12. Apart from their properties as transformations, the above transformations are of interest because of certain applications to plane curves, notably to spirals which it is hoped to bring out in a subsequent note.

Since finishing this note the writer finds that the *finite* forms of the transformations discussed were given by Laisant in the *Nouvelles Annales de Mathématiques*, 2d series, vol. 7 (1868), p. 318, in the solution of a problem proposed by Haton de la Goupillière, *Nouvelles Annales*, vol. 6 (1867), problem No. 803. The wide divergence between the properties and the points of view of the present note and the solution referred to seem to warrant its presentation to the Society. The above-mentioned volumes of the *Nouvelles Annales* are to be had in the Library of Congress.

BALTIMORE,
14 *April*, 1897.

CONTINUOUS GROUPS OF CIRCULAR TRANSFORMATIONS.*

BY PROFESSOR H. B. NEWSON.

(Read before the American Mathematical Society, at the Meeting of April 24, 1897.)

THE object of this paper is to present the outlines of a fairly complete theory of the continuous groups of linear fractional transformations of one variable. The method employed is quite different from the methods of Lie. Lie's classic theory is based upon the infinitesimal transformation; I shall make but little use of the infinitesimal transformation, but shall develop the subject from the consideration of the essential parameters of the transformation. The complex plane is chosen because it beautifully illustrates the methods. I have put together some old and some new facts and have sought to build up a general theory.

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* Several terms have been proposed to designate the linear fractional transformations of the complex plane. Möbius introduced the term "Kreisverwandtschaft." Mathews' Theory of Numbers, page 107, translates this as "Möbius' Circular Relation." Professor Cole, in *Annals of Mathematics*, vol. 5, page 137, refers to "Orthomorphic Transformation," following Cayley; this seems too general for the special case here considered, since it is applicable to all conformal transformations. Darboux, in his Theorie des Surfaces, vol. 1, page 162, uses "transformation circulaire." It seems to me that "Circular Transformation" is the best yet proposed, for the fundamental property is expressed in the name.

A circular transformation T of the complex plane is represented by

$$z_1 = \frac{az+b}{cz+d}.$$

Regarding the straight lines of the plane as circles through the one point at infinity, the fundamental property of this transformation is that it transforms circles into circles. It interchanges among themselves the circles of the plane, but leaves unchanged or invariant the configuration composed of the ∞^3 circles of the plane. (See Forsyth's Theory of Functions, pages 512–524.)

Since any two circular transformations T and T_1 each leave invariant this configuration of all circles of the plane, their product, *i. e.*, the transformation which is equivalent to the successive application of the two, must likewise leave the same configuration invariant and hence be a circular transformation.

This conclusion may be verified analytically by eliminating z_1 from two circular transformations T and T_1 as follows:

$$(1) \qquad z_1 = \frac{az+b}{cz+d}, \text{ and } z_2 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}.$$

The product of T and T_1 is T_2 given by

$$(2) \qquad z_2 = \frac{(a_1 a + b_1 c)z + (a_1 b + b_1 d)}{(c_1 a + d_1 c)z + (c_1 b + d_1 d)}.$$

This has the same form as T and therefore is a circular transformation.

But this is just what is known in modern mathematical language as the *group property*. Hence all circular transformations of the complex plane form a group.

If the coefficients a, b, c, d in (1) be made to vary continuously, all the resulting transformations belong to the above group; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in (1). Such a group is called a *continuous* group. The question of continuity will be more fully discussed later.

The circular transformation T can usually be brought to the *normal form**

* Mathews: Theory of Numbers, p. 105.

$$(3) \qquad \frac{z_1 - m}{z_1 - n} = k \frac{z - m}{z - n}$$

Where m and n are the roots of the quadratic equation $cz^2 + (d - a)z - b = 0$, and

$$k = \frac{\left(a + d - \sqrt{(a + d)^2 - 4(ad - bc)}\right)^2}{4(ad - bc)}.$$

m, n and k are called the *essential parameters* of the transformation, m and n are the two invariant points and k is the multiplier of the transformation. Since

$$k = \frac{z_1 - m}{z_1 - n} : \frac{z - m}{z - n},$$

we infer that k is the anharmonic ratio of the two invariant points and any pair of corresponding points in the transformation. Since m, n and k are complex quantities of the form $a + ib$, it follows that T involves six real variable parameters.

When the invariant points m and n coincide, T can no longer be brought to the above normal form but is then reducible to a *second normal form*

$$(4) \qquad \frac{1}{z_1 - m'} = \frac{1}{z - m'} + \alpha.$$

The condition for coincident invariant points is $(a + d)^2 = 4(ad - bc)$.

$$m' = \frac{a - d}{2c} \text{ and } \alpha = \frac{2c}{a + d}.$$

α is a constant whose properties are to be determined. When the condition for two coincident invariant points is substituted in the equation for k, we get $k = 1$. Hence the characteristic anharmonic ratio of a transformation T' of this kind is unity. Every circular transformation can be brought to the one or the other of these normal forms.

Let us consider two transformations T and T_1 which have no invariant point in common. Their equations are

$$(3') \qquad \frac{z_1 - m}{z_1 - n} = k \frac{z - m}{z - n} \quad \text{and} \quad \frac{z_2 - m_1}{z_2 - n_1} = k_1 \frac{z_1 - m_1}{z_1 - n_1}.$$

Eliminating z_1 we have the product T_2 in the form

$$(3'') \qquad \frac{z_2 - m_2}{z_2 - n_2} = k_2 \frac{z - m_2}{z - n_2}$$

where k_2, m_2, and n_2 are given as follows:

$$(5)\ \begin{cases} \dfrac{k_2+1}{\sqrt{k_2}} = \dfrac{(kk_1-k-k_1+1)}{\sqrt{kk_1}}\lambda + \dfrac{k+k_1}{\sqrt{kk_1}}; \\[2ex] m_2 = \dfrac{k_2 D + A}{C(k_2+1)}; \quad n_2 = \dfrac{k_2 A + D}{C(k_2+1)}. \end{cases}$$

A, D, C and λ stand for the following expressions:

$$(6)\ \begin{cases} A = kk_1n_1(n-m_1) - km_1(n-n_1) - k_1n_1(m-m_1) \\ \qquad + m_1(m-n_1). \\ D = kk_1m(n-m_1) - km(n-n_1) - k_1n(m-m_1) \\ \qquad + n(m-n_1). \\ C = kk_1(n-m_1) - k(n-n_1) - k_1(m-m_1) + (m-n_1). \\ \lambda = \dfrac{n_1-m}{m_1-n_1} : \dfrac{n-m}{m_1-n}; \end{cases}$$

i. e., λ is one of the anharmonic ratios of the four invariant points (mnm_1n_1).

The transformation T_2 is not independent of the order of the components T_1 and T; the value of k_2 is independent of the order of T_1 and T for λ is unaltered when m and n are interchanged with m_1 and n_1 but not so with m_2 and n_2. Hence the two transformations T and T_1 are non-commutative.

The results obtained may be formulated as follows:

THEOREM 1. *All circular transformations of the complex plane form a six parameter continuous group. The transformations of the group are non-commutative.*

Our task is now to enumerate and discuss all the subgroups of this six-parameter group, to develop their properties and to classify them according to their most characteristic properties.

Lie expounds in "Continuierliche Gruppen," page 113 an axiomatic principle which, for the purposes of this paper, is best stated in the following form: *All point transformations of the plane which leave invariant a certain figure or configuration in the plane have the group property.* The group may be either a continuous or a discontinuous group. A good example of the latter is the group of 18 linear transformations of the plane cubic into itself. We shall make frequent use of this principle in what follows; but in each case the group obtained will be seen to contain one or more continuously varying parameters and is therefore a continuous group.

According to Lie's principle all transformations leaving invariant a single point m form a group. Since there are ∞^2 points in the plane, the ∞^6 transformations of the six-parameter group G_6 are distributed into ∞^2 subgroups, one for each point. Accordingly each such subgroup should contain ∞^4 transformations and be a four-parameter group G_4 or G_m. This group leaves invariant not only the point m but also the net of circles through m. The circles of the net are interchanged among themselves, but the net as a whole is unaltered.

If we make $m_1 = m$ in (3′), (5), and (6), we find $\lambda = 1$, $k_2 = kk_1$, and $m_2 = m$. If m be a fixed point, the equations (3′) or (3″) contain only two essential variable parameters k and n and, hence, four real variable parameters; thus, it is shown analytically that the group G_m is four-parameter. The fact that $k_2 = kk_1$ is very important as will be seen later. The transformations of the group are non-commutative, for n_2 is not independent of the order of T and T_1.

THEOREM 2. *All transformations leaving a point m invariant form a four-parameter group. The law of combination of the essential parameters k of this group is given by $kk_1 = k_2$. The transformations of the group are non-commutative.*

Again by Lie's principle all transformations leaving invariant two distinct points m and n form a group. Since there are ∞^4 such pairs of points, the transformations of the six-parameter group G_6 are distributed into ∞^4 subgroups each of which contains ∞^2 transformations and is a two-parameter group. It is clear that each four-parameter group G_m contains ∞^2 of these two-parameter groups G_2 or G_{mn}, one corresponding to each point of the plane taken with the fixed point m. Such a two-parameter group leaves invariant not only the points m and n but also the pencil of circles through these points.

If we make $m_1 = m$ and $n_1 = n$ in (3′), (5) and (6), we get $\lambda = 1$, $k_2 = kk_1$, $m_2 = m$ and $n_2 = n$. Or we may eliminate z_1 from (3′) and get (3″) by multiplication and thus get directly that $k_2 = kk_1$. T in this case has only one essential variable parameter k and, hence, only two real variable parameters. Thus again the group G_{mn} is shown to be two-parameter. Since k_2 is independent of the order of k and k_1, the transformations of this group are commutative.

THEOREM 3. *All transformations leaving invariant two points m and n form a two parameter group. The law of combination of the essential parameters of this group is expressed by $k_2 = kk_1$. The transformations of the group are commutative.*

Thus far we have considered only transformations of the type T with two invariant points. The normal form of T'

(4) contains only two essential parameters m' and a and hence only four real variable parameters. There are therefore ∞^4 such transformations in the plane. Taken together do they form a group? This is readily answered in the negative. The product of two transformations T' and T_1'

$$(4') \qquad \frac{1}{z_1-m'}=\frac{1}{z-m'}+a, \text{ and } \frac{1}{z_2-m_1'}=\frac{1}{z_1-m_1'}+a_1,$$

is a transformation of the kind T with two distinct invariant points, as may easily be verified by eliminating z_1 from (4′).

But if the two transformations T' and T_1' leave invariant the same point m', they do form a group. Making $m_1'=m'$ in (4′) and eliminating z_1 by addition, we get

$$(4'') \qquad \frac{1}{z_2-m'}=\frac{1}{z-m'}+a_2;$$

where $a_2=a+a_1$. The group is evidently two-parameter and its transformations are commutative. Let it be symbolized by $G_{m'}'$ or G_2'.

THEOREM 4. *All transformations of the kind T' leaving a single point invariant form a two-parameter group. The law of combination of the essential parameters is expressed by $a_2=a+a_1$. The transformations of the group are commutative.*

The relationships of the groups thus far determined may be symbolized as follows:

$$G_6=\infty^2\, G_4=\infty^4\, G_2+\infty^2\, G_2'. \qquad G_4=\infty^2\, G_2+G_2'$$

or

$$G_6=\infty^2\, G_4=\infty^2\,(\infty^2\, G_2+G_2').$$

We now go on to examine more closely the two-parameter group G_{mn} and shall show that the transformations composing it can be distributed into one-parameter subgroups. The essential parameter k of the group G_{mn} may be written $k=\rho e^{i\theta}$. Here ρ and θ are independent parameters and may vary independently. If we put $\rho=e^{c\theta}$, where c is some constant quantity, we have $k=e^{c\theta}\cdot e^{i\theta}=e^{(c+i)\theta}$. Since in the group G_{mn} $k_2=kk_1$, we have $k_2=e^{(c+i)\theta}\cdot e^{(c_1+i)\theta_1}=e^{c\theta+c_1\theta_1}\cdot e^{i(\theta+\theta_1)}$, where c and θ are two independent parameters. But if $c=c_1$, we have $k_2=e^{c(\theta+\theta_1)}\cdot e^{i(\theta+\theta_1)}=e^{(c+i)\theta_2}$, where $\theta_2=\theta+\theta_1$. Here we have the conditions for a one-parameter group; k_2 is of the same character as k and k_1, and there is but one parameter, viz: θ. The effect of the successive transformations of the group upon a point P of the plane is to transform it into $P_1, P_2, \cdots P_n$ which lie on a curve called by Lie the Bahn-curve or path curve. By a transformation T every point P is moved along its path curve. The path curves of this group are the well-known double spirals of Holzmüller* used so extensively by Klein and others. Their properties are so well known that it is unnecessary to develop them here. For a lucid account in English see a paper by Professor F. N. Cole in *Annals of Mathematics*, vol. 5, page 121.

Within the group G_{mn} there is a sub-fold group for every real value of c and a corresponding set of double spiral path curves.

THEOREM 5. *The transformations of the group G_{mn} are distributed into ∞^1 one-parameter subgroups. The path curves of these subgroups are double spirals about the invariant points m and n.*

The chief properties of one of these one-parameter subgroups are easily deduced from the expression for the parameter, $k = e^{(c+i)\theta}$. When $\theta = 0$, $k = 1$; when the anharmonic ratio of the four points $(mnzz_1) = 1$, z coincides with z_1. Every point of the plane is unaltered by such a transformation, which is called the identical transformation of the group. The two transformations corresponding to values of θ numerically equal but of opposite signs are called inverse transformations. Their product is the identical transformation of the group. When $k = \infty$ or 0, all points of the plane are transformed respectively to the invariant points m or n. I have elsewhere called these pseudo-transformations† (*ausgeartete Transformationen*, Lie).

Within the two-parameter group G_{mn} are two one-parameter subgroups of special importance; these are the groups for which $c = 0$ and $c = \infty$ respectively; *i. e.*, for which $|k| = 1$ and for which k is real. In the first case, for which $k = e^{i\theta}$, the path curves reduce to coaxial circles having m and n for vanishing points. All transformations of this one-parameter group are elliptic. In the second case, when k is real, the path curves reduce to a pencil of circles through m and n. The transformations of this group are all hyperbolic. The other one-parameter subgroups of G_{mn} are made up chiefly of loxodromic transformations.

THEOREM 6. *Every two-parameter group G_{mn} contains one one-parameter subgroup of elliptic transformations, one one-parameter subgroup of hyperbolic transformations, and ∞^1 one-parameter subgroups of loxodromic transformations. For each group of loxodromic transformations c in the formula $k = e^{(c+i)\theta}$ is a constant.*

The continuity of the two-parameter group G_{mn} is based

* See Holzmüller: Isogonale Verwandtschaften, § 19.

† *Kansas University Quarterly*, vol. 5, page 79.

upon the continuity of the complex number system; for there is a transformation of the group corresponding to every value of k, which is a complex number. Let the values of k be represented as usual by the points of a complex plane (not to be confused with the plane of our operations). We wish to see how the values of k which give transformations belonging to a one-parameter subgroup are distributed in the plane. We have $k = e^{(c+i)\theta}$ where c is a constant and θ is variable. The locus of the point k satisfying this equation is a logarithmic spiral about the zero point cutting the axis of real numbers at an angle whose cotangent is c. This is a continuous curve from the zero point to the infinity point, and consequently our one-parameter subgroup is a continuous group.

Different values of c give us different spirals each of which corresponds to a one-parameter subgroup of G_{mn}. c varies continuously through all real values from $-\infty$ to $+\infty$ so that these spirals lie infinitely close to one another and, as we shall see, cover twice over the entire plane. These spirals all pass through the unit point. For $c = 0$ the corresponding spiral becomes the unit circle; for $c = \infty$ the spiral reduces to the straight line which is the axis of real numbers. The family of spirals for which c is positive fills the entire plane and no two of them intersect except in the unit point. The same is true of the family of spirals for which c is negative. But every spiral of one family intersects an infinite number of times every spiral of the other family. Every point in the plane not on the unit circle or the axis of real numbers lies on two of these spirals; from which we infer that every loxodromic transformation of the group G_{mn} belongs to two distinct one-parameter subgroups. Every hyperbolic transformation in G_{mn} except the involutoric transformation, for which $k = -1$, belongs to three one-parameter subgroups; for two spirals and the axis of reals pass through every point for which k is real. The elliptic transformations in G_{mn} belong only to the elliptic subgroup. The involutoric transformation is common to the hyperbolic and elliptic subgroups. The identical transformation is common to all subgroups; and the two pseudo-transformations are common to all except the elliptic subgroup. Two loxodromic subgroups for which the c's have the same signs have no transformations in common other than the identical and the two pseudo-transformations; while two loxodromic subgroups for which the c's have opposite signs have in common an infinite number of discrete transformations.

THEOREM 7. *Every one-parameter subgroup in G_{mn} is continuous. Every loxodromic transformation in G_{mn} belongs to two distinct subgroups. Every hyperbolic transformation in G_{mn}, except the involutoric transformation, belongs to three distinct subgroups.*

This same geometric representation enables us to discuss intuitively the generation of finite transformations by the repetition of an infinitesimal transformation. Every spiral passes through the unit point, and corresponding to the two points on the spiral adjacent to the unit point we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = e^{+(c+i)\delta\theta}$ and $k = e^{-(c+i)\delta\theta}$. The identical transformation divides the one-parameter group into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of a one-parameter loxodromic group can be generated by the repetition of the corresponding infinitesimal transformation. In the elliptic group, for which the spiral reduces to a circle, every transformation can be generated from either elliptic infinitesimal transformation. In the hyperbolic group, for which the spiral reduces to the axis of real numbers, the transformations for which k is negative can not be generated by the repetition of either of the hyperbolic infinitesimal transformations. Every loxodromic transformation in G_{mn} can be generated from either of two distinct infinitesimal transformations, for every loxodromic transformation belongs to two distinct subgroups. Every hyperbolic transformation for which k is positive can be generated from three infinitesimal transformations; while every hyperbolic transformation for which k is negative, except the involutoric transformation, can be generated from two distinct loxodromic transformations.

THEOREM 8. *Every hyperbolic transformation in G_{mn} for which k is positive can be generated from three distinct imfinitesimal transformations; every other transformation in G_{mn} can be generated from two distinct infinitesimal transformations.*

The two-parameter group G_m' likewise contains ∞^1 one-parameter subgroups. The law of combination of the parameters in this group is $\alpha_2 = \alpha + \alpha_1$, or in another form $r_2e^{i\theta_2} = re^{i\theta} + r_1e^{i\theta_1}$. If now we take $\theta_1 = \theta$, this becomes $r_2e^{i\theta} = (r + r_1)e^{i\theta}$. We have here the conditions for a one-parameter group; α_2 is of the same form as α and α_1 and contains only one parameter r. It is clear that we have a one-parameter group for every value of θ. The effect of successive applications of transformations of one of these one-parameter groups on a point in the plane is to move it along a path curve. The path curves of one of these groups consists of the system of circles tangent at m to each other and to the line through m which makes with the axis of reals an angle θ. All transformations of the group G_m' are parabolic. For details see the above mentioned paper by Professor Cole.

The properties of one of these one-parameter groups are easily determined. Let $a = re^{i\theta}$; when $r = 0$, we have the identical transformation of the group; the two transformations corresponding to two values of r equal but with opposite signs are inverse transformations. When $r = \infty$, all points of the plane are transformed to m and we have a pseudo-transformation. There are two infinitesimal transformations in the group given by the values $+dr$ and $-dr$. Each infinitesimal transformation generates its corresponding portion of the group. Two one-parameter subgroups of G_m' have no transformation in common except the identical and the pseudo-transformation; these are common to all subgroups of G_m'.

THEOREM 9. *All transformations of the two-parameter group G_m' are parabolic and are distributed into ∞^1 one-parameter subgroups. The path curves of a one-parameter subgroup are circles through m, touching each other at m.*

We have already shown how the four-parameter group G_m breaks up into ∞^2 two-parameter groups G_{mn}. We shall now show that the transformations of G_m may be distributed into ∞^1 three-parameter subgroups. The law of combination of the parameters k within the group G_m is expressed (Theorem 2) by $kk_1 = k_2$. Written in another form this is $e^{(c+i)\theta} \cdot e^{(c_1+i)\theta_1} = e^{c\theta+c_1\theta_1} e^{(\theta+\theta_1)i}$. If $c = c_1$, we have

$$e^{(c+i)\theta} \cdot e^{(c+i)\theta_1} = e^{(c+i)\theta_2} \text{ where } \theta_2 = \theta + \theta_1.$$

Hence, we see that if we chose from each two-parameter group G_{mn} the one-parameter group characterized by a certain value of c, the totality of the transformations comprised in these ∞^2 one-parameter groups forms a three-parameter group. It is clear at once that there is one such three-parameter group for every value of c. Thus, for example, all the elliptic transformations contained in G_m form a three-parameter subgroup. The same is true of all hyperbolic transformations.

THEOREM 10. *The ∞^3 transformations having a common invariant point at m, and for which the value of c in the formula $k = e^{(c+i)\theta}$ is the same, form a three-parameter subgroup of the four-parameter group G_m.*

A very important special case of G_n remains to be noted, viz: when the invariant point n is at infinity. All transformations of the group leave invariant the net of circles through the point at infinity. But this net of circles is the net of all straight lines in the plane. Thus the transformations of this group transform straight lines into straight lines; they are therefore projective transformations. These transformations retain the common property of all circular transformations that angular magnitudes are unchanged. The four-parameter group G_∞ is therefore identical with the projective group of similitude, whose invariant figure is the line at infinity and the two circular points.

This result can also be shown analytically. Let $n = \infty$ in equation (3), whence we have

$$z_1 - m = k(z - m). \tag{7}$$

Equating real and imaginary parts we get

$$\begin{aligned} x_1 &= k'x - k''y - k'm' + k''m'' + m', \\ y_1 &= k''x + k'y - k'm'' - k''m' + m''. \end{aligned} \tag{8}$$

This is a projective transformation the vertices of whose invariant triangle are the point (m', m'') and the two circular points.

The subgroups of G_∞ give some interesting results. The path-curves of a one-parameter subgroup of loxodromic transformations are logarithmic spirals* around the point m, and the constant of the group c in $k = e^{(c+i)\theta}$ is the cotangent of the angle between the curve and the radius vector. The path-curves of the one-parameter group of elliptic transformations are concentric circles about m; and the path-curves of a one-parameter group of hyperbolic transformations are straight lines through m.

Within G_∞ all loxodromic transformations with constant c form a three-parameter subgroup of logarithmic spiral motions with constant angle ψ. All elliptic transformations in G_∞ form the three-parameter group of all rotations in the plane. All parabolic transformations in G_∞ form the two-parameter group of all translations in the plane. Together all elliptic and all parabolic transformations in G_∞ form the three-parameter group of all Euclidian motions in the plane. All hyperbolic transformations in G_∞ form the three-parameter group of all *affine* transformations (*i. e.*, dilations) of the plane.

* Klein: Modulfunctionen, vol. 1, p. 168.

THEOREM 11. *All circular transformations leaving the point at infinity invariant are projective transformations, and the four-parameter group* G_∞ *is identical with the four-parameter projective group in the plane whose invariant figure is the line at infinity and the two circular points.*

It seems to be a favorite method with Klein to express whenever possible projective groups in terms of the complex variable both in the plane and on the Neumann sphere; see for example Nicht-Euclidische Geometrie, vol. 2, page 184 ff and Höhere Geometrie, vol. 2, page 229 ff, and many other places.

We come now to the consideration of another group type of great importance. According to Lie's principle all transformations leaving a circle invariant form a group. Consider first a group of hyperbolic transformations leaving invariant m and n and every circle of the pencil through m and n. Choose one of these circles C and another point n_1 on C. The group of hyperbolic transformations with invariant points at m and n_1 also leaves C invariant. Thus all hyperbolic transformations having one invariant point at m and the other also on C leave C invariant and form a two-parameter group. In this two-parameter group is included the one-parameter parabolic group whose invariant point is m and whose invariant line is the tangent to C at m. In like manner there is a two-parameter group for every point on C. All the transformations contained in these ∞^1 two-parameter groups form a three-parameter group leaving C but no point on C invariant. ∞^2 of these transformations are parabolic; these are distributed into ∞^1 one-parameter groups, but taken together do not form a two-parameter group.

There are also ∞^3 elliptic transformations which leave C invariant. Let m be any point within C and n its inverse point with respect to the circle C. The one-parameter group of elliptic transformations having its invariant points at m and n has C among its pencil of invariant circles. In like manner all one-parameter groups of elliptic transformations whose invariant points are a pair of inverse points with respect to C leave C invariant. There are ∞^2 such pairs of points, and hence there are ∞^3 elliptic transformations in the group leaving C invariant.

THEOREM 12. *There are* ∞^3 *circular transformations which leave invariant any given circle; these form a three-parameter group. This group is composed of all hyperbolic transformations whose invariant points are on the circle, of all elliptic transformations whose invariant points are a pair of inverse points with re-*

spect to the circle, and of all parabolic transformations whose invariant point is on the circle and whose invariant line is a tangent to the circle at the invariant point.

The transformations of this group G_C are distributed into subgroups as follows: The elliptic transformations are distributed into ∞^2 one-parameter subgroups, but not into two-parameter subgroups. The hyperbolic transformations are distributed into ∞^1 two-parameter subgroups; each of these two-parameter groups break up into ∞^1 one-parameter subgroups, one of which is parabolic.

Since a straight line is considered as a circle through the point at infinity, it follows at once that there is a three-parameter group of transformations leaving a straight line invariant. This group G_L is in all respects similar to the group G_C.

A very important special case of the group G_L is when the line L is the axis of real quantities.* G_L then becomes the group of real projective transformations of the points on a real line. The properties of the real projective group are at once known from the properties of G_C.

THEOREM 13. *The three-parameter group of real projective transformations of the points on a line is a special subgroup of the six-parameter group of circular transformations of the points of the complex plane.*

There is still another type of three-parameter group consisting entirely of elliptic transformations which is closely related to the group G_C. This is the group of transformations of the complex plane which corresponds to the three-parameter group of rotations of a sphere about its centre. Every rotation of a sphere when projected stereographically upon the equatorial plane produces an elliptic transformation in that plane. Klein discusses the group of rotations of the sphere on pages 32–36 of his Ikosaeder, and on page 35 gives an analytic proof of the group property. The relation of this group in the plane to the group G_C is shown as follows:

The invariant points of a one-parameter elliptic subgroup of G_C, being inverse points with respect to C, form a pair of corresponding points in a hyperbolic involution on the line joining the two points with the centre O of the circle. The double points of the involution are the two points where the line cuts the circle. Every line through the centre of the circle C is the bearer of such an involution; and all the

* Poincaré has investigated many of the properties of these groups G_C and G_L in his papers in *Acta Mathematica*, vols. 1 and 3, "Theorie des groupes fuchsiens" and "Theorie des groupes kleiniens."

one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions belong to the three-parameter group G_C. Let us consider a similar system of elliptic involutions on all lines of a pencil through O, such that the product of the distances from the centre of a pair of corresponding points is the same in all the involutions. Thus $OP \cdot OP' = -k^2$, constant for all the involutions. When a sphere is projected stereographically upon the equatorial plane, every pair of opposite points on the sphere project into a pair of corresponding points in one of these involutions. Thus all one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions form a three-parameter group.

This three-parameter group leaves no figure of the plane invariant; but if it were allowable to use the language of projective geometry in speaking of the complex plane, we should say that this group leaves invariant an imaginary circle with centre at O and radius equal to ki. We shall, therefore, designate this group as G_{iC}.

This completes the discussion of the subgroups of the general circular group. It remains to be shown that there are no other types of subgroups besides those discussed above. I shall attempt no formal proof, but shall only bring forward some general considerations bearing upon the question.

A circular transformation transforms points into points and circles into circles. We have considered all possible groups which leave invariant one or two points; a transformation leaving invariant more than two points is identical. We have also considered all possible groups of transformations leaving a circle invariant. If there be a continuous group characterized by the invariance of some curve other than a circle, such a curve must be the path curve of a one-parameter group. The only other path curve besides the circle is the double spiral of Holzmüller. This has two singular points and is invariant only under those transformations whose invariant points are these two singular points; hence, there is only one one-parameter group leaving invariant such a double spiral. These considerations indicate that there are no other subgroups of the general circular group.

Lie's theory of continuous groups based upon the infinitesimal transformation is better adapted than the method of this paper for determining the complete list of types of subgroups of a given group. It may be likened to a net which gathers in its meshes all types of subgroups and lets none escape. My list of groups should be verified or corrected by the application of Lie's methods.

I append here a list of the group types discussed in the foregoing pages with a brief characterization of each.

(1) The six-parameter group G_6 of all circular transformations.

(2) The four-parameter group G_4 leaving a single point invariant.

(3) The two-parameter group G_2 of type T leaving a pair of points invariant.

(4) The two-parameter group G_2' of type T' leaving a single point invariant.

(5) The one-parameter group G_{1c} of type T and constant c in $k = e^{(c+i)\theta}$ leaving two points invariant. (a) The one-parameter group of elliptic transformations for which $c = 0$. (b) The one-parameter group of hyperbolic transformations for which $c = \infty$ and $\theta = 0$.

(6) The one-parameter parabolic group G_1' with constant θ in $a = re^{i\theta}$ leaving a single point invariant.

(7) The three-parameter group G_{3c} of type T and constant c in $k = e^{(c+i)\theta}$ leaving a single point invariant.

(8) The three-parameter group G_C of elliptic, hyperbolic and parabolic transformations leaving a circle invariant.

(9) The two-parameter group G_{2C} of hyperbolic transformations leaving invariant a circle and a point on it.

(10) The three-parameter group of elliptic transformations $\overline{G}_{iC}$.

The real projective transformations of the plane and of space may be treated in the same spirit and by the same methods here employed for the circular transformations. The writer hopes to be able in the near future to publish the full results of his investigations in these fields.

UNIVERSITY OF KANSAS,
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PLÜCKER'S COLLECTED PAPERS.

Julius Plücker's gesammelte mathematische Abhandlungen. Herausgegeben von A. SCHOENFLIES. Mit einem Bildniss Plücker's, und 73 in den text gedruckten Figuren. Leipzig, Druck und Verlag von B. G. Teubner. 1895. 8vo, pp. xxxvi + 620.

The Kgl. Gesellschaft der Wissenschaften zu Göttingen, of which Plücker was a corresponding member, recently under-