ALL POSITIVE INTEGERS ARE SUMS OF VALUES OF A QUADRATIC FUNCTION OF x*

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1. *Introduction.* Fermat stated that he was the first to discover the beautiful theorem that every integer $A \geqq 0$ is a sum of $m+2$ *polygonal* numbers

$$p_{m+2}(x) = \tfrac{1}{2}m(x^2 - x) + x \tag{1}$$

of *order* $m+2$ (or $m+2$ sides), where x is an integer $\geqq 0$. The cases $m=1$ and $m=2$ state that every A is a sum of three *triangular* numbers $p_3(x) = \frac{1}{2}x(x+1)$, and also a sum of four squares $p_4(x) = x^2$.

Cauchy† was the first to publish a proof of Fermat's statement and showed that all but four of the polygonal numbers may be taken to be 0 or 1.

In this paper and its sequel we shall give a complete solution of the following more general question.

PROBLEM. Find every quadratic function $f(x)$ which takes integral values $\geqq 0$ for all integers $x \geqq 0$, such that every positive integer A is a sum of l of these values, where l depends on $f(x)$, but not on A.

2. LEMMA 1. *A quadratic function of x is an integer for every integer $x \geqq 0$ if and only if it is of the form*

$$f(x) = \tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + c, \quad m + n \ \textit{even}, \tag{2}$$

where m, n, and c are integers.

Consider ux^2+vx+c. By its values for $x=0$, 1, and 2, c, $u+v$, and $4u+2v$ are integers. Subtract the double of the

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† Oeuvres, (2), vol. 6, pp. 320–353. Pepin gave a modified proof, Atti dei Lincei, vol. 46 (1892–93), pp. 119–131. His proof requires a separate examination when $A < 110m$. For the simpler proof in §5, the limit is $A < 44m+32$.

second from the third. Hence $2u$ is an integer m. Since $\frac{1}{2}m+v$ is an integer, v is half an integer n.

3. *Positive Quadratic Functions Representing* 0 *and* 1. Let (2) be $\geqq 0$ for every integer $x \geqq 0$, whence $m>0$, $c \geqq 0$. Then

$$f(x+1)-f(x)=mx+\tfrac{1}{2}(m+n)$$

increases with x. Hence $f(x)$ does not represent every positive integer A. Thus $l>1$ in our problem, and a sum of two or more values of $f(x)$ must give $A=1$. Hence $f(u)=1$, $f(k)=0$ for certain integers $u \geqq 0$, $k \geqq 0$. We assume that k has its least value. Then

$$f(x)=f(x)-f(k)=\tfrac{1}{2}(x-k)[m(x+k)+n].$$

Since $f(u)=1$,

$$n=s-m(u+k), \quad s=2/(u-k),$$

where s is an integer. Thus $u-k=\pm 1$ or ± 2, and

$$f(x)=\tfrac{1}{2}(x-k)[m(x-u)+s].$$

If $u-k=\pm 1$,

$$f(x)=\tfrac{1}{2}(x-k)[m(x-k\mp 1)\pm 2]=p_{m+2}(\pm x\mp k).$$

If $u-k=2$,

$$f(k+1)=\tfrac{1}{2}(1-m)\geqq 0 \text{ gives } m=1,\ f(x)=p_3(x-k-1).$$

Finally, if $u-k=-2$, then $k \geqq 2$ and $f(k-1)=\frac{1}{2}(1-m)$ is zero, since it is not negative. But this contradicts the definition of k as least.

THEOREM 1. *The functions derived from* (1) *by replacing* x *by* $x-k$ *or* $k-x$ *are the only quadratic functions of* x *which are integers* $\geqq 0$ *for every integer* $x \geqq 0$, *and which take the values* 0 *and* 1 *for certain integers* $x \geqq 0$.

The values of $p_3(-x)=\frac{1}{2}(x-1)x$ coincide with the triangular numbers. Hence if $m=1$, our problem is the same when $k \geqq 1$ as when $k=0$. This is evidently true also if $m=2$. Without loss of generality, we may then henceforth take $m>2$.

4. *Polynomials with an Excess.* Let $f(x)$ have an integral value $\geqq 0$ for every integer $x \geqq 0$, and let one value be zero. Let $M_s(A)$ denote the maximum sum $\leqq A$ of s values of $f(x)$, and write $E_s(A)$ for $A - M_s(A)$. In case $E_s(A)$ has a finite maximum E_s for all integers $A \geqq 0$, every integer $A \geqq 0$ is a sum of E_s numbers 0 or 1 and s values of $f(x)$. Then E_s is called the s-excess of $f(x)$. We shall drop the subscript 4 from E_4.

Let α, β, γ, δ denote the four integral values $\geqq 0$ of x and write

$$(3) \qquad a = \Sigma\alpha^2, \qquad b = \Sigma\alpha.$$

Take (2) as $f(x)$ and insert the four values of x. Thus

$$(4) \qquad A = \tfrac{1}{2}ma + \tfrac{1}{2}nb + 4c + r, \qquad 0 \leqq r \leqq E,$$

for a suitable integer r. Cauchy proved the following result.

LEMMA 2. *If a and b are positive odd integers such that $b^2 < 4a$ and*

$$(5) \qquad b^2 + 2b + 4 > 3a,$$

equations (3) *have solutions α, β, γ, δ in integers* $\geqq 0$.

Multiply (5) by m and replace ma by its value from (4). The resulting inequality follows from that obtained by suppressing $6r$. Multiplication by $4m$ now yields the equivalent inequality

$$(6) \qquad (2mb + \tau)^2 > U, \quad U = \tau^2 + 4m(6A - 24c - 4m),$$

where $\tau = 2m + 3n$. This inequality holds if

$$(7) \qquad b > (U^{1/2} - \tau)/(2m), \quad U \geqq 0.$$

To satisfy $b^2 < 4a$, multiply by m^2 and replace ma by its value from (4). The resulting condition evidently follows from that with r replaced by E, and hence from

$$(8) \qquad (mb + 2n)^2 < 4V, \quad V = n^2 + m(2A - 8c - 2E).$$

This inequality holds if

$$(9) \qquad b < (2V^{1/2} - \quad)/m, \quad V \geqq 0,$$

and if $mb+2n\geqq 0$. For $n\geqq 0$, the latter evidently holds if $b>0$. For $n<0$, it holds by (7) if

$$(10)\qquad 4n-\tau+U^{1/2}\geqq 0$$

and hence if

$$(11)\qquad 3A\geqq 12c+2m-2n-n^2/m\quad (\text{if } n<0).$$

We desire that $b>0$. By (7), this will be true if $U^{1/2}\geqq\tau$. If $n<0$, this follows from (10). But if $n\geqq 0$, whence $\tau>0$, it holds if and only if $U\geqq\tau^2$, and hence if the quantity in the last parenthesis of (6) is $\geqq 0$:

$$(12)\qquad A\geqq 4c+\tfrac{2}{3}m\quad (\text{if } n\geqq 0).$$

There will be at least d positive integers between the limits on b stated in (7) and (9) if

$$(13)\qquad 4V^{1/2}-U^{1/2}>P,\quad P=2md-2m+n.$$

The left member is $\geqq 0$ if

$$(14)\qquad 16V\geqq U,$$

and then (13) holds* if its square holds and hence if

$$(15)\quad F\equiv(2V+W)^2-VU>0,\quad 8W\equiv U-P^2,\ P\geqq 0.$$

By the *minor conditions* we shall mean $U\geqq 0$, $V\geqq 0$, the inequality (14), and (11) or (12). Since

$$(16)\quad 16V-U=3(2m-n)^2+4n^2+8m(A-4c-4E),$$

it follows that (14) holds if

$$(17)\qquad A\geqq 4c+4E.$$

The latter implies $V\geqq 0$. Evidently $U\geqq 0$ if

$$(18)\qquad A\geqq 4c+\tfrac{2}{3}m.$$

Hence the minor conditions all follow from (17) and (18) if $n\geqq 0$, but from these two and (11) if $n<0$. We shall speak of these as the *reduced* minor conditions.

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* Automatically if $P<0$.

5. *Polygonal Numbers.* We take (1) as the function (2). Thus $n=2-m$, $c=0$. By Table I, $E(2m+3)=m-2$. Hence E is not smaller than the value in

THEOREM 2. *For the function* (1), $E_4=m-2$ *if* $m\geqq 3$.

The reduced minor conditions are all satisfied if $A\geqq 4m$. Here (4) is $A=mg+b+r$, $g=\frac{1}{2}(a-b)$. If b takes the odd values β and $\beta+2$, while r takes the values $0, 1, \cdots, m-2$, the values of $b+r$ are $\beta+j(j=0, 1, \cdots, m)$. These, with $j=m$ omitted, form a complete set of residues modulo m. Hence for any A, the preceding equation yields an integral value of g and hence an odd integral value of a.

If there are at least $d=4$ integers between the limits for b, there will exist the desired two odd values for b. Then (6), (8), (13), and (15) give

$$U = 24mA - 15m^2 - 12m + 36, \quad V = 2mA - m^2 + 4,$$

$$P = 5m + 2, \quad W = 3mA - 5m^2 - 4m + 4,$$

$$F = m^2A^2 - 44m^3A - 32m^2A + 34m^4 + 44m^3 - 56m^2 - 48m > 0.$$

Evidently $F>0$ if $A\geqq 44m+32$.

Next, let $A<44m+32$. Then $A<M\equiv 48m+21$. In Table I the entries involving the same multiple of m, together with all intervening integers, will be said to form a *block*. We suppress $29m+10-12$ and $45m+10-13$. Down to M, the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, whence $E(A)\leqq 1$ for every A within a block. For every $A<M$ not within a block, $E(A)\leqq m-2$. This will follow if proved when $A+1$ is the first number of any abridged block. Then A is the sum of $m-2$ and a number t occurring explicitly in the abridged table except as follows. If $A=10m+4$, $26m+12$, $27m+10$, or $28m+12$, then $A=m-3+t$. If $A=6m+3$, $21m+6$, $28m+7$, $30m+11$, $34m+11$, or $42m+12$, then $A=m-4+t$; while, if $m=3$, A is equal to the A for the next smaller m.

TABLE I.

Sums of Four Polygonal Numbers

0–4, m+2–5, 2m+4–6, 3m+3–7, 4m+5–8, 5m+7–8, 6m+4–9, 7m+6–9, 8m+8–10, 9m+7–10, 10m+5–11, 11m+7–9, 11, 12m+8–12, 13m+8–12, 14m+10–12, 15m+6–9, 11–13, 16m+8–13, 17m+10–13, 18m+9–14, 19m+11–14, 20m+10–14, 21m+7–13, 15, 22m+9–15, 23m+11–15, 24m+10–16, 25m+11–13, 15, 16, 26m+13–16, 27m+11–16, 28m+8–11, 13–17, 29m+10–12, 15–17, 30m+12–17, 31m+11–17, 32m+13–18, 33m+15–18, 34m+12–18, 35m+14–17, 36m+9–19, 37m+11–13, 15–19, 38m+13–15, 17–19, 39m+12–17, 19, 40m+14–20, 41m+16–20, 42m+13–20, 43m+14–17, 19, 20, 44m+16–20, 45m+10–13, 16–21, 46m+12–21, 47m+14–17, 19–21, 48m+13–21, 49m+15–21, 50m+17–22, 51m+14–17, 19–22, 52m+16–22, 53m+18–21, 54m+17–19, 21, 22, 55m+11–17, 19–23, 56m+13–23, 57m+15–21, 23, 58m+14–23, 59m+16, 17, 19–23, 60m+16–24, 61m+15–21, 23, 24, 62m+17–24, 63m+19–24, 64m+17–24, 65m+16–21, 23, 24, 66m+12–15, 17–25, 67m+14–16, 19–25, 68m+16, 17, 19–25, 69m+15–18, 20, 21, 23–25, 70m+17–19, 21–25, 71m+19–25, 72m+16–26, 73m+18–21, 23–26, 74m+20–23, 25, 26, 75m+19–25, 76m+17–26, 77m+19–21, 23–26, 78m+13–16, 20–27, 79m+15–17, 20–25, 27, 80m+17, 18, 22–27, 81m+16–21, 23–27, 82m+18–27, 83m+19–25, 27, 84m+17–28, 85m+19–21, 23–28, 86m+21–28, 87m+19–25, 27, 28, 88m+18–22, 24–28, 89m+20, 21, 23–28, 90m+20–23, 25–28, 91m+14–17, 20–25, 27–29, 92m+16–18, 22–29, 93m+18–21, 23–29, 94m+17–29, 95m+19, 20, 22–25 27–29, 96m+21–29, 97m+18–21, 23–29, 98m+20–27, 29, 30, 99m+20–25, 27–30, 100m+21–30, 101m+19–21, 23–29, 102m+21–30, 103m+22–25, 27–30, 104m+22–30, 105m+15–18, 24–29, 31, 106m+17–23, 25–31, 107m+19, 20, 22–25, 27–31, 108m+18–21, 24–31, 109m+20, 21, 23–29, 31, 110m+22–31, 111m+19–25, 27–31, 112m+21–32, 113m+23–29, 31, 32, 114m+22–27, 29–32, 115m+20–22, 24, 25, 27–32, 116m+22, 23, 25–32, 117m+23–29, 31, 32, 118m+23–32, 119m+22–25, 27–32, 120m+16–19, 21–33, 121m+18–20, 23–29, 31–33, 122m+20, 21, 25–33, 123m+19–25, 27–33, 124m+21, 22, 25–33, 125m+23, 25–29, 31–33, 126m+20–31, 33, 127m+22–25, 27–33, 128m+24–26, 28–34, 129m+23–29, 31–34, 130m+21–23, 25–27, 29–34, 131m+23, 24, 28–33, 132m+24–34, 133m+23–29, 31–34, 134m+25–34, 135m+22–24, 27–33, 136m+17–20, 24–35, 137m+19–21, 26–29, 31–35, 138m+21, 22, 25, 26, 28–35, 139m+20–23, 27–33, 35, 140m+22, 23, 26, 27, 29–35, 141m+23–29, 31–35, 142m+21–31, 33–35, 143m+23–25, 27–33, 35, 144m+25–36, 145m+24–29, 31–36, 146m+22–27, 29–36, 147m+24, 25, 27–33, 35, 36, 148m+24–27, 29–36, 149m+25–29, 31–36, 150m+25–36, 151m+23–25, 27–33, 35, 36, 152m+25–36, 153m+18–21, 27–29, 31–37, 154m+20–22, 26–37, 155m+22, 23, 28–33, 35–37, 156m+21–37, 157m+23–29, 31–37, 158m+25–30, 33–35, 37, 159m+22–25, 28–33, 35–37, 160m+24–37, 161m+26, 28, 29, 31–37, 162m+25–27, 29–38, 163m+23–25, 27–33, 35–38, 164m+25–27, 30–38, 165m+26–28, 31–37, 166m+26–38, 167m+28–33, 35–38, 168m+24–26, 29–31, 33–38, 169m+26–29, 31–37, 170m+28–38, 171m+19–22, 27–33, 35–39, 172m+21–23, 26–39, 173m+23, 24, 28, 29, 31–37, 39, 174m+22–31, 33–35, 37–39, 175m+24, 25, 27–33, 35–39, $176m+26$, 29–39, $177m+23$–26, 28, 29, 31–37, 39, $178m+25$–27, 29–39, $179m+27$, 31–33, 35–39, $180m+26$–40, $181m+24$–29, 31–37, 39, 40, $182m+26$–40, $183m+27$–33, 35–40, $184m+27$–40, $185m+29$, 31–37, 39, 40, $186m+25$–40, $187m+27$–33, 35–40, $188m+29$, 30, 32–40, $189m+27$–29, 31–37, 39, 40, $190m+20$–23, 29–31, 33–35, 37–41, $191m+22$–24, 28–33, 35–41, $192m+24$–41, $193m+23$–26, 28, 29, 31–37, 39–41, $194m+25$, 26, 30–32, 34–39, 41, $195m+27$, 29–33, 35–41, $196m+24$–27, 29–41, $197m+26$–28, 31–37, 39–41, $198m+28$–41, $199m+27$, 28, 30–32, 35, 36.

If $A=15m+5$ or $46m+11$, then $A=m-5+t$; while, if $m=3$ or 4, A is $\leqq$ the A for the next smaller m. Finally, if $A=36m+8$, then $A=m-6+t$; while, if $m=3$, 4, or 5, $A=34m+14$, 16, or 18, which belong to an earlier block. This completes the proof of Theorem 2.

By that theorem, every integer $A\geqq 0$ is a sum of $m+2$ polygonal numbers. Hence $E_s=0$ if $s\geqq m+2$. Next, let $4\leqq s<m+2$. If a sum by s of the polygonal numbers $0, 1, m+2, 3m+3, \cdots$ is $\leqq 2m+3$, at most one summand is $m+2$, whence the maximum such sum is $m+2+s-1$. Hence $E_s(2m+3)=m-s+2$. By Theorem 2, A is a sum of four polygonal numbers and $m-2$ numbers 0 or 1. Regard $s-4$ of the latter as polygonal numbers. Hence A is a sum of s polygonal numbers and $m-s+2$ numbers 0 or 1. All of these facts prove the following theorem.

THEOREM 3. *For the function* (1), $E_s=0$ *if* $s\geqq m+2$, *while* $E_s=m-s+2$ *if* $4\leqq s\leqq m+2$.

In the second case, $s+E_s=m+2$, so that the use of 5 or more polygonal numbers >1 yields no gain (but rather a loss) over the use of only four.

6. *Deductions from Table* I. We extended our table beyond the limit $48m+21$ required for the proof of Theorem 2 in order to deduce interesting facts concerning $E(A)$ for function (1), which are essential to the sequel.

LEMMA 3. *For* $54m+17\leqq A<74m+28$, $E(A)\leqq m-6$ *if* $m\geqq 7$, $E(A)\leqq 1$ *if* $m=5$ *or* 6.

From Table I we suppress $67m+14$-16. Since the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, $E(A)\leqq 1$, for every A within a block. Let f be the term free of m in the leader $qm+f$ of any abridged block. First, let $m=5$. For $q=56, \cdots, 74$, we find that $f+4$ is the term free of m in a number of the preceding abridged block. Hence $qm+f-1$ is the sum of $m-5$ and the number $(q-1)m+f+4$ in the abridged table. Finally, the E of $55m+10=53m+20$ is 1. Second, let $m\geqq 6$. Except for $q=55, 56, 62, 70$, $f+5$ is the term free of m in a number of the preceding abridged block, whence $E(A)\leqq m-6$. For the four q's, we use $f+6$ if $m>6$. If $m=6$, $56m+12=55m+18$, $62m+16=61m+22$, $70m+16=69m+22$, which fall within preceding blocks, while $55m+1<54m+17$.

LEMMA 4. *For* $74m+20\leqq A\leqq 199m+37$, $E(A)\leqq 1$ *if* $m=7$, $E(A)\leqq m-7$ *if* $m\geqq 8$, *except that* $E(80m+21)=m-6$ *if* $m=8$ *or* 9.

From each block we suppress all entries down to and including the last entry which differs by 3 or more from the next entry. As in Lemma 3, $f+6$ succeeds except for $q=80$, 106, 156, 158, 169, 195. For $m=7$, $a=80m+21$ equals $79m+28$, whose E is 1. For $m\geqq 10$, we restore the previously excluded $80m+17$-18 and then have the permissible value $E(a)=3$ and $80m+16=m-7+79m+23$. But for $m=8$ or 9, $80m+18\leqq 79m+27=a'$, and the full table includes no number numerically between a' and a, whence $E(a)=m-6$.

For $q=106$, 169, 195, we may use $f+6$ after suppressing $106m+17$-23, $169m+26$-29, and $195m+27$.

For $m\geqq 8$, $b=156m+20=m-8+155m+28$. But if $m=7$, $b=155m+27$, which was treated under $q=155$.

Finally, let $q=158$. If $m\geqq 9$ we restore the previously excluded $158m+25$-30, noting that $E(158m+32)=2$ is a permissible value. Then $158m+24=m-7+t$, where $t=157m+31$ is in the table. If $m=7$ or 8, the latter result is applicable since the missing $158m+31$-32 are equal to two of $159m+23$-25.

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