A NOTE ON "CONTINUOUS MATHEMATICAL INDUCTION."

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1. *Special case.*—Let the function $f(x)$ be defined in some interval of a real variable x.

Hyp. 1. Let there be a point a in the interval such that $f(a) = 0$.

Hyp. 2. Let there be a constant Δ for the interval, such that $f(x) = 0$ implies $f(x + \delta) = 0$, whenever $0 < \delta \leqq \Delta$.

Then for any b in the interval, where $b > a, f(b) = 0$.

Proof.—I. If $b - a \leqq \Delta$, then by Hyp. 2 the conclusion follows.

II. If $b - a > \Delta$, then first apply Archimedes' postulate, that is, there will be an integer n and a fraction $\theta (0 \leqq \theta \leqq 1)$ such that

$$b - a = (n + \theta)\Delta, \qquad \text{or} \qquad b = (a + \theta\Delta) + n\Delta.$$

Next, apply ordinary mathematical induction, thus: By Hyp. 1 and 2, since $\theta\Delta < \Delta$,

$$\therefore \quad f(a + \theta\Delta) = 0.$$

Therefore, by 2, again,

$$(1) \qquad f[(a + \theta\Delta) + 1 \cdot \Delta] = 0.$$

By 2, if $f[(a + \theta\Delta) + m \cdot \Delta] = 0$, then

$$(2) \qquad f[(a + \theta\Delta) + (m + 1)\Delta] = 0.$$

Hence, combining (1) and (2),

$$f(a + \theta\Delta + n\Delta) = 0,$$

that is,

$$f(b) = 0.$$

2. *General case.*—Let $\varphi(x)$ be any propositional function, defined in some interval of a real variable x.

Hyp. 1. Let there be a point a in the interval such that $\varphi(a)$ is true.

Hyp. 2. Let there be a constant Δ for the interval such that $\varphi(x)$ implies $\varphi(x \pm \delta)$, whenever $0 < \delta \leqq \Delta$.

Then for any b in the interval such that $b \gtreqless a$, respectively, $\varphi(b)$ is true.

The proof will be the same as for the special case, except for obvious changes of wording or sign.

Remarks.—The theorem rests essentially on Archimedes' postulate and on ordinary mathematical induction, but it is not a generalization of the latter, in the sense of including it as a special case. It is not a theorem in mathematical logic, since it is concerned with a real variable x. But it is more general than ordinary theorems dealing with equalities, in that $\varphi(x)$ may be a statement about continuity, convergence, integrability, etc., that cannot be put in the simple form of $f(x) = 0$.

The theorem is a mathematical formulation of the familiar argument from "the thin end of the wedge," or again, the argument from "the camel's nose":

Hyp. 1. Let it be granted that the drinking of half a glass of beer be allowable.

Hyp. 2. If any quantity, x, of beer is allowable, there is no reason why $x + \delta$ is not allowable, so long as δ does not exceed an imperceptible amount Δ.

Therefore any quantity is allowable.

Like all mathematical theorems, the conclusion is no surer than its hypothesis. In this case, if the argument fails, it is usually because a *constant* Δ required in the second hypothesis does not exist. Take the very wedge itself. If a wedge is driven with a constant force between two sides which are pushed together by elastic forces, it will be stopped when balanced by the component of the increasing resistance. In this case the Δ within which δ may increase for $\varphi(x + \delta)$ to continue to hold will not be "uniform for the interval," so to speak, but will become smaller and smaller as x approaches the dangerous point, beyond which the conclusion ceases to be true.

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