(and therefore for three) we should add by periods of two. Again since $8^2 + 1 = 5 \times 13$, we should test for five and thirteen (or oneate-five) by reducing to four figures by addition, and then to two figures by subtraction. Among small primes, eleven is the least adapted to the octonary system, but for this divisor we might convert the given number to the binary system, then reduce to ten figures by addition, and to five by subtraction (since $2^5 + 1 = 3 \times 11$), and finally reconvert into an octonary number of two digits.

As there is no doubt that our ancestors originated the decimal system by counting on their fingers, we must, in view of the merits of the octonary system, feel profound regret that they should have perversely counted their thumbs, although nature had differentiated them from the fingers sufficiently, she might have thought, to save the race from this error.

THE TEACHING OF ELEMENTARY GEOMETRY IN GERMAN SCHOOLS.

Inhalt und Methode des planimetrischen Unterrichts. Eine vergleichende Planimetrie. Von DR. HEINRICH SCHOTTEN. Leipzig, B. G. Teubner, 1890. 8vo, pp. iv. + 370.

WHOEVER has followed the efforts of the Association for the Improvement of Geometrical Teaching in England in the course of the last ten years will have been struck by the slowness of the progress made and the paucity of the practical results attained. In Germany there exists no such society; but a powerful agitation for the reform of geometrical teaching has been in progress there for at least sixty years, and with particular force during the last two decades. And yet, even from Germany, with its well developed and highly centralized system of education, comes the complaint that progress is slow and much remains to be done.

Recent statistics have shown, in particular, that the most widely used text-books are far from being the best. Thus, while Hubert Müller's Geometry, which may be regarded as the best representative of the "modern school," reached its third edition in 1889, after a lapse of fifteen years from its first appearance, Kambly's very inferior text-book, whose faults and mistakes have frequently been exposed and complained of, appeared in 1884 in its 74th edition.

This book of Kambly's easily leads in the list of text-books used in various schools; it is adopted in 217 schools, the next in order being another rather inferior book, by Koppe, introduced in 51 schools ; then follow Mehler's, used in 44, Reidt's in 29, etc., while there are 55 mathematical text-books used in but one school each. Similar statistics for our American schools would be both interesting and instructive.

Still the signs of improvement are not wanting. Some very good text-books of geometry have been published in recent years and are making, though slowly, their way to the front ; preparatory ("propædeutic") courses in "intuitive" geometry in connection with geometrical drawing have been introduced in many schools, and are generally recommended by the school boards ; excellent classified and graded collections of problems have appeared and are in actual use ; and, above all, the whole subject of the improvement of geometrical teaching has been ventilated and discussed with great thoroughness and completeness.

In this last respect the work done by Hoffmann's *Zeitschrift** cannot be estimated too highly. The volumes of this journal, specially devoted to the discussion of scientific instruction in the secondary schools, are replete with material for the study of this question, the editor himself being one of the principal contributors. It is to be hoped that the *Bulletin of the New York Mathematical Society* may, in the course of time, perform a similar service towards the improvement of mathematical instruction in this country.

On the other hand, the custom of many German schools of publishing scientific and educational essays in connection with the school calendar ("*Programm*") has given a welcome opportunity to many experienced teachers to express their ideas on the subject and to propose improvements.

The material that has grown up in this way is somewhat bewildering in extent, and, moreover, not very ready of access to American students. A full set of Hoffmann's *Zeitschrift* is probably to be found in but very few libraries in this country; many of the older "*Programme*" are hard to obtain ; and of the legion of German text-books of geometry that have appeared during the present century only a very small number, of course, have found their way into American libraries.

The attempt made by Dr. Schotten to sum up the results of the various efforts of reform in geometrical teaching in the secondary schools and to give a critical survey of the literature of this subject, will therefore be welcomed by all interested in elementary geometrical instruction.

The title of Dr. Schotten's work is perhaps somewhat misleading, as it does not indicate that his study is confined

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* J. C. V. HOFFMANN'S *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, published by Teubner, Leipzig.

entirely to *German* books and papers. Nor is there any indication on the title-page that the present volume is only a first instalment of the work; a second volume is announced in the preface and on the last page of the book, but even this would not seem to exhaust the subject.

There is no table of contents, and no *general* index, a serious defect in a work of this kind, which may perhaps be remedied in the second volume. The book is of course made up in a large measure of quotations, interspersed with critical remarks by the author; unfortunately, the arrangement is far from convenient, and in some instances very awkward. In general, however, the author has well accomplished his exceedingly laborious task. He shows a thorough acquaintance with the literature of his subject, as well as good judgment and discrimination in making use of it.

In an introductory essay, Dr. Schotten briefly states his views on what is desirable in the way of reform. He sustains these views, which are not over radical, not so much by argument as by a large array of quotations from various sources. They may therefore be taken to fairly represent the better thought of the day on the subject, at least in Germany.

It may be of interest to give a short account of these fundamental principles in teaching geometry which have found the approval of so large a body of experienced and well-trained teachers.

The study of mathematics in the *Gymnasium* should begin with geometry (in *Tertia*, i. e., in the fourth and fifth years of the whole nine-year course), being followed by algebra in *Secunda* (also two years), while the last two years (*Prima*) are reserved for trigonometry and a thorough review of the whole course. There is no urgent demand for increasing the *extent* of mathematical instruction; but what is taught should be taught well, that is, with thoroughness and accuracy. The object of mathematical teaching in the *Gymnasium* is not to form mathematicians, but to improve the mind, not only by training in logical thinking, but by accustoming the student to precision of language in writing and speaking, by awakening his self-activity through the solution of problems, and in the case of geometry in particular, by forming and practising the power of mental intuition ("*Anschauung*").

These objects, however, cannot be attained by the so-called Euclidean method of teaching geometry. While Euclid's arrangement of the propositions has long been abandoned in German text-books, his synthetic method of proof is still retained in many books. Here reform is most peremptory.

The "genetic" method should pervade the whole course; that is to say, the student should be led up in a natural way to each proposition, so as to see clearly its connection with what precedes, and finally conceive of it, not as a single artificial experiment in reasoning, but as an essential member of an organic whole.

The proof of a proposition should be obtained by what the Germans call the "heuristic" method, i. e., by the process that would naturally be adopted by any one trying to *find* the proof himself anew. A "synthetic" reconstruction of the proof may finally be added in some cases.

Frequent reviews are of course required to keep the student constantly alive to the conviction that he is studying a well-connected system, and not a mass of detached single facts.

The introduction of some of the ideas of modern projective geometry (symmetry, dualism, theory of rows and pencils, correspondences, etc.) will be found a great help in building up a natural system of geometry. But wherever used, these ideas must be closely interwoven with the whole system; it is decidedly objectionable to merely put these matters into an appendix at the end of the book as is sometimes done.

There is however a fundamental difficulty in introducing the ideas of projective geometry into elementary teaching. It lies in the fact that the circle is the only curved line considered in ordinary elementary geometry, while in modern geometry the circle appears as a very special case of a conic section.* This circumstance will indicate how far we may go in applying the methods of modern geometry to an elementary course, provided the study of the conic sections be excluded.

Let us now turn to the main body of Dr. Schotten's work. It is divided into five chapters: (1) Space, (2) Geometry, (3) The Space-Forms (solid, surface, line, point), (4) The Plane, (5) The Straight Line. In a second volume the author promises to treat in a similar way of (1) Direction and Distance; position of points, lines, and circles in their mutual relations; metrical relations; (2) The Axiom of Parallel Lines (Eucl. XI.); (3) The Angle; (4) Auxiliary Geometrical Ideas, such as equality, motion, dimension, concept, definition, proof, explanation, postulate, theorem, axiom, form, magnitude, position, figure, locus, symmetry, etc.; (5) Method.

The author prefaces each chapter by a brief statement of his own views, and then follow quotations from all those text-books or other works that express any original ideas on the subject of the chapter. Comments by the author on these quotations are usually given in foot-notes. But it must be

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* See O. RAUSENBERGER. *Elementargeometrie des Punktes, der Geraden und der Ebene, systematisch und kritisch behandelt.* Leipzig, Teubner, 1887, pp. 2–3.

said that the whole is not sufficiently well digested, and it requires some labor (which the author might have spared the reader) to get at the final results.

It will not be necessary to pass in review here the manifold and widely different views of the fundamental conceptions of geometry collated in Dr. Schotten's book.

The tendency in Germany seems to be at present to escape as far as possible the hidden dangers that await the teacher at the very threshold of geometry in the definitions of such ideas as space, geometry, the point, the plane, etc., by two means : (1) by requiring a preparatory course in geometrical drawing in which the student should become thoroughly familiar, in a practical way, with the fundamental geometrical ideas ; (2) by a strict adherence to Pascal's rules.

As these rules do not seem to be as widely known as they deserve to be,* they are here transcribed in full from Pascal's essay, "*De l'esprit géométrique.*"†

Rules for definitions.—"1. N'entreprendre de définir aucune des choses tellement connues d'elles-mêmes, qu'on n'ait point de termes plus clairs pour les expliquer. 2. N'omettre aucun des termes un peu obscurs ou équivoques, sans définition. 3. N'employer dans la définition des termes que des mots parfaitement connus, ou déjà expliqués."

Rules for axioms.—"1. N'omettre aucun des principes nécessaires sans avoir demandé si on l'accorde, quelque clair et évident qu'il puisse être. 2. Ne demander, en axiomes, que des choses parfaitement évidentes d'elles-mêmes."

Rules for demonstrations.—"1. N'entreprendre de démontrer aucune des choses qui sont tellement évidentes d'elles-mêmes qu'on n'ait rien de plus clair pour les prouver. 2. Prouver toutes les propositions un peu obscures, et n'employer à leur preuve que des axiomes très-évidents, ou des propositions déjà accordées ou démontrées. 3. Substituer toujours mentalement les définitions à la place des définis, pour ne pas se tromper par l'équivoque des termes que les définitions ont restreints."

Thus, conformably to Pascal's first rule on definitions, we find that some of the best German text-books do not try at all to define what is space, or what is a point, or even what is a straight line.

Strange as it may appear to some teachers, these text-books do not begin with several pages of definitions to be committed to memory, followed by a page of axioms again to be committed to memory. Nor are the demonstrations made

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* Dr. Schotten, while quoting them somewhat inaccurately in translation, says that he does not know in what work of Pascal's they occur.

† PASCAL, *Pensées*, ed. Havet, Paris, Delagrave, 1883, pp. 555–556.

to cover exactly a whole page when they can be expressed in a line. Some of these authors, although well acquainted with synthetic, and even with non-Euclidean geometry, do not at all abhor the use of the expressions "direction" and "distance." Indeed, Dr. Schotten regards these two ideas as intuitively given in the mind and as so simple as not to require definition; he therefore bases the definition of the straight line on these two ideas, or rather recommends to elucidate the intuitive idea of the straight line possessed by any well-balanced mind by means of the still simpler ideas of direction and distance.

It is interesting to compare these views deduced by Dr. Schotten mainly with regard to their pedagogical value, and as a result of practical experience in teaching, with the conclusions arrived at by Prof. G. Peano* from a purely scientific point of view and based on the principles of mathematical logic.

A more philosophical discussion of the foundations of geometry is reserved in the German schools to the review course in the *Prima* of the *Gymnasium*. Then only will the student be able to appreciate to a certain degree the niceties involved in a careful treatment of the fundamental definitions and axioms of geometry.

It is to be hoped that Dr. Schotten will continue his studies in German "comparative planimetry," and that his second volume will not be deferred *ad calendas græcas*. It would also seem desirable that somebody should give us a similar account of what has been done in other countries in the same direction, in particular in England, France, and Italy.

In conclusion, the following two text-books might be mentioned, out of a large number of others, as giving a fair idea of the reform movement in Germany:

HUBERT MÜLLER, *Leitfaden der ebenen Geometrie*, Leipzig, Teubner, 1889;

HENRICI and TREUTLEIN, *Lehrbuch der Elementar-Geometrie*, ib., 1881.

For the more scientific study of the questions involved, the reader is referred to the following works, in which ample bibliographical references will be found:

OTTO RAUSENBERGER, *Die Elementargeometrie des Punktes, der Geraden und der Ebene*, Leipzig, Teubner, 1887.

BENNO ERDMANN, *Die Axiome der Geometrie*, Leipzig, 1877.

SCHMITZ-DUMONT, *Die mathematischen Elemente der Erkenntnistheorie*, Berlin, 1878.

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*See *Rivista di Matematica*, ed. by PEANO, Turin, vol. I. (1891), pp. 24-25.

J. C. BECKER, *Abhandlungen aus dem Grenzgebiete der Mathematik und Philosophie.* Zürich, 1870.

ALEXANDER ZIWET.

ANN ARBOR, *August* 1, 1891.

PICARD'S DEMONSTRATION OF THE GENERAL THEOREM UPON THE EXISTENCE OF INTEGRALS OF ORDINARY DIFFERENTIAL EQUATIONS.

TRANSLATED BY DR. THOMAS S. FISKE.

THE cardinal proposition in the theory of algebraic equations, that every such equation has a root, holds a place in mathematical theory no more important than the corresponding proposition in the theory of differential equations, that every differential equation defines a function expressible by means of a convergent series. This proposition was originally established by Cauchy, and was introduced, with a somewhat simplified demonstration, by Briot and Bouquet in their treatise on doubly periodic functions.* A new demonstration remarkable for its simplicity and brevity has been published by M. Emile Picard in the *Bulletin de la Société Mathématique de France* for March,† and reproduced on account of its striking character in the *Nouvelles Annales des Mathématiques* for May. This demonstration requires no auxiliary propositions, and depends upon no preceding part of the theory, except the simple consideration, that any ordinary differential equation is equivalent to a set of simultaneous equations of the first order.‡ The following is a translation of Picard's demonstration.

1. Consider the system of n equations of the first order

$$\frac{du}{dx} = f_1\,(x,\ u,\ v,\ \ldots\ ,\ w),$$

$$\frac{dv}{dx} = f_2\,(x,\ u,\ v,\ \ldots\ ,\ w),$$

$$\ldots\ldots\ldots\ldots ,$$

$$\frac{dw}{dx} = f_n\,(x,\ u,\ v,\ \ldots\ ,\ w),$$

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* *Théorie des fonctions elliptiques,* p. 325.
JORDAN. *Cours d'analyse,* vol. III., p. 87.

† *Bulletin de la Société Mathématique de France,* Vol. XIX., p. 61.

‡ JORDAN. *Cours d'analyse,* vol. III., p. 4.