# ON THE CONVERGENCE OF TRIGONOMETRIC APPROXIMATIONS FOR A FUNCTION OF TWO VARIABLES*

BY ELIZABETH CARLSON

A discussion of the convergence of approximating functions for a given function of two variables $f(x, y)$, just as in the case of functions of one variable, can be based on two sets of theorems: (1) theorems on the existence of functions of closest approximation; (2) theorems on the representation of $f(x, y)$ by means of finite sums constructed in a specific way.

From the first group we shall make use of the following theorem:

THEOREM I. *Let $p_1(x,y)$, $p_2(x\ y)$, $\cdots$, $p_N(x,y)$ be $N$ functions of $x$ and $y$, continuous in the region $R$: $(a \leqq x \leqq b,$ $c \leqq y \leqq d)$, and linearly independent in this region. Let*

$$\phi(x,y) = c_1 p_1(x,y) + c_2 p_2(x,y) + \cdots + c_N p_N(x,y)$$

*be an arbitrary linear combination of the given functions with constant coefficients. Let $f(x, y)$ be a continuous function of $x$ and $y$ in $R$. Then there exists a choice of the coefficients $c_k$ in $\phi(x, y)$ such that the integral*

$$\int_a^b dx \int_c^d |f(x,y) - \phi(x,y)|^m dy, \qquad (m > 0),$$

*has its minimum value. The function $\phi(x, y)$ so determined is unique for $m > 1$. It is called an approximating function for $f(x, y)$ corresponding to the exponent $m$.*

This theorem can be proved by methods analogous to those used in proving the corresponding theorem for functions of a single variable.† In this paper we shall choose

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* Presented to the Society, April 14, 1922.

† Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128; *Note on an ambiguous case of approximation*, TRANSACTIONS, vol. 25 (1923), pp. 333–337.

the $p$'s so that a function of the form $\phi(x, y)$ with arbitrary coefficients is an arbitrary trigonometric sum of order $n$ in $x$ and of the same order in $y$, and we shall take as the region $R$ the square $0 \leqq x \leqq 2\pi$, $0 \leqq y \leqq 2\pi$.

From the second group of theorems, we shall apply the following theorems.

THEOREM II. *If $f(x, y)$, of period $2\pi$ in each argument, has a modulus of continuity $\omega(\delta)$, then $f(x, y)$ can be represented everywhere by a trigonometric sum $t_n(x, y)$, of order $n$ in each variable, with an error that does not exceed a constant times $\omega(2\pi/n)$.*

THEOREM III. *If $f(x, y)$ has continuous first partial derivatives, it can be approximately represented with a maximum error $\epsilon_n$ such that* $\lim_{n\to\infty} n\epsilon_n = 0$.

The assertion II can be proved once more by an adaptation of the reasoning used in the case of functions of a single variable.* A proof of III was communicated to the writer orally by Professor Jackson.

Theorems I and II lead to the following theorem for the convergence of $T_n(x, y)$, the trigonometric approximating function, of order $n$ in each variable, corresponding to the exponent $m$ $(m > 1)$.

THEOREM IV. *If $f(x, y)$, of period $2\pi$ in each argument, is everywhere continuous, with a modulus of continuity $\omega(\delta)$, then a sufficient condition for the uniform convergence of $T_n(x, y)$ to $f(x, y)$ is*†

$$\lim_{\delta \to 0} \omega(\delta)/\delta^{2/m} = 0.$$

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* Jackson, *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), pp. 491–515. Cf. C. E. Wilder, *On the degree of approximation to discontinuous functions by trigonometric sums*, RENDICONTI DI PALERMO, vol. 39 (1915), pp. 345–361.

† Cf. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 158–166.

The occurrence of the exponent $2/m$, where $1/m$ is found in the case of functions of a single variable, is due to the fact that the magnitude of an interval of length $1/n$ is replaced in the course of the present reasoning by the *area* of a square having a quantity of the order of $1/n$ for the length of its side.

This result is significant only if $m>2$, since otherwise the condition as stated requires that $f(x, y)$ be constant. *When* $m=2$, *it is sufficient* (in consequence of III) *that* $f(x, y)$ *have continuous first partial derivatives.* There are corresponding results for $m<2$.

The reasoning is not materially changed if the $m$th power in the integral to be minimized is multiplied by a positive measurable weight function $\rho(x, y)$ with a positive lower bound.

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# ON A TENSOR OF THE SECOND RANK IN FUNCTION SPACE*

BY DUNHAM JACKSON

In a recent paper† the writer has discussed a doubly infinite matrix of derivatives which has the properties of a tensor of the second rank in function space, and a quantity obtained by contraction of this tensor, which is analogous to a divergence. The latter concept is suggested formally by the writing of an infinite series, the general term of which in many cases does not approach zero; but an example was constructed in which the series is convergent, and defines a quantity which can be alternatively expressed in a form independent of any particular coordinate system.

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* Presented to the Society, September 9, 1926.

† D. Jackson, *Some convergence proofs in the vector analysis of function space*, ANNALS OF MATHEMATICS, vol. 27, pp. 551–567.