NOTE ON THE INVARIANTS OF n POINTS.

BY DR. EDGAR ODELL LOVETT.

(Read before the American Mathematical Society at the Meeting of April 24, 1897.)

A FAMILY of n points in ordinary space has $3n - t$ independent invariants by a t parameter Lie group. If the family is known to have p invariants there are then

$$q = p - 3n + t$$

relations among these p invariants. In particular if the group be the six parameter group of Euclidean motions,

$$p \quad q \quad r \quad yp - xq \quad zq - yr \quad xr - zp \tag{1}$$

where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial f}{\partial z}$, a system of n points has $3n-6$ independent invariants; but obviously the $\frac{n(n-1)}{2}$ mutual distances given by

$$\delta_{ij} \equiv ij = S(x_i - x_j)^2 \qquad i \neq j = 1, 2, \cdots, n. \tag{2}$$

are invariant by the group of motions; hence there are

$$q = \frac{n(n-1)}{2} - (3n - 6) = \frac{(n-3)(n-4)}{2}$$

relations among the δ_{ij}.

The invariants of the system of n points are found by the integration of the complete system of simultaneous partial differential equations

$$\sum_1^n {}^i \frac{\partial \varphi}{\partial x_i} = 0, \quad \sum_1^n {}^i \frac{\partial \varphi}{\partial y_i} = 0, \quad \sum_1^n {}^i \frac{\partial \varphi}{\partial z_i} = 0,$$

$$\sum_1^n {}^i \left(y_i \frac{\partial \varphi}{\partial x_i} - x_i \frac{\partial \varphi}{\partial y_i} \right) = 0, \quad \sum_1^n {}^i \left(z_i \frac{\partial \varphi}{\partial y_i} - y_i \frac{\partial \varphi}{\partial z_i} \right) = 0, \tag{3}$$

$$\sum_1^n {}^i \left(x_i \frac{\partial \varphi}{\partial z_i} - z_i \frac{\partial \varphi}{\partial x_i} \right) = 0;$$

and this system has at least $3n - 6$ solutions.

Let $n = 5$; then $q = 1$. An arbitrary function of the determinant

$$\triangle \equiv \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 12 & 13 & 14 & 15 \\ 1 & 21 & 0 & 23 & 24 & 25 \\ 1 & 31 & 32 & 0 & 34 & 35 \\ 1 & 41 & 42 & 43 & 0 & 45 \\ 1 & 51 & 52 & 53 & 54 & 0 \end{vmatrix},$$

where $12, 13, \cdots$, have the signification given by the identity and equation (2), is a general solution of the simultaneous system (3) for $n = 5$. In particular the vanishing of Δ satisfies the system (3) and hence expresses the relation among the mutual distances of five points in space, a result known to Lagrange. The fifth order determinant Δ_0, the minor of Δ with regard to the upper left hand corner element, equated to zero expresses the necessary and sufficient condition that five points be on a sphere. Similarly the vanishing of Δ_{00} and that of Δ_{000} give the conditions, respectively, that four points be coplanar and three points collinear.

Construct the determinant Δ for n points and call it D. $D = 0$ is then a generalization of the theorem of Lagrange expressed by $\Delta = 0$. This extension is warranted by the form of (3), the symmetry of Δ, and the fact that the invariants considered are absolute invariants.

BALTIMORE,
March 25, 1897.

NOTE ON THE FUNDAMENTAL THEOREMS OF LIE'S THEORY OF CONTINUOUS GROUPS.

BY DR. EDGAR ODELL LOVETT.

(Read before the American Mathematical Society at the Meeting of October 30, 1897.)

Lie's theory of continuous groups rests upon the following three fundamental theorems:*

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* See LIE: *Vorlesungen über continuierliche Gruppen*, herausgegeben von Scheffers, Leipzig, 1893, chapter XV; LIE: *Theorie der Transformationsgruppen*, bearbeitet unter Mitwirkung von Engel, Leipzig, Erster Abschnitt, 1888, chapters II, IV, IX, XVII; zweiter Abschnitt, 1890, chapter XVII; dritter Abschnitt, 1893, chapter XXV.