A MODERN CALCULUS OF VARIATIONS.

Lectures on the Calculus of Variations. By OSKAR BOLZA. (The University of Chicago Decennial Publications, Second Series, Volume XIV.) The University of Chicago Press, Chicago, 1904. 8vo, xv + 271 pp.

THE calculus of variations is one of the very first of the old (or formal) developments of the infinitesimal calculus; it is one of the newest conquests of the modern (or critical) school. The history of the older calculus of variations is almost trite from reiteration; to select among many, the works of Moigno-Lindelöf, Diegner, Todhunter, Carll, Jordan, and (more recently and more perfectly) Pascal, have made known the achievements of the old school from Newton to Jacobi to mathematicians of all nations. The problems and the successes of the new school — the critical investigations — have been known to few until this century opened. In 1900 the first really modern treatise — Kneser's Lehrbuch — appeared. Without gainsaying the importance of that work and the fact that it opened the doors of modern research in the calculus of variations to the general mathematical public, it must be admitted that its style and arrangement are forbidding. The articles in the Encyklopädie (by Kneser and by Zermelo and Hahn) scarcely dispel the gloom of the Lehrbuch's unilluminated interior; but this was not to have been expected in encyclopedia articles, and the circumstances which necessitated two separate articles did not particularly conduce to public enlightenment on comparative values of various methods.

The book which is the subject of the present review is of interest to us because it is the first accurate and critical presentation of all the modern methods which is thoroughly readable, and because of the natural favorable prejudice of language and country. Apart from any such bias, however, Professor Bolza's treatise is surely admirable in many ways. Let us examine some of its notable features.

The preface is illuminating concerning much that has not been generally known, and the statements of the essential differences between the older theory and the new are worthy of recapitulation. They are: (1) the critical revision, by Weierstrass, Erdmann, Du Bois-Reymond, Scheefer, Schwarz, and others, of results previously obtained; (2) Weierstrass's introduction of the parameter representation; (3) Weierstrass's discovery of the fourth necessary condition and the sufficient conditions, with many subsequent allied contributions by others; (4) Kneser's extension to extremals of the theorems on geodetics; (5) Hilbert's *a priori* existence proof. That this is given at some length here and that the body of the book is often tinged with a historical coloring is doubtless due to the fact that these Lectures are a development of the Colloquium Lectures delivered by Professor Bolza at the Ithaca meeting of the AMERICAN MATHEMATICAL SOCIETY (August, 1901), which will be remembered with pleasure by many readers.

In the opening pages (§ 1, Introduction; § 2, Notation and terminology) one is made conscious of the rigorous nature of the discussions to be expected, but is reassured by the conciseness and clearness of statement, which go to make the book throughout easily readable to one otherwise familiar with exact reasoning. The reviewer would call attention to the precise definitions of interval, region, continuum, etc., on page 5. Some general agreement concerning these terms was and is needed, and the definitions here given may serve as a basis, even if they are not generally adopted verbatim. A difference of opinion is more tenable here, however, than in the demonstrations of later pages. Thus it might be better to reserve the name region for what is called a closed region, since the term continuum already connotes an absence of boundary points, and the word domain already serves as a generic term including closed regions, continua, and the intermediate cases. An interval, as defined here, is apparently always "closed."

The crucial article on the formulation of the problem (§ 3, page 9) leaves nothing to be desired. It should be noticed that maximum (or minimum) as here defined is that which is called "improper," in contradistinction to the case of "proper" maximum (or minimum). The usage in this matter must be settled by time. Although the present definition does not materially alter the conditions discovered by others under the other definition, it seems possible that it will change the nature of the necessary conditions which remain to be discovered (see page 100), and it would necessitate the recognition of any curve whatever as a minimizing curve in the case of integrals which are independent of the path. The use of the phrase "proper minimum" (see page 30, line 29) as distinguished from "minimum" may be misleading.

The next articles contain the two older classical derivations of the condition that the first variation should vanish (§ 4), Du Bois-Reymond's lemma which reduces the preceding condition to the *first necessary condition* (§ 5), and the Du Bois-Reymond-Hilbert proof of that condition without the assumption of the existence of the second derivative (§ 6). The presentations contain nothing essentially unfamiliar, except that the last proof is here presented for the first time in a separate work, but they are especially clear and the footnotes contain valuable remarks and references. The use of Euler's name instead of Lagrange's to designate the fundamental differential equation is novel and is apparently justified (footnote 3, page 22).

The miscellaneous remarks on the integration of Euler's equation in § 7 include several important matters, for example the proof of the existence of extremals in any direction not parallel to the Y axis (page 28), the case in which Euler's equation reduces to an identity (page 29), and the inverse problem (page 31). The formal side of the latter problem is outlined here in concise and satisfactory form, but it is unfortunately not reconsidered later after the development of the sufficient conditions.

The chapter closes with discussions of Weierstrass's lemma (§ 8), including the introduction of the E function and the concept of transversality; discontinuous solutions (§ 9); and (§ 10) the conditions at a boundary.

Chapter II is for the most part an account of the treatment of the so-called *second variation* by the methods of the old school. While the general lines of this work are familiar from older works, the nice distinctions and exact statements of this presentation are characteristic of the modern point of view and are not to be found in older works. We shall mention here only the following points: (1) the exact statement of Legendre's condition (page 47); (2) the exact statement of the assumptions (§ 13, pages 54–55) made in the proof of Jacobi's theorem; (3) the exact statement of Jacobi's condition (page 67). On the whole this chapter contains less that is modern, except in point of exactness, than Chapter I, but here also the notes and references are valuable even to one familiar with the older theory.

The study of the sufficient conditions in Chapter III formally opens the modern theory, though this would be impossible without the accurate revisions of the older theory in the previous chapters.

With respect to the author's view (page 70) that the concept of "weak" extrema, for which he has just proved that Jacobi's and Legendre's conditions are sufficient, is of only temporary importance, there may be differences of opinion. The distinction between "weak" and "strong" minima is of precisely the same kind as the distinction between "strong" minima and "absolute" minima, or as the distinction between isolated minima and absolute minima on a plane curve. In case no "strong" minimizing curve exists, for example, it seems interesting at least to discover the "weak" ones. Such is actually the case in Newton's famous problem on the surface of rotation of least resistance. Again, a "strong" minimizing curve is surely also a "weak" minimizing curve; hence we need seek for "strong" minimizing curves only among the "weak" minimizing curves.

Passing to the case of a strong minimum, Bolza first proves Weierstrass's (the fourth) necessary condition by means of Weierstrass's lemma. Then follows in §§ 19–20 the exquisite modern theory due to Weierstrass and Hilbert, resulting in Weierstrass's theorem and the first set of *sufficient* conditions. The matter presented is already fairly familiar to the American mathematical public through articles by Bolza himself, Osgood, and others, to which the most detailed references are given. But this is the first systematic treatment of the theory from all these points of view, and it is probably easier to compare the different methods here than in the isolated memoirs. It goes without saying that Kneser's treatment is much more one-sided here and in many other places, if for nothing else on account of the achievements of Hilbert and others, after the appearance of the Lehrbuch, which were considered to warrant the extra article in the Encyklopädie by Hahn and Zermelo. It seems to be conceded that the simplest and most elegant proof of Weierstrass's theorem is that based upon Hilbert's invariant integral, and subsequent treatments of the calculus of variations can probably afford to limit their discussion to a single proof. It might be advantageous to make even more use of Hilbert's principle than is indicated here. In particular the matter on page 266 (addenda) and that in footnote to page 175 indicate the possible important applications of this and similar methods.

The notation employed for the various conditions (I, II, IIa, IIa', etc.) is somewhat novel but ceases to be confusing after careful examination, and the principal results are carefully stated in prominently lettered theorems. Thus the "Fundamental Theorem" (page 96), with the various appended notes (page 96 ; 1, page 99 ; and 1, p. 101), is sufficiently clear to dispel any confusion which might otherwise arise from such alternate statements as those on pages 98–101. The spirit of *c*), (page 98), shows the predilection of the author for the parameter representation of Weierstrass. Of course the omission of the equality signs from the necessary conditions to make them (apparently) coincide with the sufficient conditions is just as vital as any other difficult hiatus which might be suppressed as an "exceptional case," and a "fifth necessary condition" (see bottom of page 100) is just as indispensable in the case of parameter representation as it is here. But no such considerations are needed to establish the superiority of the parameter representation for a certain class of problems, nor could they possibly show a superiority for certain others — notably for certain problems of mechanics. The facts established in these pages (98–101) are nevertheless important and unquestionably rigorous, if we expressly repudiate any false inferences which a reader might draw upon his own account. Perhaps the theorem stated in footnote 1, page 101 is sufficiently important to have deserved a place in the text proper; but the reviewer has elsewhere* insisted upon this point sincerely enough and with possibly an excess of fervor.

Finally, the table of all the conditions on pages 101 and 102 removes any ambiguity which may remain. This table is for all practical purposes the end of the consideration of the simplest problem — namely that of minimizing the integral

$$\int_{x_0}^{x_1} f(x, y, y')dx$$

between given fixed end points.

Dividing the chapters as he does, Bolza has included in this chapter (III, Sufficient conditions) a discussion of one of the many possible generalizations — probably the first which naturally suggests itself — namely the case in which the end points are variable. The presentation is by the "method of the differential calculus," which reduces the new problem to a

* "Supplementary note on the calculus of variations," BULLETIN, vol. 9, No. 5 (February, 1905).

combination of the preceding one and that of ordinary maxima and minima, which may be assumed as known (see footnote, pages 102–103). This fact renders this discussion somewhat less interesting as an independent theory and augments the importance of the previous problem. The reviewer therefore feels that a rather disproportionate fraction of the volume has been given to this matter, in view of the fact that it is taken up again twice. The results are interesting, particularly in connection with Kneser's theory (Chapter V), of which it forms almost an integral part, but in view of the limited size of the book and the important omissions mentioned below the three separate treatments might well have been condensed into a single one, say under Chapter V.

Chapter IV is an exposition of the theory of the calculus of variations in parameter form, according to the methods of Weierstrass. The author is very frank, and apparently entirely just in his estimate (footnote, page 115) of the advantages of the method. The asymmetric forms of the preceding chapters apply in reality, as the author remarks, to those problems in which the solution $y = f(x)$ is thought of as a function rather than as a curve, but this implies that the asymmetric forms are to be preferred not only in the theory of functions of a real variable, but also in numerous applications which are not primarily geometric. One is often deceived by the convenient employment of geometric terms into a belief that the essence of a problem is geometric when it is not so, and the number of problems in which the parameter representation is superior might easily be overestimated.

On the other hand there is no possible question of the immense advantage of the parameter representation for the precise treatment of geometric problems. This advantage is shared with absolute equality, however, by the similar treatment of any geometric application which arises in any branch of mathematics whatever. This fact has been recognized but a short time even in the very elements of geometry — even in the definition of a curve. It is for those nice considerations which become possible in an accurate critical revision of a theory, and not for the relatively crude work of discovery of new and general principles, however, that these forms are valuable even in geometry, and there is many a geometric problem which has not yet been stated in this manner. The calculus of variations is no more a geometric subject than is many another branch of mathematics, and the stimulus which the success of these methods in the calculus of variations will doubtless produce will surely cause the production of supplementary theories in these other subjects which will make applications to geometric problems as safe as are the applications of the results of this chapter. As a fitting example I might mention first of all the theory of differential equations, in which the ordinary theorems are no better suited to geometric applications than are those of the first three chapters of this book. A supplement to the theory of differential equations along the lines of the present chapter would be valuable and is indispensable to just the extent that the Weierstrass parameter representation is in the calculus of variations. As a matter of fact, on account of the perfect one-to-one correspondence of the real number system with the points upon a line and that of the assemblage of all pairs of real numbers with the points of a plane, we have been too unconscious of the discrepancy between curves on the one hand and equations of the form $y = f(x)$ on the other. It is perhaps not too much to say that geometric problems in general constitute by no means the nearest application of any well-known mathematical theory in the form in which it is at present usually given.

The development of the various conditions is wholly analogous to the work of previous chapters, and pages 115–146 and 153–156 involve few considerations which are *essentially* new; there are indeed new *facts*, but these are such as we should expect in any discussion in passing from non-homogeneous to homogeneous forms. In many cases omissions are made when the generalizations are rather obvious (see page 115, footnote 1). The following points deserve at least mention: (1) Weierstrass's form of Euler's equation (page 123) and the commentary upon it (page 124); (2) the proof (page 125) of the existence of an extremal in any direction; (3) the statement and proof of the necessary and the sufficient conditions (page 143); (4) the proof of the existence of a minimum "im Kleinen" (pages 146–147); (5) the conditions at a boundary (pages 148 ff.); and (6) the extension of the meaning of a definite integral (pages 156–163).

With regard to the statement of the sufficient conditions on page 143, the possibility of misunderstanding has already been mentioned. The reader should be careful to note that Bolza does not even imply that there is any one set of conditions given which is at once necessary *and* sufficient, though he seems to use those words. The four sufficient conditions stated on that page have nowhere been shown to be necessary, nor are they so (see footnote 2, page 148). While not wishing to criticize the real content of these statements, the reviewer feels that the actual gap between the necessary and the sufficient conditions should not only not be lost sight of, it should be emphasized and brought home by every possible device; the conscious or unconscious failure to do this in the past has been the bane of exact mathematical reasoning. For this reason the reviewer would object to the notation which apparently identifies two different conditions, one of which is necessary and the other sufficient, and he would object to the phraseology (in *d*, page 143) which bears out this false impression. As a matter of fact the necessary and the sufficient conditions are not appreciably nearer together than they are in the other — non-geometric — problem; they are not appreciably nearer than they have been for several years.

The proof of the existence of a minimum "im Kleinen," which follows Bliss, *Transactions of the American Mathematical Society*, volume 5 (1904), is very important in many applications and constitutes one of the several important and wholly new topics treated. It is regretable that no mention is made of Hilbert's concept of "Stückweise Variation,"* which is intimately connected with this topic and which has important applications.

The treatment of the extension of the meaning of a definite integral is also particularly noteworthy. The results of Weierstrass, Hilbert, Osgood, Peano and others are presented in their essentials and compared. The subject is again taken up in Chapter VII and forms an integral part of the author's excellent presentation of Hilbert's *a priori* existence proof.

The proof just mentioned is given in detail in Chapter VII, which we shall consider here on account of the connection mentioned above. The whole chapter is carefully written and contains the first accurate detailed proof of Hilbert's theorem. There is first a precise statement of the theorem and of the conditions under which it is to be proved. Then follows a supplementary investigation of extensions of the meaning of an integral, which together with the treatment just mentioned forms one of the important contributions of the book. I shall not enter into the details of the proof. It is rather long, but seems as concise as possible, and it actually proves, with apparently only necessary restrictions, all that was ever

*The reviewer has elsewhere translated this as "limited variation." See BULLETIN, vol. 9, p. 12.

claimed for the Hilbert proof in the case of a simple integral. In its accuracy and generality this chapter really amounts to an original contribution to mathematics, since neither Hilbert nor anyone else has published in any place an accurate proof of the theorem in its most general form, although portions of the proof and special cases of it have been proved by Hilbert and others and the truth of the theorem rather generally accepted. Leaving aside any other merit, this chapter alone would make this book a pioneer work of the first importance.

Chapters V and VI are devoted, respectively, to Kneser's transversal theory and the isoperimetric problem. Since the main features of both these chapters follow Kneser's Lehrbuch, they will be passed over rapidly.

Chapter V begins with the general explanation of Kneser's transversal theory as an extension of the ordinary theory of geodetic lines, and it is pointed out that this theory applies with peculiar force to the case in which the end points are variable. Kneser's principal theorems (on transversals, page 172 ; and on the envelope of a set of extremals, page 174) are given in § 33. It is shown in a note (page 175) that these theorems can be derived in a manner wholly analogous to Hilbert's original considerations which led to Hilbert's invariant integral theorem. These considerations are given in a note in the Addenda (page 266) in almost the same form in which Hilbert led up to his theorem in the course of lectures mentioned on pages 246 and 268. It should be mentioned that this course was formally entitled "Ausgewählte Capitel aus der Flächentheorie" and dealt with the calculus of variations *via* the problem of geodetic lines. The proof of the existence of a field (cf. Kneser, Lehrbuch, § 14) is made rigorous by the considerations of § 34, and the way is prepared for the introduction of Kneser's curvilinear coördinates. A number of important conclusions are stated in § 36, including Osgood's theorem "concerning a characteristic property of a strong minimum" (*Transactions of the American Mathematical Society*, volume 2, 1901). Kneser's proof of Weierstrass's theorem is compared with Hilbert's in § 37 and the unnecessary assumption in Kneser's proof is discussed in an enlightening manner. The chapter closes with a discussion of focal (or critical) points for the case of variable end-points.

The treatment of the isoperimetric problem follows Kneser's Chapter IV and Weierstrass's lectures. The essential facts —

Euler's rule, Meyer's law of reciprocity, the various necessary and sufficient conditions — are presented with the usual clearness and accuracy. In both these chapters which follow Kneser, the advantage over Kneser in clearness and conciseness is indisputable.

The reviewer is not one who would consciously conceal the faults, much less the errors, of a treatise under his criticism. But if there are serious errors in this book, they have escaped the reviewer either through ignorance or through insufficient scrutiny. The faults of a book are often matters of opinion, and I have pointed out a very few minor points in which I should venture to disagree with the notation or with the method of presentation. There are two other general criticisms of this nature. The arrangement of the matter into chapters was probably the result of careful consideration, but it seems that space might have been saved had the chapters been arranged according to subject matter rather than according to authors or the nature of the conditions to be proved. Thus Weierstrass's theorem, the case of variable end points, and several other matters are all considered in each of three distinct chapters. Again the repeated proofs of certain theorems by several methods consumes space, but this is made up for by the value of the comparisons which become possible. A future treatise will be spared this burden by reason of the existence of this book. As a result, perhaps, of the policy just mentioned, many important topics have been omitted altogether. For example, no attention is paid to the extension of the theory to double integrals, nor to the case in which higher derivatives enter in the integrand; and the consideration of the important inverse problem is restricted to the formal work on the two pages 31–32, which gives only the results due to Darboux.

These suggestions regarding arrangement and selection of topics do not alter the opinion already expressed that the treatment of the topics which are considered is excellent and that the book as a whole is an important contribution to mathematical knowledge not only in America but throughout the world. That Bolza's work is of international importance is shown by the speedy recognition it has found in at least one foreign quarter: the chapter on the calculus of variations in the second volume of Goursat's very recent Cours d'Analyse contains explicit acknowledgment of the indebtedness of its author to this treatise.

May this be a pioneer, not only in the subject which it represents, but also in the publication of accurate works on advanced mathematical topics in America.

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TWO BOOKS ON ANALYTIC GEOMETRY.

The Elements of Analytic Geometry. 8vo. 424 pp. *Introduction to Analytic Geometry.* 8vo. 217 pp. By PERCEY F. SMITH and ARTHUR SULLIVAN GALE. New York, Ginn and Company, 1904.

It is a good sign for the university instruction in mathematics of the day that text-books are appearing more frequently in series than formerly. As a rule it means a better individual text-book, for it is the attempt of not one, but several members of the department collaborating to meet in the best possible way what actual experience has taught them to be the need of the student; and it certainly means a more organic, better coördinated mathematical curriculum in the institution which properly uses them. There is furthermore a considerable time economy for the student in the use of such a series. Take for instance the subjects: functions, graphs, partial fractions, and so forth, occurring in different lights in algebra, analytic geometry and calculus; considerable space and time can be gained in their second and subsequent treatments. Still another important gain is to be found in the clearing away for the student of all non-essential difficulties of a new subject, such as becoming accustomed to a new style, a new method of exposition, or a new notation. The books before us are two of a series called into being by needs at Yale University and now appearing under the editorial direction of Professor P. F. Smith.

In those of the works which are at present complete, the main object seems to have been to create drill books, which should be clear and minute in the exposition and analysis of method and rich in exercises, so that a course in one of them would leave the student a ready master of the more usual problems of the subject. If this is a correct conception of their