THE DECOMPOSITION OF MODULAR SYSTEMS OF RANK n IN n VARIABLES.

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I.

THEOREM A. *If in the realm* $\mathfrak{R}$ *of integrity-rationality* $\mathfrak{R} = [x_1, \cdots, x_n]\ (\mathfrak{R}_1', \ldots, \mathfrak{R}_\nu')$, *where the* $x_1 \cdots x_n$ *are independent variables and the realm* $\mathfrak{R}' = (\mathfrak{R}_1', \cdots, \mathfrak{R}_\nu')$ *is independent of the* $x_1 \cdots x_n$, *the modular system*

$$(1) \qquad \mathfrak{L} = [L_1[x_1, \cdots, x_n], \cdots, \ L_m[x_1, \cdots, x_n]]$$

is contained in the coefficient modular system $\mathfrak{F}$

$$(2) \qquad \mathfrak{F} = [\cdots, \underset{(k_1 \cdots k_n \mid t)}{f_{k_1, \ldots k_n}}, \cdots]$$

of the form

$$(3) \qquad F[u_1, \cdots, u_n] = \sum_{k_1 \ldots k_n \mid t} f_{k_1 \ldots k_n} u_1^{k_1} \cdots u_n^{k_n}$$

$$= \prod_{h=1, s} \Big(\sum_{i=1, n} (x_i - \xi_{hi})\, u_i \Big)^{e_h} \qquad \Big(t = \sum_{h=1, s} e_h\Big)$$

where the $f_{k_1 \ldots k_n} = f_{k_1 \ldots k_n}[x_1, \cdots, x_n]$ *belong to* $\mathfrak{R}$ *and the* ξ_{hi} *belong to* $\mathfrak{R}'$ *or to a family-realm containing* $\mathfrak{R}'$, *and where the* s *linear forms* $\sum_{i=1, n} (x_i - \xi_{hi})\, u_i\ (h = 1, 2, \cdots, s)$ *are distinct, then in the realm* $\mathfrak{R}^* = [x_1, \cdots, x_n]\ (\mathfrak{R}_1', \cdots, \mathfrak{R}_\nu', \xi_{hi}\ {}^{h=1, 2, \ldots s}_{i=1, 2, \ldots n})$ *the system* $\mathfrak{L}$ *decomposes (in the sense of equivalence) into relatively prime factors* $[\mathfrak{L}, \mathfrak{D}_h^{e_h}]$,

$$(4) \qquad \mathfrak{L} \sim \prod_{h=1, s} [\mathfrak{L}, \mathfrak{D}_h^{e_h}],$$

where $\qquad \mathfrak{D}_h = [x_1 - \xi_{h1}, \cdots, x_n - \xi_{hn}]$, *so that*

$$(5) \qquad [\mathfrak{D}_h, \mathfrak{D}_{h'}] \sim [1] \quad (h \neq h';\ h, h' = 1, 2, \cdots, s).$$

Every such modular system $\mathfrak{L}$ *is of rank* n *in* n *variables.*

Every modular system $\mathfrak{L}$ *of rank* n *in* n *variables decomposes in this way in particular with respect to its* resolvent *form*

$$\mathrm{F}[u_1, \cdots, u_n].$$

1. *Kronecker* * in connection with his general theory of elimination effected (*l. c.*, § 20) the decomposition of modular systems of rank n in n variables with non-vanishing discriminant.

In elucidation and extension of certain of the Kronecker Festschrift theories Mr. *Molk* † wrote the elaborate paper, *Sur une notion* ···

In Ch. IV., § 1 (*l. c.*, pp. 79–107) Mr. Molk discusses the general modular system ‡

$$(6) \qquad \mathfrak{L} = [L_1[x, y], \cdots, L_m[x, y]]$$

of rank 2 in 2 variables $[x, y]$. The resolvent form $F[u, v]$ of this system $\mathfrak{L}$

$$(7) \quad F[u, v] = \sum_{i=0,\, t} f_i u^i v^{t-i} = \prod_{h=1,\, s} \left((x - \xi_h) u + (y - \eta_h) v\right)^{e_h}$$
$$(t = \sum_{h=1,\, s} e_h)$$

is a certain homogeneous form in the adjoined indeterminates $u\,v$, which factors into s distinct linear factors $\left((x - \xi_h) u + (y - \eta_h) v\right)$ each to its proper multiplicity e_h. The $\xi_h\,\eta_h$ are independent of the $x\,y$. These factors correspond to the distinct solution systems $(x, y) = (\xi, \eta)$ of the system of equations $L_j[x, y] = 0$ $(j = 1, 2, \cdots, m)$, and their multiplicities are the multiplicities of those solution systems.

Now *in all cases* the coefficient modular system $\mathfrak{F}$ contains the system $\mathfrak{L}$,

$$(8) \qquad \mathfrak{F} = [f_0, f_1, \cdots, f_t] \equiv 0 \qquad [\mathfrak{L}],$$

and conversely, *if the system $\mathfrak{L}$ has a non-vanishing discriminant*, that is, if every multiplicity e_h is 1, then $\mathfrak{L}$ contains $\mathfrak{F}$,

$$(9) \qquad \mathfrak{L} \equiv 0 \qquad [\mathfrak{F}],$$

so that $\mathfrak{L}$ and $\mathfrak{F}$ are equivalent,

$$(10) \qquad \mathfrak{L} \sim \mathfrak{F}.$$

Mr. Molk's highly involved algebraic proof (*l. c.*, pp. 91–97)

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* KRONECKER: *Grundzüge einer arithmetischen Theorie der algebraischen Grössen, Festschrift* ... (1882; reprinted, *Journal für Mathematik*, vol. 93, pp. 1–122, 1882).

† MOLK: *Sur une notion qui comprend celle de divisibilité et sur la théorie générale de l'élimination* (*Acta Mathematica*, vol. 6, pp. 1–166, 1885).

‡ I use the notations of this paper.

of this converse is not above criticism. Then the decomposition of $\mathfrak{L}$

$$(11) \qquad \mathfrak{L} \sim \mathfrak{F} \sim \prod_{h=1,s} [x-\xi_h, y-\eta_h]^{e_h=1}$$

follows (*l. c.*, p. 104) by resolvent considerations.

Similarly Kronecker for the general n makes the decomposition of the system $\mathfrak{L}$ with non-vanishing discriminant depend upon the equivalence of $\mathfrak{L}$ with the resolvent system $\mathfrak{F}$.

It is, however, possible, *by pure-arithmetic process, for the general n and whether the discriminant vanish or not, to effect first a decomposition of $\mathfrak{F}$ and then a corresponding decomposition of $\mathfrak{L}$, from which, if the discriminant does not vanish follows the equivalence of $\mathfrak{L}$ and $\mathfrak{F}$.* I proceed to prove the caption theorem A, from which these results follow easily.

2. *A realm* $\mathfrak{R}$ *of integrity-rationality** $\mathfrak{R} = [\mathfrak{R}_1, \cdots, \mathfrak{R}_\mu] (\mathfrak{R}_{\mu+1}, \cdots, \mathfrak{R}_{\mu+\nu})$ consists of all functions $F[\mathfrak{R}_1, \cdots, \mathfrak{R}_\mu] (\mathfrak{R}_{\mu+1}, \cdots, \mathfrak{R}_{\mu+\nu})$ integral in $\mathfrak{R}_1 \cdots \mathfrak{R}_\mu$ and rational in $\mathfrak{R}_{\mu+1} \cdots \mathfrak{R}_{\mu+\nu}$, the coefficients being integers. The realm is closed under addition, subtraction, and multiplication, and likewise under division by any function not 0 of $\mathfrak{R}' = (\mathfrak{R}_{\mu+1} \cdots, \mathfrak{R}_{\mu+\nu})$.

Any set of functions $F_1, \cdots, F_m$, of a realm $\mathfrak{R}$ constitutes a *modular system* $\mathfrak{F} = [F_1, \cdots, F_m]$ of that realm. The whole theory of such modular systems relates to this underlying realm.

Any set of modular systems $\mathfrak{F}_i = [F_{i1}, \cdots F_{im_i}]$ $(i = 1, 2, \cdots, n)$ determines a modular system $[F_{ij_i} \, {}_{j_i = 1, 2, \cdots, m_i}^{i = 1, 2, \cdots, n}]$ for which we use the notation $[\mathfrak{F}_1, \cdots, \mathfrak{F}_n]$.

3. The very useful theorem: *If* $[\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}] \sim [1]$, *then* $[\mathfrak{F}_1, \mathfrak{F}]\, [\mathfrak{F}_2, \mathfrak{F}] \sim [\mathfrak{F}_1 \cdot \mathfrak{F}_2, \mathfrak{F}]$: may readily be proved by the use of the fundamental theorems concerning the composition and the equivalence of modular systems.

4. The decomposition (4) of theorem A depends upon the decomposition (12) in the same realm $\mathfrak{R}^*$,

$$(12) \qquad \mathfrak{F} \sim \prod_{h=1,s} \mathfrak{D}_h^{e_h}.$$

[This is indeed a particular case of (4), viz., for $\mathfrak{L} = \mathfrak{F}$: for $\mathfrak{F} \equiv 0$ $[\mathfrak{F}]$ and $\mathfrak{F} \equiv 0$ $[\mathfrak{D}_h^{e_h}]$ and so $[\mathfrak{F}, \mathfrak{D}_h^{e_h}] \sim \mathfrak{D}_h^{e_h}$ $(h = 1, 2, \cdots, s)$]. This decomposition (12) will appear below as the third corollary to the theorem B (II., § 7).

We have (5) $[\mathfrak{D}_h, \mathfrak{D}_{h'}] \sim [1]$ $(h \neq h'; h, h' = 1, 2, \cdots, s)$, and hence (§3)

* A convenient refinement of Kronecker's *realm of rationality*.

$$(13) \qquad [\mathfrak{D}_h^{e_h},\ \mathfrak{D}_{h'}^{e_{h'}}] \sim [1] \qquad (h \neq h'\,;\ h, h' = 1, 2, \cdots, s).$$

Further since by hypothesis

$$(14) \qquad \mathfrak{F} \equiv 0 \quad [\mathfrak{L}]$$

we have from (14, 12, 13) by §3 the desired decomposition (4)

$$(15) \qquad \mathfrak{L} \sim [\mathfrak{L}, \mathfrak{F}] \sim [\mathfrak{L}, \prod_{h=1,\,s} \mathfrak{D}_h^{e_h}] \sim \prod_{h=1,\,s} [\mathfrak{L}, \mathfrak{D}_h^{e_h}].$$

The s factor systems $[\mathfrak{L}, \mathfrak{D}_h^{e_h}]$ $(h = 1, 2, \cdots, s)$ are by pairs relatively prime (13).

The system $\mathfrak{D}_h^{e_h}$ consists of the totality of homogeneous products of degree e_h of the n differences $x_1 - \xi_{h1}, \cdots, x_n - \xi_{hn}$. If the m functions $L_i\,[x_1, \cdots, x_n]$ of $\mathfrak{L}$ be arranged each according to these n differences, then the system $[\mathfrak{L}, \mathfrak{D}_h^{e_h}]$ is equivalent to the system obtained by retaining in each function of $\mathfrak{L}$ only those terms of degree less than e_h. Hence, in particular $[\mathfrak{L}, \mathfrak{D}_h^{e_h}] \sim [1]$, unless $\mathfrak{L} \equiv 0\ [\mathfrak{D}_h]$.

On another occasion I shall develop the theory of modular systems capable of such decomposition into relatively prime factors.

5. A modular system $\mathfrak{L}$ of rank n in n variables has (Kronecker, *l. c.*, § 20) a form $F[u_1, \cdots, u_n]$—its *resolvent* form —of the kind called for by the hypothesis of theorem A, and indeed every system $\mathfrak{L}$ to which theorem A applies is of rank n. For this form F we have further

$$(16) \qquad \mathfrak{L} \equiv 0 \quad [\mathfrak{D}_h] \qquad (h = 1, 2, \cdots, s).$$

Thus the system $\mathfrak{L}$ decomposes with respect to the resolvent F according to theorem A.

For the particular case of non-vanishing discriminant we have Kronecker's decomposition and equivalence,

$$(17) \qquad \mathfrak{L} \sim \prod_{h=1,\,s} [\mathfrak{L}, \mathfrak{D}_h] \sim \prod_{h=1,\,s} \mathfrak{D}_h \sim \mathfrak{F}.$$

6. Let e denote the largest multiplicity e_h. Let D denote any function $D[x_1, \cdots, x_n]$ of $\mathfrak{N}^*$ for which

$$(18) \qquad D \equiv 0 \quad [\mathfrak{D}_h] \qquad (h = 1, 2, \cdots, s).$$

Then, from (5, 18) and § 3,

$$(19) \qquad [D, \prod_{h=1,\,s} \mathfrak{D}_h] \sim \prod_{h=1,\,s} [D, \mathfrak{D}_h] \sim \prod_{h=1,\,s} \mathfrak{D}_h.$$

Hence

$$(20)\quad D \equiv 0\ [\prod_{h=1,s} \mathfrak{D}_h],\quad D^e \equiv 0[\prod_{h=1,s} \mathfrak{D}_h^{\ e}],\quad D^e \equiv 0\ [\prod_{h=1,s} \mathfrak{D}_h^{\ e_h}].$$

Then from (20, 12, 14) we have

$$(21)\qquad D^e \equiv 0\quad [\mathfrak{L}].$$

This theorem for the case $n = 2$ is due to Mr. Netto.*

II.

THEOREM B. *In any realm* $\mathfrak{R}$ *of integrity-rationality the product* $\mathfrak{F}$ *of the coefficient modular systems* $\mathfrak{D}$, $\mathfrak{E}$ *of two homogeneous* n*-ary forms* $D[u_1, \cdots, u_n]$, $E[u_1, \cdots, u_n]$ *of the realm* $\mathfrak{R}$ *is equivalent to the coefficient modular system of their product form* $F = DE$, *if for any certain system of* n *integers*† $a_1, \cdots, a_n$ *whose greatest common divisor is* 1 *in the realm* $\mathfrak{R}$

$$[D[a_1, \cdots, a_n],\ E[a_1, \cdots, a_n],\ \mathfrak{F}] \sim [1].$$

1. We set, calling m_d, m_e the degrees respectively of D, E,

$$(1)\qquad D[u_1, \cdots, u_n] = \sum_{i_1, \ldots, i_n \mid m_d} d_{i_1 \ldots, i_n} u_1^{\ i_1} \cdots u_n^{\ i_n},$$

$$E[u_1, \cdots, u_n] = \sum_{j_1, \ldots, j_n \mid m_e} e_{j_1 \ldots, j_n} u_1^{\ j_1} \cdots u_n^{\ j_n}$$

$$(2)\qquad F[u_1, \cdots, u_n] = \sum_{k_1, \ldots, k_n \mid m_f} f_{k_1 \ldots, k_n} u_1^{\ k_1} \cdots u_n^{\ k_n}$$

$$= D[u_1, \cdots, u_n] . E[u_1, \cdots, u_n] \qquad (m_f = m_d + m_e)$$

so that

$$(3)\qquad f_{k_1 \ldots k_n} = \sum_{\substack{i_1, \ldots, i_n \mid m_d \\ j_1, \ldots, j_n \mid m_e \\ \hline k_1, \ldots, k_n \mid m_f}} d_{i_1 \ldots i_n} e_{j_1 \ldots j_n} \qquad (k_1, \cdots, k_n \mid m_f)$$

where the summation remarks of (1, 2; 3) have the definitions (4; 5)

$$(4)\quad h_1, \cdots, h_n \mid m_c \sim h_1, \cdots, h_n = 0, 1, \cdots, m_c;\ h_1 + \cdots + h_n = m_c$$

* NETTO: *Zur Theorie der Elimination* (*Acta Mathematica*, vol. 7, pp. 101–104, 1885).

† Or, more generally, the $a_1, \ldots, a_n$ may be any column of an unimodular matrix $(a_{ss'})$ $(s, s' = 1, 2, \ldots, n)$ of the realm $\mathfrak{R}$, $|\, a_{ss'} \,| = 1$. The proof then needs change only in § 3.

$$(5)\qquad \frac{\begin{matrix} i_1, \cdots, i_n \mid m_d \\ j_1, \cdots, j_n \mid m_e \end{matrix}}{k_1, \cdots, k_n \mid m_f} \sim \begin{matrix} i_1, \cdots, i_n \mid m_d;\ j_1, \cdots, j_n \mid m_e; \\ i_s + j_s = k_s \ (s = 1, 2, \cdots, n) \end{matrix}$$

For the corresponding coefficient modular systems we write

$$(6)\qquad \mathfrak{D} = [\cdots, \underset{i_1, \ldots, i_n \mid m_d}{d_{i_1 \ldots i_n}}, \cdots], \ \mathfrak{E} = [\cdots, \underset{j_1, \ldots, j_n \mid m_e}{e_{j_1 \ldots j_n}}, \cdots],$$

$$\mathfrak{F} = [\cdots, \underset{k_1, \ldots, k_n \mid m_f}{f_{k_1 \ldots, k_n}}, \cdots];$$

and in general we denote the coefficient modular system of any form $G[u_1, \cdots, u_n]$ of the realm $\mathfrak{R}$ by the corresponding Gothic capital letter $\mathfrak{G}$.

We are to prove that under a certain hypothesis H

$$(7)\qquad \mathfrak{D}\,\mathfrak{E} \sim \mathfrak{F}$$

2. Under an unimodular linear homogeneous substitution

$$(8)\qquad u_s = \sum_{s'=0,\,n} a_{ss'} u'_{s'} \qquad |\,a_{ss'}\,| = 1 \qquad (s, s' = 1, 2, \cdots, n)$$

whose coefficients $a_{ss'}$ belong to the realm $\mathfrak{R}$, the form $G[u_1, \cdots, u_n]$ of the realm is transformed into the form $G'[u_1', \cdots, u_n']$, and the corresponding coefficient modular systems are equivalent, $\mathfrak{G} \sim \mathfrak{G}'$.

Since identities in the u's transform into identities in the u''s in order to prove for the two forms D, E under the hypothesis H the equivalence (7) $\mathfrak{D}\,\mathfrak{E} \sim \mathfrak{F}$ it is sufficient to prove for the two transformed forms D', E' under the transformed hypothesis H' the corresponding equivalence (7) $\mathfrak{D}'\,\mathfrak{E}' \sim \mathfrak{F}'$.

3. By hypothesis H there exists a system of n integers $a_1, \cdots, a_n$ of greatest common divisor 1 such that in $\mathfrak{R}$

$$(9)\qquad [D[a_1, \cdots, a_n], E[a_1, \cdots, a_n], \mathfrak{F}] \sim [1].$$

There exists* then a substitution (8) with integral coefficients in which

*We can pass from $(a_1, a_2, \ldots, a_n)$ to $(1, 0, \ldots, 0)$ by a sequence of *elementary transformations*, *i. e.*, interchange of two elements with change of sign of one and addition to one element of another element. The application of the reverse sequence simultaneously to the n *columns* of the identity matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 \\ \vdots & & & \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

carries us to the matrix $(a_{ss'})$ desired.

This determination of $(a_{ss'})$ is suggested by *Kronecker's Reduction der Systeme von* n^2 *ganzzahligen Elementen* (*Journal für die Mathematik*, vol. 107, pp. 135–136, 1891).

$$(10) \qquad a_{s_1} = a_s \qquad (s = 1, 2, \cdots, n).$$

For this substitution (8), since

$$(11) \quad (u_1, u_2, \cdots, u_n) = (a_1, a_2, \cdots, a_n) \sim (u_1', u_2', \cdots, u_n') = (1, 0, \cdots, 0),$$

the transformed hypothesis H' affirms the equivalence in $\mathfrak{R}$

$$(12) \quad [D'[1, 0, \cdots, 0], E'[1, 0, \cdots, 0], \mathfrak{F}] \sim [D'[1, 0, \cdots, 0], E'[1, 0, \cdots, 0], \mathfrak{F}'] \sim [d'_{m_d 0 \cdots 0}, e'_{m_e 0 \cdots 0}, \mathfrak{F}'] \sim [1].$$

4. Thus the theorem holds if it holds for the special case $(a_1, a_2, \cdots, a_n) = (1, 0, \cdots, 0)$, when

$$(13) \qquad [d_{m_d 0 \ldots 0}, e_{m_e 0 \ldots 0}, \mathfrak{F}] \sim [1],$$

so that, by I. §3,

$$(14) \qquad [d^{m_e+1}_{m_d 0 \cdots 0}, e^{m_d+1}_{m_e 0 \cdots 0}, \mathfrak{F}] \sim [1].$$

The equivalence

$$(15) \qquad \mathfrak{D}\,\mathfrak{E} \sim \mathfrak{F}$$

in $\mathfrak{R}$ is nothing but the two congruences

$$(16) \qquad \mathfrak{D}\,\mathfrak{E} \equiv 0 \quad [\mathfrak{F}], \quad \mathfrak{F} \equiv 0 \quad [\mathfrak{D}\,\mathfrak{E}].$$

Of these the second holds by (3), and the first holds by (14) if

$$(17) \qquad \mathfrak{D}\,\mathfrak{E}[d^{m_e+1}_{m_d 0 \ldots 0}, e^{m_d+1}_{m_e 0 \ldots 0}, \mathfrak{F}] \equiv 0 \quad [\mathfrak{F}],$$

and this holds if simultaneously

$$(18) \qquad \mathfrak{D}[e^{m_d+1}_{m_e 0 \ldots 0}] \equiv [\cdots, \underset{i_1, i_2, \ldots, i_n \mid m_d}{d_{i_1 i_2 \ldots i_n} e^{m_d+1}_{m_e 0 \ldots 0}}, \cdots] \equiv 0 \quad [\mathfrak{F}],$$

$$(19) \qquad \mathfrak{E}[d^{m_e+1}_{m_d 0 \ldots 0}] \equiv [\cdots, \underset{j_1, j_2, \ldots j_n \mid m_e}{e_{j_1 j_2 \cdots j_n} d^{m_e+1}_{m_d 0 \ldots 0}}, \cdots] \equiv 0 \quad [\mathfrak{F}].$$

We prove that (18) holds; the similar proof applies to (19). We have from (3) for $d_{i_1 i_2 \ldots i_n}$, $i_1 = m_d$ (20), $i_1 < m_d$ (21):

$$(20) \qquad d_{m_d 0 \ldots 0}\, e_{m_e 0 \ldots 0} = f_{m_f 0 \ldots 0} \equiv 0 \quad [\mathfrak{F}],$$

$$(21) \qquad d_{i_1 i_2 \ldots i_n}\, e_{m_e 0 \ldots 0} = f_{i_1 + m_e i_2 \ldots i_n} - \Sigma^* d_{h_1 h_2 \ldots h_n} e_{j_1 j_2 \cdots j_n}$$

$$\left(* \sim \frac{\begin{matrix} h_1, h_2, \cdots, h_n & m_d \\ j_1, j_2, \cdots, j_n \mid m_e \end{matrix}}{i_1 + m_e, i_2, \cdots, i_n \mid m_f}, \quad \begin{matrix} h_1 > i_1 \\ j_1 < m_e \end{matrix} \right)$$

$$(21') \qquad d_{i_1 i_2 \ldots i_n}\, e_{m_e 0 \ldots 0} \equiv -\Sigma^* d_{h_1 h_2 \ldots h_n} e_{j_1 j_2 \ldots j_n} \quad [\mathfrak{F}].$$

Hence, applying (21′) $m_d - i_1$ times and (20) once, we see that

$$(22) \qquad d_{i_1 i_2 \ldots i_n}\, e_{m_e 0 \ldots 0}^{m_d - i_1 + 1} \equiv 0 \quad [\mathfrak{F}],$$

and so that (18) does hold.

5. Cor. 1. The product $\mathfrak{F}$ of the coefficient modular systems $\mathfrak{D}_1, \cdots, \mathfrak{D}_t$ of t n-ary forms $D_1, \cdots, D_t$ of the realm $\mathfrak{R}$ is equivalent to the modular system of their product-form F, if for any certain system of n integers $a_1, \cdots, a_n$ with greatest common divisor 1

$$(23) \qquad [D_g[a_1, \cdots, a_n], D_{g'}[a_1, \cdots, a_n]] \sim [1]$$
$$(g \neq g'; g, g' = 1, 2, \cdots, t)$$

6. Cor. 2. The s linear forms

$$(24) \qquad D_h[u_1, \cdots, u_n] = \sum_{i=1,n} d_{hi} u_i \quad (h = 1, 2, \cdots, s)$$

belong to the realm $\mathfrak{R}$ and have leading coefficients by pairs relatively prime

$$(25) \qquad [d_{h1}, d_{h'1}] \sim [1] \qquad (h \neq h'; h, h' = 1, 2, \cdots, s).$$

Then, setting

$$(26) \qquad D_h[u_1, \cdots, u_n]^{e_h} = F_h[u_1, \cdots, u_n], \qquad (h = 1, 2, \cdots, s),$$

$$(27) \qquad \prod_{h=1,s} F_h[u_1, \cdots, u_n] = F[u_1, \cdots, u_n],$$

we have the equivalence in $\mathfrak{R}$

$$(28) \qquad \prod_{h=1,s} \mathfrak{D}_h^{e_h} \sim \prod_{h=1,s} \mathfrak{F}_h \sim \mathfrak{F}$$

This appears from Cor. 1 for $(a_1, a_2, \cdots, a_n) = (1, 0, \cdots, 0)$ since obviously for any linear form D_h and its power $D_h^{e_h} = F_h$ we have $\mathfrak{D}_h^{e_h} \sim \mathfrak{F}_h$ and since from (25) by I § 3 $[d_{h1}^{e_h}, d_{h'1}^{e_{h'}}] \sim [1]$ $(h \neq h'; h, h' = 1, 2, \cdots, s)$.

7. Cor. 3. We consider the realm $\mathfrak{R}$ of integrity-rationality

$$\mathfrak{R} = [x_1, \cdots, x_n] \; (\xi_{hi} \; {}^{h=1,2,\ldots,s}_{i=1,2,\ldots,n}) \tag{29}$$

where the $x_1, \cdots, x_n$ are indeterminates and where the ξ_{hi} belong to a realm $\mathfrak{R}^*$ not containing the indeterminates x and in that realm are such that the s forms

$$D_h[u_1, \cdots, u_n] = \sum_{i=1,n} (x_i - \xi_{hi}) u_i \qquad (h=1, 2, \cdots, s) \tag{30}$$

are distinct linear forms. Then we have (in the notations of Cor. 2) the equivalence (28).

The particular case, in which

$$\xi_{h1} \neq \xi_{h'1}, \;.\dot{}.\; [x_1 - \xi_{h1}, x_1 - \xi_{h'1}] \sim [1] \tag{31}$$
$$(h \neq h';\; h, h' = 1, 2, \cdots, s),$$

follows at once from Cor. 2.

The general case is reduced to this particular case by transformation of the $u_1 \cdots u_n$ by a properly chosen unimodular substitution in the realm [1]

$$u_i = \sum_{i'=1,n} a_{ii'} u'_{i'} \qquad (i = 1, \cdots, n) \tag{32}$$

and simultaneously of the $x_1 \cdots x_n$ and the $\xi_{hi} \cdots \xi_{hn} (h=1, \cdots, s)$ by the substitutions contragredient to (32)

$$x_i' = \sum_{i'=1,n} a_{i'i} x_{i'} \qquad (i = 1, \cdots, n), \tag{33}$$

$$\xi_{hi}' = \sum_{i'=1,n} a_{i'i} \xi_{hi'} \qquad (i = 1, \cdots, n). \tag{34}$$

Since the forms D_h (30) are distinct we can determine integers $a_1 \cdots a_n$ with greatest common divisor 1 such that $\sum_{i'=1,n} \xi_{hi'} a_{i'} \neq \sum_{i'=1,n} \xi_{h'i'} a_{i'}$ $(h \neq h'; h, h' = 1, \cdots, s)$. Then any unimodular matrix $(a_{ii'})$ in [1] having $a_{i'1} = a_{i'}$ $(i' = 1, \cdots, n)$ will yield satisfactory reducing substitutions (32, 33, 34).

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