The cases marked $^+$ give

$$\tfrac{1}{2}(u+v) = 1323536760x + 1160932384.$$

Sifting with 23, 37, 41, 53, 61, I find $x = 287$ remaining. A few further tests suffice to show that the goal is near. And, in fact,

$$287.1323536760 + 1160932384 = 381015982504,$$

and

$$\begin{aligned} 2^{67} - 1 &= 381015982504^2 - 380822274783^2 \\ &= 193707721 \times 761838257287. \end{aligned}$$

COLUMBIA UNIVERSITY,
October, 1903.

NOTE ON THE p-DISCRIMINANT OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, August 31, 1903.)

IN a fundamental memoir * Darboux has proved that the resultant $g(x, y) = 0$ obtained by the elimination of $p = dy/dx$ between

$$\phi(x, y, p) = 0 \ \ (1), \quad \text{and} \quad \partial\phi/\partial p \ \ (2),$$

where ϕ is a polynomial in x, y, p, represents in general the locus of the cusps of the integral curves.†

For this purpose Darboux first proves that the resultant obtained by the elimination of p between

$$\phi(x, y, p) = 0 \ \ (3), \quad \text{and} \quad \frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0 \ \ (4)$$

represents in general the locus of the points of inflexion of the integral curves.

By purely geometric reasoning, applying the principle of duality, Darboux then concludes that to this theorem corresponds dualistically the theorem in connection with equations (1) and (2) as stated above.

* "Sur les solutions singulières des équations aux derivées ordinaires du premier ordre." *Bull. des Sciences Math. et Astron.*, vol. 4, pp. 158–176 (1873).

† For an analytic proof see Picard's Traité d'Analyse, vol. 3, pp. 529–534.

In this note I shall prove directly that to the condition (4) corresponds dualistically the condition (2).

Assume the circle

$$x^2 + y^2 = 1,$$

and the transformation by reciprocal polars with respect to this circle. Designating the coördinates of an arbitrary point of the plane of the circle by (x, y) and of its transformed by (x_1, y_1) the relations hold *

$$x = \frac{-p_1}{y_1 - x_1 p_1}, \quad y = \frac{1}{y_1 - x_1 p_1}, \quad p = -\frac{x_1}{y_1} \tag{5}$$

$$(p = dy/dx, \quad p_1 = dy_1/dx_1).$$

By this transformation $\phi(x, y, p) = 0$ is transformed into another linear differential equation of the same order

$$\phi_1(x_1, y_1, p_1) = 0,$$

and

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial y} = 0$$

into an expression resulting next from

$$\frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial x} + p\frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial y} = 0,$$

or

$$\frac{\partial \phi}{\partial \phi_1}\left\{\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial \phi_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x} + \frac{\partial \phi_1}{\partial p_1} \cdot \frac{\partial p_1}{\partial x}\right.$$
$$\left. + p\left(\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial \phi_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial y} + \frac{\partial \phi_1}{\partial p_1} \cdot \frac{\partial p_1}{\partial y}\right)\right\} = 0. \tag{6}$$

But according to (5)

$$\frac{\partial x_1}{\partial x} = -\frac{p^2}{(y - xp)^2}, \quad \frac{\partial y_1}{\partial x} = \frac{p}{(y - xp)^2}, \quad \frac{\partial p_1}{\partial x} = -\frac{1}{y},$$

$$\frac{\partial x_1}{\partial y} = \frac{p}{(y - xp)^2}, \quad \frac{\partial y_1}{\partial y} = -\frac{1}{(y - xp)^2}, \quad \frac{\partial p_1}{\partial y} = \frac{x}{y^2}.$$

Substituting these values in (6) and reducing, we get from (6)

$$\frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial p_1}\left(-\frac{1}{y} + p\frac{x}{y^2}\right) = 0. \tag{7}$$

* S. Lie, Berührungstransformationen, vol. 1, p. 23.

But generally $\partial\phi/\partial\phi_1$ and $-1/y + px/y^2$ do not vanish, so that (7) reduces to the condition $\partial\phi_1/\partial p_1 = 0$. Hence the theorem:

To the simultaneous equations

$$\phi(x, y, p) = 0, \quad \frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0$$

correspond dualistically, the equations

$$\phi_1(x_1, y_1, p_1) = 0, \quad \partial\phi_1/\partial p_1 = 0.$$

The same result would be obtained by operating with the analytic expressions for the most general dualistic transformation. The special transformation (5) was chosen for the sake of simplicity.

UNIVERSITY OF COLORADO,
August, 1903.

HYDRODYNAMIC ACTION AT A DISTANCE.

Vorlesungen über hydrodynamische Fernkräfte nach C. A. BJERKNES'S *Theorie.* Von V. BJERKNES. 2 Bände, 8vo. Leipzig, J. A. Barth, 1900–1902. Bd. I, xvi + 338 pp., 40 figs. Bd. II, xvi + 316 pp., 60 figs.

C. A. BJERKNES is dead. The news is scarcely yet spread over the scientific world. No more fitting time could be found for calling attention to his life-work on hydrodynamic action at a distance. Pupil of Dirichlet at Göttingen, professor of mathematics and physics at the university in Christiania, ardent admirer and follower of Faraday and Maxwell, Bjerknes more than twenty years ago had developed practically to completion a theory which never has received much attention owing to the manner of its publication. That the work is now before the public in a complete, accessible, and easily intelligible form is due to the editorial patience of the son, V. Bjerknes.

The first volume, designed primarily for mathematicians, contains the theoretical development of the mutual actions of pulsating and oscillating spheres immersed in a common incompressible perfect fluid. Fortunately the mathematical analysis is so elementary and so carefully explained that the most meager training amply suffices for its comprehension. In