, vol. 123 (1914), pp. 801–819. See also a reference to this article in *An acknowledgement,* by R. L. Moore, Fundamenta Mathematicae, vol. 8, p. 374.

‡ A class D of Fréchet is a class of elements which satisfy the following conditions:

1. With every pair of elements A and B there is associated a number $(A, B)=(B, A)\geqq 0$.

2. $(A, B)=0$ if, and only if, $A=B$.

3. If A, B and C are any three elements, then $(A, C)\leqq(A, B)+(B, C)$.

4. The sequence of elements $P_1, P_2, P_3, \cdots$ converges to a limit P if and only if the distance (P, P_n) approaches zero as n becomes infinite. A class in which conditions 1, 2 and 4 hold but in which 3 need not hold is a class E.

The following example, due to P. Alexandroff, shows that none of the first four theorems stated above is true without change in all classes E. Consider the space composed of two parallel lines A and B. Let the écart of any two distinct points x and y both of which are on A or on B be unity while the écart of a point x on A and a point y on B is the ordinary distance from x to the projection of y on A or vice versa; if x and y are not distinct their écart is zero. In this space the limit points of a set entirely on one line are all on the other line. The space thus formed is a separable class E. Every uncountable subclass has a limit point but is not always separable and does not contain a point of condensation. Theorem 4 is not true in this space.

Theorems 1′, 2′, 4′, and 5′ which follow, are modifications respectively of Theorems 1, 2, 4, and 5, and can be shown to hold in classes E (not at the same time classes D).

THEOREM 1′. *In order that every subclass of a given class E should be separable, it is necessary and sufficient that every uncountable subclass of that class E should contain a limit point of itself.*

The proof of this theorem is the same as the proof of Theorem 1 in Professor Moore's paper.

THEOREM 2′. *If every subclass of a given class E is separable, then every uncountable subclass of that class E contains a point of condensation of itself.*

The proof of this theorem follows from the first part of the proof of Theorem 2.

Theorem 3 does not hold in all classes E as the above example shows.

THEOREM 4′. *In order that every uncountable subclass of a given class E should contain a point of condensation of itself, it is necessary and sufficient that every uncountable subclass of the given class should contain a limit point of itself.*

This theorem follows at once from Theorems 1′ and 2′.

THEOREM 5′. *In order that every ascending sequence of distinct closed subsets of a given class E in which every derived set is closed should be countable, it is necessary and sufficient that every descending one should be.*

This theorem is a consequence of Theorems 1′ and 4′ and the result of Sierpinski, used by Professor Moore in the proof of Theorem 5.

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NOTE ON A SCHOLIUM OF BAYES

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In his fundamental paper on a posteriori probability,* Bayes considered a certain event M having an unknown probability p of its occurring in a single trial. In deriving his a posteriori formula he assumed that all values of p are equally likely, and he recommended this assumption for similar problems in which nothing is known concerning p. In the corollary to proposition 8 he derives the value

$$\int_0^1 \binom{n}{x} p^x(1-p)^{n-x}dp = \frac{1}{n+1}$$

for the probability of x successes in n trials. This result is independent of x; in a scholium he observes that this consequence is what is to be expected, on common sense grounds, from complete ignorance concerning p, and this concordance is considered to justify the assumption that all values of p are equally likely.†

In order to complete the argument of the scholium it is necessary to show that no other frequency distribution for p has the same property.

More precisely, given that a cumulative frequency function $f(p)$ has the property that for $0 \leqq x \leqq n$, x, n being integers,

$$\int_0^1 \binom{n}{x} p^x(1-p)^{n-x}df(p) = \frac{1}{n+1},$$

* Bayes, *An essay towards solving a problem in the doctrine of chances*, Philosophical Transactions of the Royal Society, vol. 53 (1763), pp. 370–418.

† In other words, the assumption "all values of p are equally likely" is *equivalent* to the assumption "any number x of successes in n trials is just as likely as any other number y, $x \leqq n$, $y \leqq n$." It has been suggested verbally by Mr. E. C. Molina that this proposition has a possible importance in certain statistical questions.