ON THE SEPARATION OF THE PLANE BY IRREDUCIBLE CONTINUA†

BY W. A. WILSON

1. *Introduction.* This question was investigated by A. Rosenthal‡ in 1919. His principal results may be stated as follows. If F is the union of two bounded continua, C_1 and C_2, which are irreducible between the points a and b and have no other common points, then the complement of F with respect to its plane consists of two principal component regions (*Hauptgebiete*), each of which has F as its frontier, and possibly of a number of secondary component regions (*Nebengebiete*), each of which has its frontier wholly in either C_1 or C_2.

It is a simple matter to construct two irreducible continua which together constitute the frontier of precisely two of the complementary regions, but which have more than two points in common. For example, let bmc and bnc be two complementary arcs of a circumference, and let am^* and an^* be two wavy lines intersecting only at a and approaching asymptotically bmc and bnc, respectively. Then $C_1 = am^* + bmc$ and $C_2 = an^* + bnc$ are both irreducible between a and b and divide the plane into three regions, of which two only have $C_1 + C_2$ as their frontier. But C_1 and C_2 have in common three points, not two.

In this particular case, to be sure, Rosenthal's theorem can be used, for $C_1 + C_2$ can be expressed as the union of two continua, irreducible between a pair of points and intersecting only in these points. This is possible here because C_1 and C_2 can each be separated into two irreducible continua having only one common point. Such a decomposability,

† Presented to the Society, October 30, 1926.

‡ (A) A. Rosenthal, *Teilung der Ebene durch irreduzibele Kontinua*, Sitzungsberichte der Münchener Akademie, 1919, pp. 91–109.

however, is not a general property of irreducible continua, even when no indecomposable continua are involved. Therefore a removal of the requirement that the two irreducible continua have but two points in common is a real extension of the theorem quoted above.

It is the purpose of the present article to give some general conditions under which this extension is possible and also to investigate the frontiers of the secondary regions. The principal results will be found in §§ 5 and 8.

In certain parts of the work free use is made of the oscillatory set of a continuum about a point and its properties, which have been developed by the author in the papers listed below.* The notation of these papers is also used.

2. *Some Generalizations.* For what follows it is convenient to extend the notion of a continuum irreducible between two points in two ways which must be kept distinct.

DEFINITION I. *If A is a closed sub-set of the continuum C, then C is irreducible about A if no proper sub-continuum of C contains A.*

DEFINITION II. *If α and β are closed sets without common points and the continuum C contains one or more points of both sets, then C is irreducible between α and β if no proper sub-continuum of C contains points of both sets.*

The second definition is due to Miss Anna M. Mullikin.† It is readily seen that if A or $\alpha+\beta$ consists of two points, both definitions yield the ordinary continuum irreducible between two points. The following properties will be used.

LEMMA I. *If C is a bounded continuum irreducible about the sum of the two continua A and B, which have no common points, then $C-(A+B)$ is connected.*

* (B) *On the oscillation of a continuum*, Transactions of this Society, vol. 27.

(C) *Some properties of the irreducible continuum*, ibid., vol. 28.

(D) *On the structure of the irreducible continuum*, American Journal, vol. 48.

† (E) Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24.

The proof given by Rosenthal (loc. cit., p. 104) for the case where A and B are points is immediately applicable.

LEMMA II. *Let C be bounded and closed and contain points of both the closed sets α and β. Let $\alpha \cdot \beta = 0$. Then C can be separated into two closed sets C_1 and C_2, such that $C_1 \cdot C_2 = \alpha \cdot C_2 = \beta \cdot C_1 = 0$, or else C contains a sub-continuum irreducible between α and β.*

This theorem is stated and proved in an equivalent form by Miss Mullikin (loc. cit., p. 147).

LEMMA III. *Let C be a bounded continuum irreducible between the closed subsets α and β, which have no common points. Let A be the oscillatory set of C about some point a of α. Then A contains α and, if D is a sub-continuum of C containing a point x of $C-A$ and a point a' of α, D contains A.*

PROOF. We omit the trivial case that C is indecomposable. If A is complete, $C-A$ is a semi-continuum by a theorem proved elsewhere.* Obviously it contains β and hence can contain no point of α. Let E be a sub-continuum of $C-A$ joining x and β. Then $D+E$ joins α and β, and hence equals C. As $A \cdot E = 0$, D contains A.

If A is not complete, it is indecomposable and is not a continuum of condensation.† Then $\overline{C-A}$ is a proper sub-continuum‡ of C and contains β. Hence it contains no point of α. Thus A contains α in this case too. As x is a point of $C-A$, $D+\overline{C-A} = C$; hence D contains $\overline{C-\overline{C-A}}$. Since $\overline{C-\overline{C-A}}$ is a sub-continuum of A and contains all points of A whose distance from a is less than some positive δ, it is identical with A. Thus D contains A.

3. *A Special Type of Frontier Set.* Let $F = H_1 + H_2$, where H_1 and H_2 are bounded continua and $F - H_2 = H_1$

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* See reference (C), p. 543.

† See reference (D), p. 155.

‡ (F) C. Kuratowski, *Théorie des continus irréductibles*, Fundamenta Mathematicae, vol. 3, Theorem 3.

$-H_1 \cdot H_2$ and $F-H_1=H_2-H_1 \cdot H_2$ are connected sets. Certain properties of this type of set and its complementary regions are easily deduced.

If Z denotes the plane, it follows from the fact that $F-H_1$ is connected that $F-H_1$ lies in some one component G_1 of $Z-H_1$. Likewise, $F-H_2$ lies in one component G_2 of $Z-H_2$.

Now let G' be any other component of $Z-H_1$ and G'' any other component of $Z-H_2$. We first show that, if $G' \cdot G_2 \neq 0$, then G_2 contains G'. For otherwise G' would contain points of H_2, the frontier of G_2, since G' is connected. This is impossible, for G_1 is the only component of $Z-H_1$ containing points of H_2. Likewise, either $G'' \cdot G_1=0$ or G_1 contains G''.

A second property regarding these components is that either $G' \cdot G''=0$ or else $G'=G''$ and the frontier of G' is a part of some component of $H_1 \cdot H_2$. To prove this let us assume that $G' \cdot G'' \neq 0$. Then, if G'' contained points not in G', it would contain points of the frontier of G', which is a part of H_1. This is impossible by the second paragraph above. Likewise, G' contains no point not in G'' and hence $G'=G''$. Let g' denote the frontier of G'. Then $g' \subseteq H_1H_2$. As g' is a continuum, it is a part of some component of H_1H_2.

With these preliminaries, we are in a position to prove the following lemma.

4. LEMMA IV. *Let F be the union of two bounded continua H_1 and H_2 having these properties: $F-H_1$ and $F-H_2$ are connected;** $H_1 \cdot H_2=A+B$, *where A and B are continua and $A \cdot B=0$; H_1 and H_2 contain sub-continua C_1 and C_2, respectively, such that $\alpha=C_1 \cdot C_2 \cdot A \neq 0$, $\beta=C_1 \cdot C_2 \cdot B$, C_1 and C_2 are irreducible between α and β, and $F=C_1+C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.*

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* In view of §2, Lemma I, this requirement may be replaced by the hypothesis that H_1 and H_2 are irreducible about A and B.

PROOF.* It was shown in §3 that, if Z denotes the plane, one component G_1 of $Z-H_1$ contains $H_2-(A+B)$ and one component G_2 of $Z-H_2$ contains $H_1-(A+B)$. Let $K_1=Z-G_1$ and $K_2=Z-G_2$. Then K_1 and K_2 are continua which do not cut the plane.

Now consider $K_1 \cdot K_2$. As K_1 consists of H_1 and the components G' of $Z-H_1$ differing from G_1, and K_2 is similarly constituted, it is evident from §3 that $K_1 \cdot K_2$ consists of $A+B$ and such components G' of $Z-H_1$ as coincide with components G'' of $Z-H_2$. In §3 it was shown that, if $G'=G''$, then g', the frontier of G', is a part of either A or B. In consequence of these facts $K_1 \cdot K_2$ is the sum of two continua, which may be denoted by A' and B', where A' contains A and may contain one or more regions whose frontiers form a part of A, B' contains B and may contain one or more regions whose frontiers form a part of B, and $A' \cdot B'=0$.

Two cases must now be considered: (a) one of the regions G_1 and G_2 is unbounded; (b) neither G_1 nor G_2 is unbounded. We shall complete the proof for the first case and then return to the second.

If G_1 is unbounded, K_1 is bounded. For otherwise a point in K_1 could be joined to one in G_1 by a broken line not cutting the bounded set H_1, which is the frontier of G_1. Likewise K_2 is bounded, if G_2 is unbounded. Thus either K_1 or K_2 is bounded. We have seen that neither of these continua separates the plane and that their divisor is two continua without common points. We can therefore apply a theorem of Miss Mullikin,† which shows that K_1+K_2 cuts the plane into exactly two regions, R_1 and R_2. Let F_1 and F_2 be the frontiers of these regions. If we can show that $F_1=F_2=F$, the theorem is proved. To this end we first observe that, since no point of F_1 or F_2 can be an inner point of K_1 or K_2, then F_1 and F_2 are parts of $H_1+H_2=F=C_1+C_2$.

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* (G) This proof is a modification of the proof of Rosenthal's theorem given by S. Straszewicz in his paper *Über die Zerschneidung der Ebene durch abgeschlossene Mengen*, Fundamenta Mathematicae, vol. 7, p. 187.

† See reference (E), p. 160; also reference (G), §§18, 19.

It is easily seen that $R_1+R_2=G_1 \cdot G_2$. Now if m is a point of R_1 and n is one of R_2, there is a broken line in G_1 joining m and n and not cutting H_1, for $G_1 \cdot H_1=0$. Thus H_1 (and in like fashion H_2) is not an $S(m, n)$.*

There are two sub-cases: either F_1 contains C_1 or C_2, or it does not. To fix the ideas, let $C_1 \subset F_1$. Then $F_1 \cdot C_2$ is not void by the previous paragraph. Since C_2 is irreducible between α and β, it follows from §2, Lemma II, that, if $F_1 \cdot C_2 \neq C_2$, then $F_1 \cdot C_2$ is the sum of two closed sets M and N, such that $M \cdot N=M \cdot \beta=N \cdot \alpha=0$. (One of these, but not both, may be void.) Then $F_1=M+C_1+N$. Since $M+A \subseteq H_2$ and $A+C_1 \subseteq H_1$, neither $M+A$ nor $A+C_1$ is an $S(m, n)$. Hence by a theorem of Janiszewski† $M+A+C_1$, and consequently $M+C_1$, is not an $S(m, n)$. Likewise, C_1+N is not an $S(m, n)$ and a second application of the theorem referred to gives the contradiction that F_1 is not an $S(m, n)$. Hence $F_1 \cdot C_2=C_2$, and $F_1=C_1+C_2=F$.

Now suppose that neither C_1 nor C_2 is a part of F_1. Since C_1 is irreducible between α and β, $F_1 \cdot C_1$ is the sum of two closed sets M and N, such that $M \cdot N=N \cdot \alpha=M \cdot \beta=0$. Either M or N, but not both, may be void. For the same reason $F_1 \cdot C_2=P+Q$, where P and Q are closed and $P \cdot Q=P \cdot \beta=Q \cdot \alpha=0$. Then $F_1=(M+P)+(N+Q)$ and $(M+P)(N+Q)=0$, which is a contradiction, as F_1 is a continuum, unless both M and P, or N and Q, are void. If $M=P=0$, $F_1=N+Q \subseteq N+B+Q$. As neither $N+B$ nor $B+Q$ is an $S(m, n)$, this is a contradiction.

Thus in both sub-cases $F_1=C_1+C_2=F$, and in like fashion $F_2=F$.

It now remains to take care of the situation that arises when neither G_1 nor G_2 is unbounded. If ν is a point of G_1

* A set H is an $S(m, n)$ if every continuum joining m and n cuts H.

† (H) Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A: "If P and Q are bounded closed sets, if $P \cdot Q$ is connected, and neither P nor Q is an $S(m, n)$, then $P+Q$ is not an $S(m, n)$." See also Straszewicz, Fundamenta Mathematicae, vol. 4, p. 129.

not on H_2 and the plane is inverted with respect to ν as a center, the image F^* of F will satisfy the conditions of the first case. This follows from the facts that the correspondence between F and F^* is homeomorphic, that the image G_1^* of G_1 is unbounded, that the images of the frontiers of the component regions of $Z-F$ are the frontiers of the images of these regions, and that the property of irreducibility is an invariant of analysis situs. Then by the proof given above there are precisely two components, R_1^* and R_2^*, of $Z-F^*$ which have F^* as their frontier. Inverting again. we have the inverse images R_1 and R_2 of R_1^* and R_2^*, respectively, as the only components of $Z-F$ which have F as a frontier.

5. *A General Theorem.* It is obvious that Rosenthal's theorem is the special case of the above lemma obtained when A and B are points, in which case $H_1=C_1$ and $H_2=C_2$. We can, however, obtain from this lemma the following theorem, which is, aside from one exceptional case, more general than the theorem in question.

THEOREM I. *Let F be the union of two bounded continua* C_1 *and* C_2 *having these properties:* $C_1 \cdot C_2=\alpha+\beta$, *where* α *and* β *are closed and* $\alpha \cdot \beta=0$; *both* C_1 *and* C_2 *are irreducible between* α *and* β; *and either* C_1 *and* C_2 *are both decomposable, or* C_1 *is indecomposable and* C_2 *is decomposable and not the union of two indecomposable continua. Then F cuts the plane and is the frontier of exactly two components of its complement.*

PROOF. This theorem is a corollary of §4. If we let A_1 and A_2 denote the oscillatory sets of C_1 and C_2, respectively, about some point a of α, and let B_1 and B_2 have the same meaning for a point b of β, it follows from §2, Lemma III, that $\alpha=A_1 \cdot A_2$ and $\beta=B_1 \cdot B_2$.

If both continua are decomposable, take $A=A_2$ and $B=B_1$, and set $H_1=A+C_1+B=A_2+C_1$ and $H_2=A+C_2+B=C_2+B_1$. Then $H_1 \cdot H_2=A+B=A_2+B_1$. If C_1 is indecomposable, take $A=A_2$ and $B=B_2$, and set $H_1=A+C_1+B=A_2+C_1+B_2$ and $H_2=A+C_2+B=C_2$. Again $H_1 \cdot H_2=A+B$.

In both cases $A \cdot B=0$, H_1 and H_2 are irreducible about $A+B$, and $F=H_1+H_2=C_1+C_2$. Hence §4 gives the theorem.

6. *The Secondary Regions.* Let R_1 and R_2 be the two principal regions, which have the frontier $F=C_1+C_2$, and let R' denote any one of the other components of $Z-F$, that is, one of the secondary regions defined by F if any exist.

Since R' is a component of $Z-F$, it is a part of a component of $Z-H_1$ and of a component of $Z-H_2$. It is not a part of both G_1 and G_2, for, as seen in §4, $G_1 \cdot G_2=R_1+R_2$. To fix the ideas let $R' \subseteq G'$, where G' is a component of $Z-H_1$ different from G_1. By definition of G' and G_1, $G' \cdot H_2=0$; hence $\overline{G}' \cdot F \subseteq H_1$. Then, if F' is the frontier of R', $F' \subseteq H_1$. Likewise, if R' is a part of some component of $Z-H_2$ different from G_2, $F' \subseteq H_2$.

Since $F' \subset F$ and $R_1 \cdot R'=0$ by hypothesis, it follows that, if m is a point of R' and n one of R_1, then F' is an $S(m, n)$. It is also an irreducible $S(m, n)$.* For, if K is a closed sub-set of F' and x is a point of $F'-K$, then for a sufficiently small positive δ there is a circular region U_δ containing x and points of both R' and R_1, but no point of K. Then m and n can be joined by a broken line not cutting K. Thus no proper closed part of F' is an $S(m, n)$.

If $F' \subseteq H_1$, but F' is not a part of C_1, there are two cases to consider. If C_1 is decomposable, it follows from the definition of H_1 (§5) that F' contains points of A_2 not in C_1. If $F' \cdot (C_1-A_1) \neq 0$, we have a contradiction. For then by §2, Lemmas II and III, $F' \supset A_1$. Then $F' \cdot C_1$ and $F' \cdot (A_1+A_2)$ have in common the continuum A_1. As both of these sets are proper closed parts of F', neither is an $S(m, n)$, which gives the contradiction that F' is not an $S(m, n)$. Thus we have $F' \cdot (C_1-A_1)=0$ and in this case $F' \subseteq A_1+A_2$.

If C_1 is indecomposable, then C_2 is decomposable and not the sum of two indecomposable continua. Then $A_2 \cdot B_2=0$ and the argument used in the previous paragraph shows

* A set P is called an irreducible $S(m, n)$ if no proper closed part of P sian $S(m, n)$.

that we arrive at a contradiction if we assume that F' contains points of both A_2 and B_2 not in C_1. As C_1 is indecomposable, $A_1=B_1=C_1$; hence either $F'\subseteq A_1+A_2$, or $F'\subseteq B_1+B_2$.

The same sort of discussion is employed if $F'\subseteq H_2$. Thus we have found that, *if R' is a secondary region defined by the continuum F of* §5, *then the frontier of R' is a part either of C_1, or of C_2, or of A_1+A_2, or of B_1+B_2.* For a closer determination of this frontier we need the following theorem.

7. THEOREM II. *Let C be a bounded plane continuum irreducible between the points a and b. Let m and n be two points not on C and let K be a closed sub-set of C which is an irreducible $S(m, n)$. Then K is a part or the whole of the oscillatory set of C about one of its points**.

PROOF. CASE I. *K is a continuum of condensation.* Let K_a and K_b be the saturated semi-continua of $C-K$ containing a and b, respectively, and let neither be void. Since K is a continuum of condensation, $C=\overline{K}_a+\overline{K}_b$. Then $K\cdot\overline{K}_a$ and $K\cdot\overline{K}_b$ have a common point c, which is a limiting point of both K_a and K_b. Then the reasoning in §13 of the paper mentioned in reference (C) shows that K is a part of the oscillatory set about c. Similar reasoning establishes the theorem for the case that either K_a or K_b is void.

CASE II. *K is not a continuum of condensation.* Let us assume that K contains neither a nor b. Then there are subcontinua A_k and B_k, irreducible between $\underline{K \text{ and } a \text{ and}}$ b, respectively, and $A_k\cdot B_k=0$.* Then $L=C-(A_k+B_k)$ is irreducible‡ between A_k and B_k, and $L\subseteq K$. We first show that L is indecomposable or is the union of two indecomposable continua. For, if not, there is an irreducible de-

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* If every oscillatory set of C is complete, the theorem can be deduced easily from Theorem 9 of a paper by R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

† See reference (F), Theorems II and IV.

‡ See reference (C), §4.

composition* of L into two proper sub-continua M and N such that

$$M \cdot B_k = N \cdot A_k = 0$$

and either M or N, say N, is decomposable. Obviously N is irreducible between $A_k + M$ and B_k and there is an irreducible decomposition of N into two proper sub-continua P and Q such that

$$P \cdot B_k = (A_k + M) \cdot Q = 0.$$

Then we have

$$C = (A_k + M + P) + (P + Q + B_k),$$
$$(A_k + M + P) \cdot (P + Q + B_k) = P.$$

Since $K \supseteq L$,

$$K = K(A_k + M + P) + K(P + Q + B_k)$$

is a decomposition of K into two closed proper sub-sets whose common part is a continuum. This is a contradiction, by reference (H). There are, then, two sub-cases to discuss: (*a*) when L is the union of two indecomposable continua; (*b*) when L is indecomposable.

(*a*) Let $L = M + N$, where M and N are indecomposable and $M \cdot B_k = N \cdot A_k = 0$. Reasoning similar to the above shows that in this case K contains no points not on L. Let x be a point of $M \cdot N$; then it is not a point of A_k or B_k. It is easily seen that $A_k + M$ is irreducible between a and x, and $N + B_k$ between x and b. Then, since M and N are indecomposable, M and N are the oscillatory sets of $A_k + M$ and $N + B_k$ respectively about x. Hence

$$K = L = M + N$$

is the oscillatory set of C about x.†

(*b*) Let L be indecomposable. From the definition of oscillatory sets it is evident that L is the oscillatory set of C about any point of $C - (A_k + B_k)$; hence the theorem holds if $K = L$. If $K \neq L$, it follows by reasoning similar to

* See reference (D), p. 156.

† See reference (D), p. 153.

that used above that either $(K-L) \cdot A_k$ or $(K-L) \cdot B_k$ is void, say the latter. Then $K \subseteq A_k+L$. Let y be a point of $A_k \cdot L$; then A_k is irreducible between a and y, and $L+B_k$ is irreducible between y and b. Let Y' be the oscillatory set of A_k about y. Since y does not lie on B_k, the oscillatory set of $L+B_k$ about y is L. Then $Y=Y'+L$ is the oscillatory set of C about y.*

If $K \subseteq Y$, the theorem is proved; if not, we arrive at a contradiction. For K will contain a point z on A_k-Y'. Then, if Y_a' is the saturated semi-continuum of A_k-Y' containing a, $\overline{Y}_a' \supset A_k-Y' \supset z$.† Then $\overline{Y}_a' = A_k$, since A_k is irreducible between a and K. Hence Y' is complete‡ and z is a point of Y_a'. As K joins Y_a' and B_k, this shows that $K \supset Y'$. Then $K \cdot A_k$ and $Y'+L$ are closed proper parts of K and their divisor is the continuum Y'. This is a contradiction, as their union is K, which is an irreducible $S(m, n)$.

Thus the proof is complete, except for the special cases where K contains a or b, or both. The above demonstration holds for these cases, if we merely replace A_k by a if K contains a and B_k by b if K contains b.

8. THEOREM III. *Let F be the union of two bounded continua C_1 and C_2 having these properties: $C_1 \cdot C_2=\alpha+\beta$, where α and β are closed and $\alpha \cdot \beta=0$; both C_1 and C_2 are irreducible between α and β; and either C_1 and C_2 are both decomposable, or C_1 is indecomposable and C_2 is decomposable and is not the union of two indecomposable continua. Then the frontier of each secondary region determined by F is a part or the whole of some oscillatory set of C_1 or C_2, or it is a part or the whole of the union of the oscillatory sets of C_1 and C_2 about some point of α or β.*

PROOF. Let R' be any secondary region and let F' be its frontier. Let A_1, A_2, B_1, and B_2 be the oscillatory sets of C_1

* See referrence (D), p. 153.

† See reference (C), §10.

‡ See reference (C), §15.

and C_2 about a point a of α and a point b of β, respectively.

It was shown in §6 that, if F' is not a part of A_1+A_2 or B_1+B_2, then F' is a part of C_1 or C_2. Since C_1 and C_2 are each irreducible between α and β, each of them is irreducible between a point a of α and a point b of β. Hence, if $F' \subseteq C_1$, the theorem of §7 shows that F' is a part of some oscillatory set of C_1. Thus the theorem is proved.

9. *Conclusion.* A consequence of the previous theorem is that under the hypotheses there stated the frontier of a secondary region is a part of either a continuum of condensation, an indecomposable continuum, a pair of indecomposable continua, or the union of a continuum of condensation and an indecomposable continuum. This follows at once from the proof of §7 and the fact that the oscillatory set of a bounded irreducible continuum about one of its end points is either a continuum of condensation or an indecomposable continuum.* Obviously there are no secondary regions unless A_1+A_2, or B_1+B_2, or some oscillatory set of C_1 or C_2 cuts the plane.

The conditions imposed on the character of the continua C_1 and C_2 in §5 take care of all possible cases with two exceptions. These are that both continua are indecomposable or that one is indecomposable and the other is the union of two indecomposable continua. With regard to the first it has been shown by Kuratowski† and Knaster‡ that the extended theorem of Rosenthal (§5) need not hold in this case. Whether or not it holds in the second case remains to be proved.

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* See reference (D), p. 155.

† C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6, p. 138.

‡ B. Knaster, *Quelques coupures singulières du plan*, Fundamenta Mathematicae, vol. 7, p. 281.