$$\frac{2}{\pi}\int_0^\infty \frac{\sin\phi}{\phi}\cos\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right)\phi\,d\phi$$

has the constant value unity for every point within the ellipsoid; it becomes zero for points outside and has the value one half for points on the surface. The potential of the ellipse is now employed, and imaginary parameters are introduced, by means of which the integrals are readily reduced to elliptic forms, similar to those obtained in the preceding chapter.

In the last chapter the discontinuous factor is employed to reduce an extended type of multiple integrals to gamma functions. The results are applied to the determination of volumes and moments of inertia of certain solids. The substance of this chapter was published in *Liouville's Journal* in 1839, but has been materially improved in its present form.

In a series of notes appended to each part every slight variation in the text from the form originally given by Dirichlet is pointed out. Occasionally further steps of a proof are there reproduced, which are now known to be unnecessary. On the other hand, many earlier authors had contented themselves with simpler proofs which in many cases are shown to be insufficient.

Typographically, the book is uniform with the well known Theory of numbers of Dirichlet-Dedekind, and the books of H. Weber, published by the same house. Apart from the errors which are mentioned at the end of the volume, the book is remarkably free from typographical errors.

VIRGIL SNYDER.

THE THETA FUNCTIONS.

Lehrbuch der Thetafunktionen. Von ADOLF KRAZER. Leipzig, Teubner, 1903. 8vo. xxiv + 509 pp.

Theorie der Riemann'schen Thetafunction. Von GEORG ROST. Leipzig, Teubner, 1901. 4to. iv + 66 pp.

The masterly treatise of Professor Krazer provides the mathematical public for the first time with an adequate and satisfactory account of the theta functions. These functions being the elements out of which uniform periodic functions of any number of variables, without finite essential singularities, may be constructed, they afford the most powerful instrument for the investigation of such periodic functions and suffice for a complete theory of them. Periodic functions of the kind specified, including as they do the elliptic, hyperelliptic, and abelian functions, undoubtedly constitute the most important class of special functions at present known.

The sources of information on the theta functions hitherto most readily accessible have been some very excellent chapters included in various books on general or particular function-theory, together with three small volumes containing investigations by Krazer and Prym which have been the direct precursors of the present work. The most recent of these, the Neue Grundlagen einer Theorie der allgemeinen Thetafunktionen, a splendid piece of work in the remodeling of the general theory and extending the treatment to functions with rational characteristics, has for more than a decade done good service in lieu of a text or reference book.

The delay in the appearance of a suitable text in this important subject is compensated for by the admirable treatise now before us. It is in every respect a model. The author has devoted his life to this particular field, and no one is better qualified to write upon it than he. A vast amount of material has been systematized and condensed into a moderate compass. Many of the details, especially those relating to the transformation theory, are of necessity quite intricate, but the entire theory has been presented in such a lucid and attractive form, and the formulas are derived with such directness, simplicity, and absence of artifice, that the efforts of the reader are reduced to a minimum. The reader is materially aided, too, by the remarkable freedom from typographical errors. There are over 2,100 numbered formulas in the book, many of the numbers representing several equations. The reviewer has verified the greater part of these, and has found only the following corrections needed.

On page 4 insert the term $+ k/m$ in the exponent of the last member of (10); on page 8 the last A in (23) should have the subscript 0; on page 36, fourth line from the end of §5, read (XXXIII) instead of (XXIII); on page 41 the first g in (121) should have an accent; on page 154 read

$$B = (N^2M^2NM)^3$$

in (110); on page 164 insert g in the characteristic symbol in (136); on page 165 the c after the first summation sign in (141) should be raised above the level of its pair of subscripts; on page 173 the last x in (163) should have the subscript ν; on page 180 the factor G [cf. (XXXI) and (XXXII) page 181] should appear in the right member of (203); on page 254 interchange "ungeraden" and "geraden" in lines 4 and 5 from the top; on pages 502–3, insert 485, 356, 356, 356, 125 in the references to Biermann, Krazer, Noether, Prym, Wirtinger respectively; on page 502 change 362 to 361 in the reference to Cayley; on page 503 change 23 to 22 in the references to Riemann and Weierstrass.

We proceed to give a brief outline of the subject matter.

The first chapter treats of the p-fold theta series with rational characteristics, and its functional properties.

The next two chapters deal with the general problem of transforming a q-fold infinite series by means of a linear substitution, with rational coefficients, of the q letters of summation. The formulas so obtained are applied to the derivation of a number of relations between theta functions with different arguments and moduli.

The fourth chapter deals with the properties of uniform periodic functions without finite essential singularities, and gives a proof that such functions can always be expressed rationally in terms of theta functions. The complete proof of this theorem, announced by Riemann in 1860, has only recently been obtained. The chief difficulty has been to show that the $2p$ systems of periods of any such periodic function in p variables must always satisfy conditions equivalent to those to which the theta moduli are subject.

On the other hand the abelian functions were at first supposed to form a special class of periodic functions, since such functions of genus $p > 3$ depend on fewer parameters than the general periodic function. That the one class is coextensive with the other is proved in chapter XI, which is devoted to a discussion of the conditions for the reducibility of abelian integrals and the associated theta functions. It is shown that special classes of abelian functions of genus $q > p$ can be found whose associated theta functions break up, after a suitable transformation, into products of two theta functions, one of p and the other of $q - p$ variables, of which the first is a theta function of the most general kind.

Chapter V deals with the most difficult part of the subject, the transformation of the theta functions. It is shown how every transformation may be compounded of certain linear and non-linear transformations. The general formula for a linear transformation in which the determinant of the coefficients $c_{\mu, \nu+p} (\mu, \nu = 1, 2, \cdots, p)$ occupying the second quadrant is not zero, is completely worked out. A transformation in which this determinant is zero is later shown to be expressible as a combination of transformations of the preceding kind.

Chapter VI deals with the complex multiplication of the theta functions.

Chapter VII, containing 130 pages, treats in detail of functions with half-integer characteristics. It deals at some length with Frobenius's theory of groups and systems of characteristics. The Riemann theta formula and various relations obtainable from it are next derived, following which the special formulas for the cases $p = 1, 2$ are deduced and then the addition theorem for $p > 2$, closing with a general discussion of theta relations.

The two succeeding chapters give the familiar Riemann theory of the abelian and hyperelliptic ($p \geqq 2$) theta functions.

The author's systematic method of fully defining all the symbols used in the enunciation of each theorem greatly enhances the value of the book for reference purposes. The historical and bibliographical remarks are given in smaller type at the ends of the sections to which they relate. They are very full and suggestive, and add greatly to the value and completeness of the work.

Two indexes are given, one of the authors quoted and another of the subject matter, while the fourteen-page table of contents furnishes quite a minute analysis and a systematic résumé of the volume.

The Habilitationsschrift of Dr. Rost is a critical revision of the Riemann theory, having for its leading aim to close certain gaps which have heretofore existed. The memoir is divided into two parts of equal length. The first part is preparatory to the second and gives a clear and systematic account of algebraic functions, with especial reference to properties of point systems on the Riemann surface which can serve as infinites for such functions. There is nothing new in this portion of the subject except the form in which it is presented.

The second part gives a complete discussion of the theta function regarded from Riemann's point of view as a function of position in the Riemann surface. The interest centers in the point systems at which the theta function vanishes. To complete the theory in certain respects, in which it is still open to criticism on account of the possibility of the occurrence of point systems of special character, is the chief aim of this part of the paper. This, together with the critical remarks and illustrative examples collected into an appendix of four pages at the end of the work, form a new and useful contribution to the Riemann theory.

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MATHEMATICAL CRYSTALLOGRAPHY.

Mathematical Crystallography and the Theory of Groups of Movements. By HAROLD HILTON Oxford, The Clarendon Press, 1903. 262 pp.

THE problem of crystallography is to establish a correspondence between chemical composition or certain abstract aspects of it and geometric form in crystals.

Setting aside the work, usually assigned to the mineralogist, of reducing the actual forms of crystals to idealized representatives, and the chemist's problem, of establishing the molecular configuration, and showing why given molecules are more likely to be piled in one way than another, the subject matter of mathematical crystallography is the formal problem of showing a correspondence between the idealized forms of the mineralogist and the geometrically possible methods of piling, that is of filling space of three dimensions with interpenetrant homogeneous assemblages. Mr. Hilton's work undertakes to set this out as it has been done by Bravais, Sohncke, Schoenflies and others, and also to supply such geometric material as may be pertinent.

Dealing first with the main issue the definition of subject matter (page 11) amounts to this: The idealized forms of natural crystals are polyhedra with rational indices; more explicitly polyhedra whose faces are parallel to the set of planes

$$\frac{hx}{a}+\frac{ky}{b}+\frac{lz}{c}=0,$$

where a, b, c are any real positive quantities and h, k, l, the "indices," are integers, usually small.