THE MADISON COLLOQUIUM.

THE Seventh Colloquium of the American Mathematical Society was held in connection with its twentieth summer meeting at the University of Wisconsin, Madison, Wis. At the April meeting of 1911 the Council appointed a committee consisting of Professors Van Vleck, Moore, Osgood, and the Secretary to make the arrangements for the meeting and colloquium. The courses of lectures were announced in the preliminary circular of May, 1913, and printed syllabi were issued in advance of the meeting. The colloquium opened on Wednesday morning, September 10, and extended through the rest of the week. The following fifty-one persons were in attendance, a larger number than at any previous colloquium:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Professor W. H. Bussey, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Mr. H. J. Ettlinger, Professor W. W. Hart, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Mr. Barnem Libby, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Miss Lulu Runge, Dr. Mildred Sanderson, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry."

II. Professor W. F. OSGOOD: "Selected topics in the theory of analytic functions of several complex variables."

Abstracts of the lectures follow below. The lectures will soon be published in full by the Society as Volume IV of the Colloquium Series.

I. In the first lecture, Professor Dickson discussed the invariants of quadratic forms

$$q_m \equiv \beta_{11}x_1^2 + 2\beta_{12}x_1x_2 + \cdots + \beta_{mm}x_m^2, \qquad \beta_{ij} = \beta_{ji},$$
$$D \equiv | \beta_{ij} | \neq 0,$$

for the case in which the β_{ij} are complex numbers (the algebraic case), and also for the case in which they are integers, considered as equivalent if congruent modulo p (the number-theory case); forms of the latter type are called modular forms.

The invariants are obtained in both cases by a general method, based upon the classification of the forms in such a way that any two forms of the same class shall be convertible into each other by means of a transformation

$$x_j = \sum_{i=1}^{n} \alpha_{ji} X_i,$$

where in the algebraic case the α_{ji} are complex numbers and $| \alpha_{ji} | = 1$, and in the number-theory case they are integers, reducible modulo p, and $| \alpha_{ji} | \equiv 1 \pmod{p}$. It is then clear that any function $\varphi(\beta_{11}, \cdots, \beta_{mm})$ which takes the same value for all the forms of any one class must be an invariant of q_m. It was shown that in the algebraic case the value D of the determinant $| \beta_{ij} |$ and the rank r of this determinant form a fundamental system of invariants, i. e., that every invariant $\varphi(\beta_{11}, \cdots, \beta_{mm})$ is a single-valued function of D and r; furthermore, every rational integral invariant was shown to be a polynomial in D. In the number-theory case, it was found that, if the principal minors of order r of the determinant $| \beta_{ij} |$ are designated by $M_1, \cdots, M_n$, then a fundamental system of rational integral invariants is given by

$$D, I_0, A_1, \cdots, A_{m-1},$$

where

$$A_r = \left[M_1^{\frac{p-1}{2}} + \sum_{i=2}^{n} M_i^{\frac{p-1}{2}} (1 - M_1^{p-1}) \cdots (1 - M_{i-1}^{p-1}) \right] \Pi(1 - d^{p-1}),$$

and

$$I_0 = \prod_{i,j=1}^{m} (1 - \beta_{ij}^{p-1}),$$

d ranging over all the principal minors of $|\beta_{ij}|$ of order greater than r. An outline was given of a general theory of modular invariants of a system of forms.

Lecture II was devoted to an application of the methods developed in the first lecture to a study of the algebraic and the modular seminvariants of a binary form, i. e., of such functions of the coefficients a_i of a binary form

$$f \equiv \sum_{i=0}^{n} C_{n,i}\, a_i x^{n-i} y^i$$

as are invariant under the transformation $x = x' + ty'$, $y = y'$. The work was carried through in detail for the case $n = 4$, leading to the well-known result that if we put $t = -a_1/a_0$ $(a_0 \neq 0)$, then the form f goes over into a new form, whose coefficients, multiplied by proper powers of a_0, together with certain combinations I and J of them, form a fundamental system of seminvariants for f. It was found that the algebraic seminvariants were not sufficient to characterize all the different classes of forms into which the binary quartic forms were grouped; the modular seminvariants were then constructed and thereby a complete characterization of these classes was secured.

In the third lecture, Professor Dickson took up a discussion of the invariants of the group of transformations

$$x' = ax + by, \quad y' = cx + dy,$$

a, b, c, d being integers, such that $ad - bc \equiv 1 \bmod p$.

The fundamental system of invariants for this case was found to be

$$L \equiv y \prod_{i=0}^{p-1} (x - iy) \equiv yx^p - xy^p$$

and

$$Q \equiv (yx^{p^2} - xy^{p^2}) \div L.$$

In the case of n variables, there are n fundamental invariants. The "form problem" for the case of two variables,

i. e., the problem of determining x and y when $L(x, y)$ and $Q(x, y)$ are given, was then briefly discussed. It was shown that the form problem is completely solved when we find two linearly independent solutions of the congruence

$$z^{p^2} - \kappa z^p + \lambda^{p-1} \equiv 0 \pmod{p},$$

where $\lambda \equiv L^{p-1}(x, y)$, and $\kappa \equiv Q(x, y)$.

The lecturer then passed on to a presentation of the work of Hurwitz on formal invariants and its connection with the modular invariants. Considering for the purpose of illustration the form

$$f \equiv a_0x^2 + a_1xy + a_2y^2,$$

where a_0, a_1, a_2 are arbitrary variables, and subjecting this form to a transformation $x = ax' + by'$, $y = cx' + dy'$, $ad - bc \not\equiv 0 \pmod{p}$, which brings f to the form $b_0x'^2 + b_1x'y' + b_2y'^2$, a formal invariant may be defined as a function $F(a_0, a_1, a_2)$ of the coefficients of such character that

$$F(a_0, a_1, a_2) \equiv F(b_0, b_1, b_2) \pmod{p},$$

identically in a_0, a_1, a_2.

Of especial interest and importance here is the general theorem, due to Miss Sanderson, that from any modular invariant a formal invariant may be constructed, which will reduce to the modular invariant for integral values of the coefficients. This theorem is of great value in the construction of covariants. The lecture closed with a discussion of formal seminvariants.

The last two lectures were concerned with modular geometries, i. e., geometries in which only points with integral coordinates are considered as real points and in which two points are considered as identical if their coordinates are congruent. This work belongs entirely in the theory of numbers; the terminology, borrowed from geometry, suggests material and mode of treatment, and leads to results new and important in the theory of numbers. Particular reference was made to the recent work in this field of Veblen and Bussey, and to the work of Coble, who obtained an entry into the theory of theta-functions by carrying over into modular geometry some of the notions of projective geometry.

Some of the results of work in this field as stated by Professor Dickson are: All the polars of a conic in the space of a modular geometry of n dimensions pass through a fixed point, if n is odd. This point is called the *apex* of the conic. All the lines through the apex meet the conic in only one point. The tangential equation of the conic is *linear*.

The fundamental system of covariants of the ternary quadratic form were found, and the case of m variables was briefly considered.

Professor Dickson discussed in the closing lecture a theory of plane cubic curves with a real inflexion point. The number-theory character of the work led him to discover a rational reduction of the general cubic to the following normal form:

$$x^2y + gy^3 + hy^2z + \delta z^3 = 0,$$

where the point (1, 0, 0) is the real point of inflexion the existence of which among the real points of the curve was preassumed, and in which the coefficients g and h are simply related to the ordinary invariants of the cubic. The discussion of the points of inflexion, as to their reality and their configuration was carried on for various cases and for modular geometries with different moduli, one case being found in which the cubic has 9 real points of inflexion.

II. Professor Osgood's first lecture gave a general survey of the field, treating the following topics: Analytic functions of several variables; the factorial function and analytic continuation; existence theorems; Weierstrass's theorem of factorization; Jacobi's problem of inversion and the abelian functions; periodic functions; theta functions with several arguments; the theta theorem.

The second lecture was occupied with a number of general theorems: Rational and algebraic functions; sufficient conditions that a function of several variables be analytic; sufficient conditions that a function be rational, algebraic; on the associated radii of convergence of a power series.

The third lecture was devoted to singular points and analytic continuation: Introduction; non-essential singularities; essential singularities; removable singularities; analytic continuation by means of Cauchy's integral formula; application to the distribution of singularities; Levi's memoir of 1910; lacunary spaces; the boundary of the domain of definition of $f(x, y)$; representation of certain meromorphic functions as quotients.

The fourth lecture dealt with implicit functions under the following headings: Weierstrass's theorem of factorization; a tentative generalization of the foregoing theorem; algebroid configurations; single-valued functions on an algebroid configuration; connectivity and the Riemann manifold; parametric representation im Kleinen; solution of a system of analytic equations, Weierstrass's theorem; a general theorem; special cases of the foregoing; the inverse of an analytic transformation.

In the final lecture, Professor Osgood discussed the prime function, introduced by Klein in the theory of the algebraic functions and their integrals. That function is not a function on the given algebraic configuration, but depends on the homogeneous coordinates of an allied configuration. It is possible, however, to obtain a function on the given configuration which has a single zero and is completely analogous to the elliptic θ or σ function.

ARNOLD DRESDEN.

THE VIENNA MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-fifth convention of the association of German naturalists and physicians at Vienna, September 22–25, under the presidency of Professor K. Rohn.

The Germans fully recognize the importance of the social side of these annual meetings, and make ample provision for the pleasant entertainment of the guests. Although the program was a long one, frequent excursions were arranged and a reception or concert each evening.

The session of Thursday afternoon was devoted to the administrative affairs of the society. Reports of the status of the Encyclopedia, the Euler commission, and the International commission were read, and a briefer statement was submitted concerning the publication of various other works supported in whole or in part by the society.