KLEIN'S COLLECTED PAPERS, VOLUME III

Felix Klein: *Gesammelte mathematische Abhandlungen. Vol.* III: *Elliptische Funktionen, insbesondere Modulfunktionen*; *Hyperelliptische und Abelsche Funktionen*; *Riemannsche Funktionentheorie und automorphe Funktionen.* Edited by R. Fricke, H. Vermeil, and E. Bessel-Hagen. Berlin, Julius Springer, 1923. IX + 774 + 36 pp.

The editors of the third volume are three in number; those of the second having continued, and been joined by the labors of a third collaborator. The plan of the preceding volumes has been carefully preserved. To give some idea of the care with which the work of the editors has been done, it may be mentioned that every formula has been rederived, and in cases of any discrepancies in notation or in result, these are fully explained in appropriate footnotes. The volumes are practically free of typographical errors.

The memoirs in the first part, mentioned under the first subtitle, are fourteen in number, being memoirs LXXXI to XCIV inclusive. Of these, the first eight were written during the period 1877–1880, while the author was in Munich; the next four, one from 1881, the other three in 1885, were written in Leipzig, and the last two, 1893, 1896, in Göttingen. This first part occupies 314 pages.

When Klein went to the technical school in Munich from Erlangen, in 1875, his main interest was in the use of elliptic functions in the solution of the quintic equation; in particular, what part the theory of the icosahedron could play in such solution. His courses in the technical school included a repeated cycle in the theory of numbers, elliptic functions, and algebraic equations. In the seminary he had such men as Pick, Gierster, Dyck, Fricke, Hurwitz, and for one semester, Bianchi. Not only did these men contribute material assistance at the time, but except the first and last, wrote their dissertations under Klein's direction, and have very considerably enriched the literature by their own contributions.

The first memoir is an extract of a letter, in Italian, to Professor Brioschi, showing how certain formulas used by Klein could be reduced to those previously used by Kronecker, Brioschi and Hermite. The second, on the transformation of elliptic functions and the solution of the quintic equation, contains the systematic development of the method, and the result, in abstract form, that the general solution of the quintic equation can be expressed in terms of elliptic functions. The author apologizes that his paper contains so much that was already known, but it was necessary to reproduce much of the known theory, to show the relations between the various parts, and in particular the meaning for his method of procedure. The result is a systematic presentation which serves splendidly to introduce the reader to the whole Klein scheme of the theory of functions, in particular the conformal mapping, extended from Schwarz's paper on the hypergeometric series.

The next paper considers the reduction of the modular equation by appropriate transformations. In this paper it appears that transformations of elliptic functions of orders $n = 5, 7, 11$ offer particularly promising results. The details for $n = 5$ are included in the paper itself. Besides the modular equation of order 6 and its known quintic resolvent, we also have, as Galois resolvent of the latter, the icosahedron equation of order 60.

The next paper, transformations of order seven of elliptic functions, brings the case $n = 7$ to the same degree of completion. The Galois resolvent is now of order 168; its roots have the property of remaining invariant under the linear fractional substitutions of the ratio η of the periods of the elliptic function, the coefficients of which are congruent to the identity, modulo 7, and only these. The next problem is to determine the relation between this root and the absolute invariant. We know that the genus of this equation is 3, and that it cannot be hyperelliptic. Hence the equation can be reduced to represent a non-singular plane quartic curve which remains invariant under a birational group of order 168. Moreover, the p adjoints of order $n - 3$ go over into the p adjoints of order $n' - 3$ by every birational transformation which sends the given curve of order n into one of order n'; hence in our case, the operations of the group are linear. But from Jordan's list of finite groups of ternary linear substitutions it appeared that no such group existed. Either Klein's premises were wrong, or Jordan had made an error. Subsequent consideration established the existence of the group, which should therefore be added to Jordan's list.*

From the singular values of the crossratio, in the linear fractional substitutions of η, it appears that our curve can have but three sets of points which have fewer than 168 conjugate positions; these exceptions appear by sets of 2, or 3, or 7. The first are the sextactic points, the second the points of tangency of the bitangents, and the last are the points of inflexion. But there is only one set of 24 points, hence the residual intersection of each inflexional tangent with the curve must be another point of inflexion. By choosing three associated inflexional tangents for the sides of the triangle of reference, the

* C. Jordan, *Mémoire sur les équations différéntielles lineaires à intégrale algébrique,* CRELLE'S JOURNAL, vol. 84 (1878), pp. 89-215. The error is on page 167, line 8 from below. The expression Ω should be divided by 3φ instead of 9φ.

equation of the curve is readily determined. The arrangements of the double tangents, the points of inflexion and sextactic points complete this part of the problem. The later sections of the paper are devoted to the construction of the Riemann surface, and of the curve itself, the above coordinate system being an equilateral triangle.

This paper is a gem of mathematical writing. Although it contains a wealth of new, and many unexpected results, the thought is so developed that each is derived directly and naturally, and each step suggests how the next should be taken. Instead of being lost in a maze of formulas — particularly likely to be the case in this branch of mathematics, — all the results are obtained with a minimum of manipulation, and by steps that leave nothing to be desired as to rigor or conclusiveness. The most striking feature is the ready and easy use of so many different branches of mathematics; here we find projective geometry, binary and ternary invariant theory, theory of groups, elliptic functions, conformal mapping, differential equations, algebraic curves — brought into service, each contributing its part in the general problem. This is everywhere characteristic of Klein's writings, but perhaps it reached its highest manifestation in this and the following papers. The case $n=11$ was treated immediately, but the generalization to n any odd prime appeared several years later, after the author had been intensely occupied with more general questions.

Each of these memoirs is followed by a few pages of comments, largely concerned with subsequent studies by other authors. These are followed by two general reports on the theory of elliptic functions and modular functions, written near the end of the Leipzig professorship, then follows a short note on the composition of binary quadratic forms, and a longer outline of the autographed notes on the course on the theory of numbers given at Göttingen in 1895–96. It is shown that by proper interpretation of certain non-euclidean concepts of measurement the definite and the indefinite forms can be treated in the same way. Later the forms with negative discriminant are applied to elliptic functions.

The second part consists of three memoirs, two on hyperelliptic functions, and the third on abelian functions. The latter has two rather distinct parts, the first being general, and the second, which carries certain considerations much further, is restricted to $p=3$. All these memoirs were written in Göttingen, 1886–1889. Beginning with the summer of 1887, Klein lectured two semesters on hyperelliptic functions, and the three following ones on abelian functions. During the last two years at Leipzig Klein had directed seven doctor's theses in this field, among them Fricke, and during the first few years at Göttingen as many more. These assisted materially in providing confirmations of details which Klein felt were correct, but which he had not taken the time or trouble to verify himself. Even then a great number of formulas were left undeveloped, and many numerical factors unprovided, which were supplied later by Burkhardt for the hyperelliptic case in two splendid papers.*

In the case of the abelian functions, the author mentions that he received the greatest assistance from the two students W. F. Osgood and H. S. White. The purpose of these papers is to bring the theory of hyperelliptic and of abelian functions into systematic and natural relation with the theory of invariants. In order to make this possible, the number of homogeneous variables could not be prescribed, and the domain of rationality had to be defined in each case.

The main result in the first paper is a systematic treatment of the theta functions, in particular for $p = 2$, and to determine the theta constants (for zero arguments) as modular functions. The associated hyperelliptic curves can be expressed in a definite canonical form, which suggests and makes possible a uniform treatment. No corresponding form exists for the non-hyperelliptic curves, and a general treatment is probably not possible. In the former, Klein developed the entire theory of the sigma functions; for abelian functions this was possible only for $p = 3$. The general form of the associated plane quartic curve is presupposed, and all the steps are taken in terms of invariants and covariants, rather than as applicable only to a particular system.

In the second part, a great deal of the paper is taken up with the systems of contact lines (double tangents), conics and cubics, and the necessary modifications when the curve has a double point. While this work is very interesting, and perhaps of considerable value in the study of the theta functions, and their geometric meanings, practically all these theorems on the quartic curve can be established directly from the study of the (1, 2) correspondence associated with the Geiser involutorial transformation. An interesting application is to hyperelliptic surfaces of genus 3, as there are 36 non-equivalent systems of canonical sections that make the surface simply connected. The author mentions that he was indebted to H. D. Thompson for appropriate graphical schemes to show the relations between these various systems.

The third division of the volume occupies about three hundred pages.

---

* *Beiträge zur Theorie der hyperelliptischen Sigmafunktionen,* MATHEMATISCHE ANNALEN, vol. 32 (1888), pp. 381–442, and *Grundzüge einer allgemeinen Systematik der hyperelliptischen Funktionen erster Ordnung. Nach Vorlesungen von F. Klein,* MATHEMATISCHE ANNALEN, vol. 35 (1889), pp. 198–296. The later publications in volumes 36, 38, 41 of the same periodical represent further developments of Burkhardt's own ideas.

Chronologically, it precedes the one just mentioned, as practically all of it was written while Klein was at Leipzig. We are told in the introduction that Klein regards this as the most productive period of his activity, as to quantity and as to importance. The beautiful results obtained in the study of the transformation of elliptic functions led the author to a more ambitious program of an exhaustive study of the groups of linear fractional substitutions of one variable, and then of the functions which remained invariant under such substitutions. The method was to be that of Riemann, with a minimum of analysis and a systematic application of physical ideas, strengthened by a highly developed intuition, both of spatial relations and of physical phenomena. Although himself a pupil of Plücker, and later in Göttingen greatly assisted and directed by Clebsch, Klein emphasized throughout his career that he was more influenced by Riemann than by any other source.

At Leipzig he at once began a course on the geometric theory of functions, to extend over several semesters. The lecture notes of the first semester were not published, but several copies were in the hands of students, and the dissertations of Gierster, Hurwitz, Staude, Lange and Weichold were begun. The substance of the second semester's lectures was put into book form and published under the title *Über Riemann's Theorie der algebraischen Funktionen und ihrer Integrale* by Teubner in 1882. It is given in full in this third division. There are no additional comments on the text, but at its end appears a note which calls attention to the similarity of methods here developed and those employed by F. Schottky. In reponse to an inquiry of Klein as to the source of his ideas, Schottky replied that it was the study of the circulation of an incompressible fluid—that is, Riemann's idea. Attention is also called to the fact that in Weyl's book (*Die Idee der Riemannschen Fläche*) most of the concepts used by Klein have been confirmed by the rigid requirements of modern analysis.

An algebraic function was defined by means of its Riemann surface, and the latter was interpreted as any closed surface in space. The rational functions associated with the rotations which leave the regular bodies invariant, trigonometric, and elliptic functions were all well known examples of functions belonging to the group to be studied. The modular functions furnished a large advance, as they belonged to infinite groups in which the coefficients are integers with determinant unity.

While these ideas were rapidly maturing, but before the broad lines of the general theory were published, there appeared, in rapid succession, a number of brilliant notes in the COMPTES RENDUS, 1881, by H. Poincaré, on certain phases of the same subject. These notes prompted Klein to write to Poincaré, primarily to protest against the name (*Sur les fonctions fuchsiennes*) and to call to his attention a considerable number of earlier memoirs in which he (Klein) had developed the same ideas. The name *Fuchsian* had been applied by Poincaré to those functions which have a fundamental circle; the matter was not made more pleasant when he proposed and later used the adjective Kleinéens, to describe the functions which do not have a fundamental circle. Now follow twenty-five letters, all written during the years 1881, 1882. Klein invited Poincaré to prepare a synoptic paper on his work, and proposed that he would also prepare a similar paper. They both appear in volume 19 of the MATHEMATISCHE ANNALEN, that of Klein appearing in the present volume, covering eight pages. It was followed, six months later, by the principal memoir Klein wrote on this subject, *Neue Beiträge zur Riemann'schen Funktionentheorie,* which appeared in the MATHEMATISCHE ANNALEN, vol. 21 (1882–83), and occupies 80 pages. Of these, the last six are occupied with modifications and refinements of the existence theorem that on every given Riemann surface there exists one and but one principal function. The proof is not completed; indeed in various parts of the paper the argument is somewhat sketchy. It was at this time that Klein's health broke down. He suspended work for some time, and when he could resume, it was only on a restricted scale, for several years. Apart from a few notes, this is the last memoir on automorphic functions written over his own name, but in the preface to the treatise on *Automorphe Funktionen*, Fricke explained that he was directed by Klein through most of the great undertaking. Near the end of the present volume appears a critical analysis of the various proofs of this theorem, prepared by Klein and Bessel-Hagen, and a similar criticism of the origin and development of the *Automorphe Funktionen.*

In the appendix of 36 pages a number of lists are collected, as follows: University courses and seminar themes, 1871–1923, complete list of the 48 doctor dissertations written under his direction, together with the time and place of publication of each, list of his assistants, and a list of his 151 memoirs, 5 commemorative address, 3 reviews, 10 books, 10 autographed volumes, 33 reports on the organization of the teaching of mathematics, and his work as editor of the MATHEMATISCHE ANNALEN, the ENCYKLOPÄDIE, the International Commission, and the works of Plücker, of Möbius, and of Gauss.

These three magnificient volumes, clearly printed on high quality paper, filled with instructive personal notes, generously stocked with references to the work of others, even when cherished ideas of the author were first published by others, all skillfully compiled by patient and sympathetic hands, almost completely free from typographical errors, furnish an almost invaluable commentary on the development of mathematics during the last half century.

VIRGIL SNYDER