A CHARACTERISTIC PROPERTY OF MINIMAL SURFACES*

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1. *Statement of Theorems.* By the mean curvature of a surface at a point we shall understand $1/R_1+1/R_2$, where $1/R_1$, $1/R_2$ are the principal curvatures at the point. There is some divergence in the literature with respect to this nomenclature; some authors call $(1/R_1+1/R_2)/2$ the mean curvature.‡

Imagine that about any point O of an arbitrary surface Σ as center we describe a sphere S of infinitesimal radius r. Then S is intersected by Σ in a curve C infinitely close§ to the great circle cut from S by the tangent plane to Σ at O. The curve C divides the surface of S into two areas nearly equal to each other. Also, the portion of Σ lying within S divides S into two nearly equal volumes.

The present paper is devoted to the proofs of the following three theorems.

THEOREM I. *Let the difference between the two areas into which C divides S be denoted by η. Then, in the limit as $r\to 0$,*

$$\frac{\eta}{V}=\frac{3}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

where V denotes the volume of S.

THEOREM II. *Let v denote the difference of the two volumes into which S is divided by Σ. Then, in the limit as $r\to 0$,*

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‡ Bianchi, Eisenhart, and Scheffers use the first notation; Blaschke and Darboux the second.

§ Even in comparison with the radius of the sphere, that is, the deviation of C from the great circle is an infinitesimal of the second or higher order.

$$\frac{v}{rV} = \frac{3}{16}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

When Σ is a minimal surface, or one of mean curvature zero, these theorems yield the following corollary.

THEOREM III. *A minimal surface bisects the area and volume of an infinitesimal sphere about any point of the surface as center.*

The sense of this statement is that the bisection occurs to infinitesimals of higher order for a minimal surface than for a non-minimal surface.

2. *Proof of Theorem* I. The proof of these theorems is a straightforward problem in calculus. We choose for axes the tangents to the lines of curvature at O and the normal to the surface. Then, for all our purposes, the surface may be replaced by the paraboloid

$$z = ax^2 + by^2 \tag{1}$$

having second order contact with it at O. Here

$$a = \frac{1}{2R_1}, \quad b = \frac{1}{2R_2} \cdot \tag{2}$$

The equation of the sphere S is

$$x^2 + y^2 + z^2 = r^2. \tag{3}$$

We use cylindrical coordinates ρ, θ, z, in terms of which the equations of Σ, S are respectively

$$z = \rho^2(a\cos^2\theta + b\sin^2\theta), \tag{4}$$

$$\rho^2 + z^2 = r^2. \tag{5}$$

The polar equation of the projection C_1 of C on the tangent plane at O is

$$\rho_1^2 + \rho_1^4(a\cos^2\theta + b\sin^2\theta)^2 = r^2.$$

Expanding ρ_1 in powers of r, we need retain only the first two terms, and have

$$\rho_1 = r - \tfrac{1}{2}r^3(a\cos^2\theta + b\sin^2\theta)^2 \tag{6}$$

for the (approximate) polar equation of C_1. The area of the portion of the sphere above C is

$$A_1 = 4\int_0^{\pi/2}\int_0^{\rho_1} \frac{r}{(r^2-\rho^2)^{1/2}}\,\rho d\rho d\theta.$$

Performing the first integration, we get

$$A_1 = 4r\int_0^{\pi/2} \left\{ -(r^2-\rho^2)^{1/2}\right\}_0^{\rho_1} d\theta.$$

Now we have

$$\left\{ -(r^2-\rho^2)^{1/2}\right\}_0^{\rho_1} = r - r^2(a\cos^2\theta + b\sin^2\theta)$$

where, as will be our rule hereafter, we neglect all infinitesimals beyond those which would influence the final result. Performing the second integration we find

$$A_1 = 2\pi r^2 - \pi r^3(a+b).$$

The area of the portion of the sphere to the other side of C is the complement of A_1 as to $4\pi r^2$, or

$$A_2 = 2\pi r^2 + \pi r^3(a+b).$$

The difference

$$\eta = A_2 - A_1 = 2\pi r^3(a+b) = \pi r^3\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

by (2). Introducing $V=\frac{4}{3}\pi r^3$, we have

$$\frac{\eta}{V} = \frac{3}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

or Theorem I.

3. *Proof of Theorem* II. We first obtain the volume V_1 above Σ and below the sphere by calculating the difference $V_1 = V_S - V_\Sigma$, where

V_S = volume under portion of sphere above C,

V_Σ = volume under portion of Σ within C.

The volume V_S turns out to be that of a hemisphere up to infinitesimals of the sixth order. For, with the use of (5) and (6),

$$\begin{aligned} V_S &= 4\int_0^{\pi/2}\int_0^{\rho_1}(r^2-\rho^2)^{1/2}\rho d\rho d\theta \\ &= 4\int_0^{\pi/2}\tfrac{1}{3}\left\{r^3-(r^2-\rho_1^2)^{3/2}\right\}d\theta \\ &= \frac{4}{3}\int_0^{\pi/2}\left\{r^3-r^6(a\cos^2\theta+b\sin^2\theta)^3\right\}d\theta = \frac{2}{3}\pi r^3, \end{aligned}$$

neglecting the term in r^6.

$$V_\Sigma = \iint_{\Sigma_1} z\rho d\rho d\theta$$

where Σ_1 is the area enclosed by C_1, the projection of C. By (4),

$$\begin{aligned} V_\Sigma &= 4\int_0^{\pi/2}\int_0^{\rho_1}\rho^3(a\cos^2\theta+b\sin^2\theta)d\rho d\theta \\ &= \int_0^{\pi/2}\rho_1{}^4(a\cos^2\theta+b\sin^2\theta)d\theta. \end{aligned}$$

By (6), $\rho_1^4=r^4$ up to terms in r^6 and higher powers. Therefore $V_\Sigma \pi r^4(a+b)/4$. It follows that

$$V_1 = \frac{2}{3}\pi r^3 - \frac{\pi}{4}r^4(a+b).$$

The complementary volume of V_1 as to the sphere is

$$V_2 = \frac{2}{3}\pi r^3 + \frac{\pi}{4}r^4(a+b).$$

Dividing the difference $v=V_2-V_1$ by $rV=4\pi r^4/3$, we obtain

$$\frac{v}{rV} = \frac{3}{8}(a+b) = \frac{3}{16}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

that is, Theorem II.

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