I conclude this review by expressing the hope that the important new views on the foundations of geometry opened up in this memoir may soon become generally known and be introduced into the teaching of elementary geometry.

J. SOMMER.

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## KOENIGS' LECTURES ON KINEMATICS.

*Leçons de Cinématique* professées à la Sorbonne par GABRIEL KOENIGS, avec des notes par M. G. DARBOUX et par MM. E. et F. COSSERAT. Paris, Hermann, 1897. 8vo., x + 499 pp.

WITH this book Professor Koenigs begins the publication of a treatise consisting of two or three volumes, which is to present the development of a course of lectures on kinematics delivered annually either at the École Normale or the Sorbonne for the last eight years. The first volume, the first ten chapters of which were printed in 1894, is devoted to theoretical kinematics ; the rest of the work will be occupied with applied kinematics.

Kinematics as a distinct science is of comparatively recent origin. The formulæ which give the variations of the coördinates of the points of a movable solid in space were published by Euler in 1750. D'Alembert suggested the importance of studying the laws of movements separately. Ampère drew a definite demarcation between mechanics and the geometry of movement, but his object was to develop kinematical science solely for its use in the theory of mechanisms ; the term kinematics is due to Ampère. Previously, in his geometry of position, Carnot predicted a much wider career for this science then to be, by calling attention to the fact that mechanics and hydromechanics would be infinitely simplified if the theory of geometrical motions were thoroughly investigated, since then the analytic difficulties encountered in the study of equilibrium and motion would be reduced to the general principle of the communication of motions, which is only another form of the principle of action and reaction. In 1838 Poncelet included the geometric properties of moving bodies in his course at the Faculty of sciences of Paris ; with the exception of the notions of Chasles, we owe to Poncelet the theory of the continuous motion of a solid in space. Willis, of

Cambridge University, published a treatise on kinematics in 1841, and twenty-one years later Resal's treatise on pure kinematics appeared.

The present form assumed by this body of doctrine in the exposition of mathematical science in France is due largely to the lectures of Bouquet, Darboux, and Tannery, who have treated kinematical questions with that absolute precision and unflinching rigor demanded by mathematics. The influence of Tannery's course, given for several years at the Faculty of sciences of Paris, is especially notable; here we find the time relegated to its proper rôle as an auxiliary variable, and kinematics itself interpreted as the geometry of displacement.

Koenigs' treatise promises to be a classic. In facing the formidable array of researches and methods of Poinsot, Chasles, Bonnet, Ribaucour, Darboux, and a host of others the author must have experienced no little difficulty in choosing a method of exposition of the subject matter. A treatise that is to be both scientific and didactic must consider the demands both of the student and of the scientist. Knowledge is of most use when accompanied by a kit of tools, and the instructor's art is a double one—he must not only present the facts of a subject with reasons for faith in them, but also employ those methods which promise results at the hands of the independent investigator. Further, the method should have the necessary breadth and unity, and be possessed of the clearness and directness of geometry and the power and generality of analysis. This multiplicity of demands Professor Koenigs has met admirably by basing his exposition on the geometry of the straight line and employing the mobile trieder of reference ; the latter in the hands of Ribaucour and Darboux has proved itself to be the most certain and powerful implement yet used in infinitesimal geometry and it naturally lends itself with equal facility and elegance to the geometry of displacement.*

The first chapter is taken up with preliminary geometric notions, principally the theory of segments. The author regrets in his preface that the latter theory has not yet found a place in elementary mathematics. Its introduction in a work on kinematics is as annoying as that of kinematics in a treatise on dynamics. The theory belongs to geometry and, as Koenigs pertinently remarks, the business

* It may be remarked that the English writers on mechanics have also used the method of moving axes for a long time with success ; see, for example, Routh's treatise on rigid dynamics.

of the geometer is not alone to discover new facts, but also to classify results already found and to group together ideas of the same kind. The chapter considers in turn projections, moments, systems of segments, couples, the theory of the right line, and concludes with a beautiful application of the theory of complexes to systems of segments.

The notions of motion, velocity, and acceleration are presented in the second chapter. The velocity is defined both as an algebraic and as a geometric function of the time. The hodograph is introduced to define the acceleration and deduce its properties. Among the applications are harmonic, uniform helicoidal, and uniformly varied rectilinear motions.

The third chapter is devoted to the change of the system of reference and the study of relative motion. The fundamental relation between the absolute velocity, the relative velocity, and the velocity of restraint is followed by the derivation of the fundamental formulæ of kinematics which express the projections of the absolute velocity on the axes of the mobile trieder. These are applied to the composition of velocities and lead directly to the theorems of Poinsot and Roberval for the construction of tangents and normals.

The author gives an interesting historical note relative to Roberval's method for constructing tangents. It was discovered in 1636 and communicated by Roberval to Fermat in 1640. It was published in the sixth volume of the memoirs of the Paris academy of sciences. Roberval's statement of the theorem is incorrect; this error was reproduced by Montucla in his history and by Monge in his descriptive geometry; the latter also gave faulty applications. These errors were rectified by Duhamel in a note inserted in the fifth volume of the *Savants étrangers*.

In concluding the chapter Koenigs shows how the composition of velocities may be utilized for the construction of tangents to conchoids of the most general form.

The fourth chapter analyzes in detail the nature of the motion of one invariable system with respect to another; it is one of the most successful parts of the volume. Its sections bear the captions: distribution of velocities in a moving solid, helicoidal motion, direct demonstrations of the helicoidal form of every infinitesimal displacement, relations between the theory of linear complexes and that of the displacement of a solid, determination of a continuous motion when the rotations are known, and inverse movement. The reviewer regrets that he has space only to call attention to the author's presentation of the geometric method of Chasles which has been treated with fulness in the work of Schoenflies, to the analogy between the ideas of Chasles and the method of infinitesimal transformations of Sophus Lie, and to the application to space curves of the method of the mobile trieder as developed analytically in the theory of surfaces of Darboux.

The fifth chapter defends Koenigs' choice of method against all comers; in less than seven pages we find the complete theory of acceleration in relative motion developed, the fundamental formulæ of Bour established and interpreted geometrically, the theorem of Coriolis demonstrated, and the distribution of accelerations in a moving solid studied.

The sixth and seventh chapters are devoted to the motion of a plane figure in its plane, the latter being taken up largely with examples. Among the many details of these two chapters we observe, with interest, Koenigs' generalization of Savary's formula, the study of trajectories in the vicinity of a point, the simple presentation of the principle of planimeters, and Koenigs' generalization of Steiner's theorem relative to roulettes.

Of the various movements of a body, two are particularly important: the first is that in which a plane of the body slides upon itself; the second is that in which a spherical shell of the body slides upon itself, or what amounts to the same thing, in which the body has a fixed point. The two preceding chapters considered the first case; the eighth chapter studies the second. The theorem of Rivals and the analogue of Savary's theorem are established by the same method employed in the case of plane motion; the formulæ of Euler and Olinde Rodrigues terminate the chapter.

The ninth chapter considers the most general continuous motion of a solid body. Koenigs studies first the curves connected with a moving figure which are possessed of envelopes and shows by the method of Darboux that these curves can be obtained by quadratures. In the most general motion of an invariable figure there is always a ruled surface $\Delta_m$ which joins at each instant a ruled surface $\Delta_f$ along a rectilineal generator $\Delta$, which is found to be the axis of the tangent helicoidal motion; this particular rolling of $\Delta_m$ upon $\Delta_f$, Reuleaux designated by the term viration; the author studies the distinctive character of the viration in the general rolling of ruled surfaces. He considers the applicability and deformation of ruled surfaces, the rolling of developables and space curves, and concludes with certain propositions relative to the acceleration in the motion of a solid body.

The heading : degrees of liberty of a moving system, movements with several parameters, gives the key to the tenth chapter. The various sections are occupied with the degrees of liberty of a solid body, the motion of a body subject to four conditions, and movements dependent on three parameters.

The eleventh chapter devoted to articulate systems is one of the most extensive of the volume. It is introduced with a valuable historical sketch recounting the researches* of Scheiner, Watt, Peaucellier, Kempe, Hart, Lipkine, Tchebicheff, Sylvester, Clifford, Roberts, Cayley, Saint-Loup, Laisant, Lemoine, and Darboux. The general reader may be interested in Neuberg's résumé of his elementary conferences on linkwork and Liguine's bibliography† to which the author refers. This chapter is admirably constructed and is enriched by Koenigs' own contributions which have been of both a practical and theoretical kind. The material is arranged in the following order—systems having four members, link transformers, exact or approximate determination of the rectilineal motion of a point, motions and transformations which can be realized by the same linkwork, applications of linkworks to the resolution of equations and the representation of functions, linkworks in space, general theorems on linkworks ; among these last mentioned theorems there appears the very remarkable one of the author that every algebraic space curve or surface can be described by means of linkwork.

The last chapter, on displacement as a particular case of homography, introduces a new chapter in kinematics and is of itself a superb geometrical memoir. The reviewer must content himself however with a mere index of the sections —movement of a plane figure, displacement of a figure in space, imaginaries in the kinematics of the plane, linear substitutions of one variable and displacements about a fixed point.

The volume is happily supplemented by the notes of MM. Darboux, E. and F. Cosserat, and the author.

In his first note Professor Darboux presents a new demonstration of the formulæ of Euler and Olinde Rodrigues, which is direct and yields an immediate interpretation of the parameters.

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* The names follow in the order taken up by Koenigs.

† *Bulletin des sciences mathématiques*, 2d series, vol. 7.

The second note of Darboux studies the reversions and plane inversions. By reversion (*renversement*) Darboux means the rotation of a figure through 180° about a straight line in space, and by plane inversion a symmetry with regard to a plane. These two transformations simplify the study and composition of displacements.

Darboux's third note is an elaborate memoir on algebraic movements. It consists of three divisions: movement of which all the trajectories are plane, use of the formulæ of Olinde Rodrigues in the study of algebraic movements, motions in which certain points of the body describe planes or plane curves.

The note of MM. Cosserat is an extract of a memoir published in the *Toulouse Annales* on the kinematics of a continuous medium. The introduction of this note is peculiarly fortunate for it is high time that kinematics should comprehend the study of deformation and of deformable spaces. The authors have included in their extract certain generalities on curvilinear coördinates, the deformation of a continuous medium in general, infinitely small deformation, use of the mobile trieder, and the case where the non-deformed medium is referred to any curvilinear coördinates.

Koenigs' own notes are eleven in number and on the following subjects: tetraedral coördinates of segments, the theory of Grassmann, infinitisimal properties of linear complexes, the expression of the virtual work of forces applied to a solid body, the volumes generated by a closed contour, the problem of centers of curvature in the movement of a plane figure, accelerations, Ball's theory of screws, the cylindroid, the composition of rotations and quaternions, and graphical representation.

The mechanical execution of the book is excellent and the marginal headings and leaders are especially useful. The mathematical public would welcome an announcement from M. Hermann that the second volume of Koenigs' work is ready.

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