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Comment on Article by Page and Quintana*

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Abstract. Page and Quintana (2016) introduce the novel methodology of spatial product partition models in order to explicitly model the partitioning of spatial locations, with the aim of balancing local and global spatial dependence. Here we first discuss Gibbs-type partitions and their connection to exchangeable product partition models and their possible use as building blocks of spatial product partition models. Then, adopting the viewpoint of extreme value theory, we focus on two approaches for modeling spatial extremes, namely hierarchical modeling based on a latent stochastic process and modeling based on max-stable processes. Additional insights and interesting findings may arise by developing the approach of Page and Quintana (2016) along these lines.

Keywords: asymptotic independence, extreme value theory, Gibbs-type partition, hierarchical modeling, max-stable random field, upper-tail dependence coefficient function, two parameter Poisson-Dirichlet partition.

We would like to congratulate the authors for an innovative and stimulating article on spatially correlated clustering of data via a clever extension of the popular product partition models first introduced in Hartigan (1990). Our contribution to the discussion considers first Gibbs-type random partitions (Gnedin and Pitman, 2005), which are closely connected to exchangeable product partition models. See De Blasi et al. (2015) for a recent review. We highlight some of their implications in terms of clustering and point out their potential usefulness as building blocks of spatial product partition models. Moreover, we consider some potential developments of Section 3.3 for modeling extremes of a spatially observed real-life process. Our modeling perspective relies on the extreme value theory (see e.g. Coles 2001; Beirlant et al. 2004; de Haan and Ferreira 2006). In this field one can consider different approaches to modeling spatial extremes, which have been recently reviewed in Davison et al. (2012). Here we focus on two cases: hierarchical modeling based on a latent stochastic process and modeling based on max-stable processes. In the sequel, we denote the maximum computed over a large number m of independent replicates of a random field by $Y(\boldsymbol{s})$, where $\boldsymbol{s}$ represents the location.

1 Gibbs-Type Partitions

Gibbs-type priors (Gnedin and Pitman, 2005), as argued from a predictive viewpoint in De Blasi et al. (2015), may be seen as the most natural generalization of the Dirichlet process. In terms of the induced random partition, this class is characterized by an exchangeable partition probability function (EPPF) of product form, a feature which

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is crucial for guaranteeing mathematical tractability. More precisely, an exchangeable random partition is said to be of *Gibbs–type* if, for any $n \geq 1$, $1 \leq k \leq n$ and $(n_1, \ldots, n_k)$ s.t. $n_i \geq 1$, $i = 1, \ldots, k$ and $\sum_{i=1}^k n_i = n$, the corresponding EPPF can be represented as

$$\Pi_k^{(n)}(n_1, \ldots, n_k) = V_{n,k} \prod_{i=1}^k (1-\sigma)_{n_i - 1},$$

for $\sigma \in (-\infty, 1)$, a set of non-negative weights $\{V_{n,k} : n \geq 1, 1 \leq k \leq n\}$ satisfying the recursion $V_{n,k} = V_{n+1,k+1} + (n - \sigma k)V_{n+1,k}$ with $V_{1,1} = 1$ and $(a)_q = \Gamma(a+q)/\Gamma(a)$ is the q-th ascending factorial of a.

Hence, a Gibbs-type random partition is completely determined by the parameter $\sigma \in (-\infty, 1)$ and the weights $V_{n,k}$'s. Popular models are recovered by particular choices. For instance, if $V_{n,k} = \frac{\prod_{i=1}^{k-1}(\theta + i\sigma)}{(\theta+1)_{n-1}}$ with $\sigma \in [0, 1)$ and $\theta > -\sigma$ or $\sigma < 0$ with $\theta = m|\sigma|$ for some positive integer m, one obtains Pitman's two parameter Poisson-Dirichlet family characterized by an EPPF of the form

$$\Pi_k^{(n)}(n_1, \ldots, n_k) = \frac{\prod_{i=1}^{k-1}(\theta + i\sigma)}{(\theta+1)_{n-1}} \prod_{i=1}^k (1-\sigma)_{n_i - 1}. \tag{1}$$

Moreover, if $\sigma = 0$ it reduces to the Dirichlet process random partition, whereas, if $\sigma \in (0, 1)$ with $\theta > -\sigma$, (1) corresponds to the Pitman-Yor process random partition.

The role of σ is crucial since it determines the clustering structure of Gibbs-type models. To see this, consider first the asymptotic behavior of the random number of clusters K_n. Define

$$c_n(\sigma) = \begin{cases} 1 & \sigma < 0 \\ \log n & \sigma = 0 \\ n^\sigma & \sigma \in (0, 1) \end{cases}$$

for any $n \geq 1$, then

$$\frac{K_n}{c_n(\sigma)} \xrightarrow{\text{a.s.}} S_\sigma$$

as $n \to \infty$ and the limiting random variable S_σ is termed σ-*diversity*. See Pitman (2006) for details. In the Dirichlet process case the σ-diversity is degenerate on the total mass $\theta > 0$ and $K_n \sim \theta \log n$, for n large enough, almost surely. The larger σ, the faster the rate of increase of K_n or, in other terms, the more new values are generated. Clearly, the case where $\sigma < 0$ corresponds to a model accommodating for a finite number of distinct clusters. Moreover, as shown in Lijoi et al. (2007); De Blasi et al. (2015), a reinforcement mechanism driven by σ takes place. Indeed, assuming the first n values have generated k clusters, the ratio of the probabilities that value $n+1$ belongs to i-th and j-th cluster, respectively, can be shown to be $(n_i - \sigma)/(n_j - \sigma)$. If $\sigma = 0$, the previous quantity reduces to the ratio of the sizes of the two clusters. Therefore the probability is proportional to the size of the cluster. On the other hand, if $\sigma > 0$ and $n_i > n_j$, the ratio is an increasing function of σ. Hence, as σ increases the mass is reallocated from the j-th to i-th cluster. This corresponds to reinforcing among the observed clusters, more than proportionally those that have higher frequency. If $\sigma < 0$, the reinforcement mechanism works in the opposite direction in the sense that the coincidence probabilities are less than proportional to the cluster size. See De Blasi et al. (2015) for details. Combing these two aspects one notes that a larger σ produces more clusters while at the same time inducing a stronger reinforcement of the large ones. The latter feature can be equivalently described by saying that for $\sigma < 0$ "the richer get poorer", for $\sigma = 0$ "the richer get richer" proportionally to the cluster size and for $\sigma > 0$ "the richer get richer" more than proportionally.

As first noted in Lijoi et al. (2007), there is also a close connection between Gibbs-type partitions and exchangeable product partition models (Hartigan, 1990). A product partition model corresponds to a probability distribution for the random partition of the form

$$\mathbb{P}(\{S_1, \ldots, S_k\}) \propto \prod_{i=1}^{k} c(S_i) \tag{2}$$

where $c(\cdot)$ is termed *cohesion function*. Now, let $|S| = \mathrm{card}(S)$ and impose the cohesion function $c(\cdot)$ to depend only on the cardinality of the set S, that is $c(S_i) := c(|S_i|) = c(n_i)$. This is a natural and reasonable choice for a cohesion function in an exchangeable framework. Then (2) is, for any $n \geq k \geq 1$, the random partition induced by an exchangeable sequence if and only if $c(n_i) = (1-\sigma)_{n_i - 1}$ for $i = 1, \ldots, k$ and $\sigma < 1$. This is equivalent to saying that (2) is of Gibbs-type. Such a statement follows immediately from Gnedin and Pitman (2005). Therefore, exchangeable product partition models with cohesion function depending solely on the cardinality coincide with the family of Gibbs-type priors.

In order to define their spatial product partition models Page and Quintana (2016) clearly have to move beyond the exchangeable framework. Nonetheless in defining their models they still use the partition distribution corresponding to the Dirichlet process as a building block of their cohesion functions in order to avoid a large number of singleton blocks through the "the richer get richer" property of the Dirichlet partition. However, when seen within the framework of Gibbs-type priors, the Dirichlet partition is a very specific case corresponding to a logarithmic increase of the number of clusters and proportionality of "the richer get richer" property. Therefore, it would be interesting to consider also more general Gibbs-type partitions as building blocks of spatial product partition models. For instance, one could resort to the two parameter Poisson-Dirichlet family and put a prior on the key parameter σ so that the data can select the appropriate rate of cluster generation and intensity of the reinforcement. If putting a prior on the full range of σ turns out to be computationally too intensive, a convenient alternative might be to allow σ to take only 3 values, say $-1, 0, 1/2$. In this way the model would still incorporate all three types of reinforcement regimes.

2 Hierarchical Modeling

A simple approach for modeling a vector $Y(\boldsymbol{s}_1), \ldots, Y(\boldsymbol{s}_n)$ of maxima consists in assuming that

$$Y(\boldsymbol{s}_i)|\{\mu(\boldsymbol{s}_i), \sigma(\boldsymbol{s}_i), \xi(\boldsymbol{s}_i)\} \stackrel{iid}{\sim} GEV\{\mu(\boldsymbol{s}_i), \sigma(\boldsymbol{s}_i), \xi(\boldsymbol{s}_i)\}, \quad i = 1, \ldots, n,$$

where $GEV(a, b, c)$ denotes a Generalised Extreme Value distribution with location, scale and shape parameters $a \in \mathbb{R}$, $b > 0$ and $c \in \mathbb{R}$, respectively. See e.g. Coles (2001, Ch. 3). Furthermore, it is assumed that $\mu(\boldsymbol{s})$, $\sigma(\boldsymbol{s})$ and $\xi(\boldsymbol{s})$ change over space according to some random fields. For instance, we can consider specification (11) in Davison et al. (2012)

$$\mu(\boldsymbol{s}) = \boldsymbol{x}(\boldsymbol{s})'\boldsymbol{\beta} + \theta(\boldsymbol{s})$$

where $\theta(\boldsymbol{s})$ is a stationary Gaussian process with zero mean, $\boldsymbol{x}(\boldsymbol{s})$ is a deterministic vector-valued function and $\boldsymbol{\beta}$ is a vector of coefficients.

Within this framework complex spatial patterns can be accommodated and Bayesian inference can be naturally performed by means of Markov chain Monte Carlo algorithms. See e.g. Casson and Coles (1999); Gaetan and Grigoletto (2004); Sang and Gelfand (2009). The previous hierarchical model can be viewed as analogous to the likelihood (12) in Page and Quintana (2016, Proposition 3.2) when the measured responses are maxima and belong to the same cluster.

A benefit of this approach is that it captures the local spatial variations of extremes well, as shown empirically by Davison et al. (2012) in the case of heavy rainfall.

On the other hand one can argue that the spatial global dependence might not always be fully described. For simplicity, assume that $\mu(\boldsymbol{s}) = \theta(\boldsymbol{s})$ is a second order stationary Gaussian random field with zero mean and a correlation function $\rho(\boldsymbol{v})$, where $\boldsymbol{v} \in \mathbb{R}^2$ is the spatial lag, and let σ and ξ be constants. As an example, consider the exponential correlation function

$$\rho(\boldsymbol{v}) = \tau^2 \exp\left(\|\boldsymbol{v}\|/\phi\right), \quad \phi, \tau > 0, \quad \boldsymbol{v} \in \mathbb{R}^2, \tag{3}$$

where $\|\cdot\|$ is the Euclidean norm.

The correlation coefficient is not well suited for measuring the pairwise dependence at the extreme level. Instead a common measure is the coefficient of upper tail dependence (see e.g. Coles 2001, Ch. 8), i.e.

$$\chi(\boldsymbol{v}) := \lim_{u \to 1} \chi(u, \boldsymbol{v}) = \lim_{u \to 1} \mathbb{P}[F_{\boldsymbol{s}+\boldsymbol{v}}\{Y(\boldsymbol{s}+\boldsymbol{v})\} > u | F_{\boldsymbol{s}}\{Y(\boldsymbol{s})\} > u], \tag{4}$$

where $F_{\boldsymbol{s}}$ and $F_{\boldsymbol{s}+\boldsymbol{v}}$ denote the distribution function of Y at the locations $\boldsymbol{s}$ and $\boldsymbol{s}+\boldsymbol{v}$, respectively. One has $0 \leq \chi(\boldsymbol{v}) \leq 1$ with $\chi(\boldsymbol{v}) = 0$ when the marginal distributions are asymptotically independent in the upper tail, while $\chi(\boldsymbol{v}) > 0$ when they are asymptotically dependent. The larger the coefficient the stronger the dependence.

Moreover, when the marginal variables are asymptotically independent, the dependence at a sub-asymptotic level can be measured by the following alternative coefficient (Coles, 2001, Ch. 8),

$$\bar{\chi}(\boldsymbol{v}) := \lim_{u \to 1} \bar{\chi}(u, \boldsymbol{v}) = \lim_{u \to 1} \frac{2\,\mathbb{P}[F_{\boldsymbol{s}}\{Y(\boldsymbol{s}+\boldsymbol{v})\} > u]}{\mathbb{P}[F_{\boldsymbol{s}}\{Y(\boldsymbol{s})\} > u | F_{\boldsymbol{s}+\boldsymbol{v}}\{Y(\boldsymbol{s}+\boldsymbol{v})\} > u]}, \tag{5}$$

where $-1 \leq \bar{\chi}(\boldsymbol{v}) \leq 1$. When $\chi(\boldsymbol{v}) = 0$, the cases $\bar{\chi}(\boldsymbol{v}) > 0$, $\bar{\chi}(\boldsymbol{v}) = 0$ and $\bar{\chi}(\boldsymbol{v}) < 0$ correspond to positive association, (near) independence, and negative association, respectively.

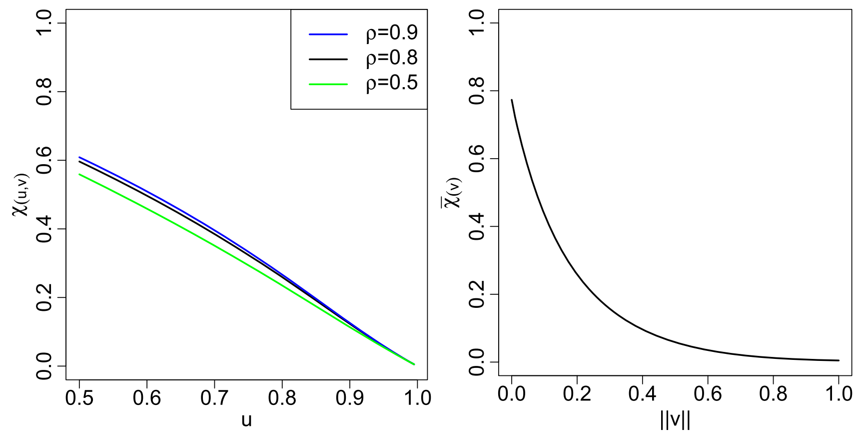

Figure 1: Coefficients $\chi(u, \boldsymbol{v})$ and $\bar{\chi}(\boldsymbol{v})$ of tail dependence.

Under the present modeling setup, the analytical expressions of (4) and (5) are infeasible to derive, due to some intractable integrals. Nonetheless, they can be evaluated numerically. Assuming $\sigma = 1$, $\xi = 0.3$ and $\tau^2 = 1$, $\phi = 0.2$ in (3), the left panel of Figure 1 displays the limiting behavior of $\chi(u, \boldsymbol{v})$ for $u \to 1$ for different values of the correlation ($\rho(\boldsymbol{v}) = 0.5, 0.8, 0.9$). The marginal variables are clearly asymptotically independent in the upper tail regardless of the value of the correlation. It can be checked that varying σ and ξ the same conclusion is attained.

However, the marginal variables can exhibit positive association. Indeed, the right-panel of Figure 1 displays the plot of the coefficient $\bar{\chi}(\boldsymbol{v})$ against the distance $\|\boldsymbol{v}\|$, for the case that $\sigma = 0.2$, $\xi = 0$ and the previous correlation setting. There is a remarkable positive association of the marginal distributions at short distances and the degree of association reduces as the distance increases, reaching (near) independence at large distances.

The spatial partition model proposed by the authors adds a supplementary layer in the hierarchy that can strengthen the dependence between the observations. This aspect deserves further investigation. Moreover, it would be interesting to study what type of dependence is obtained when also σ and ξ vary according to a random field.

3 Max-Stable Random Fields

Another popular approach for modeling spatial extremes is by means of max-stable random fields. Loosely speaking, the point-wise partial-maximum of independent replicates of a random field, appropriately normalized, converges for large samples to a max-stable random field. See de Haan and Ferreira (2006, Ch. 9) for a rigorous account. Then, a sample of maxima observed at n spatial locations can be drawn from a finite dimensional distribution of a max-stable random field.

A max-stable random field has a constructive spectral representation (de Haan and Ferreira, 2006; Schlather, 2002). Assume that r_i, $i \geq 1$, are points of a Poisson process on $(0,\infty)$ with intensity dr. Let W_i, $i \geq 1$, be independent and identically distributed (iid) copies of a real valued random field $\{W(\boldsymbol{s})\}$, independent from the $\{r_i\}$ and such that $E[W^+(\boldsymbol{s})] = \mu \in (0,\infty)$, where $W^+(\boldsymbol{s}) = \max(W(\boldsymbol{s}), 0)$. Then

$$Y(\boldsymbol{s}) = \mu^{-1} \max_{i \geq 1} W_i^+(\boldsymbol{s})/r_i \tag{6}$$

is a max-stable process with unit Fréchet marginals, i.e. $\mathbb{P}(Y(\boldsymbol{s}) \leq y) = \exp(-1/y)$, $y > 0$.

Choosing a particular expression for W_i leads to known examples of max-stable processes (Davison et al., 2012). In the following we consider the extremal-Gaussian max-stable random field (Schlather, 2002), where W_i is a stationary Gaussian random field with zero mean, unit variance and correlation function $\rho(\cdot)$. Such a construction theoretically justifies its use for describing complex phenomena such as heavy rainfall. A drawback is that its coefficient of upper tail dependence never reaches the case when the marginal variables are asymptotically independent. Indeed,

$$\chi(\boldsymbol{v}) = 1 - \sqrt{\frac{1-\rho(\boldsymbol{v})}{2}}, \quad \boldsymbol{v} \in \mathbb{R}^2, \tag{7}$$

and when the correlation, as e.g. (3), is such that $\rho(\boldsymbol{v}) \to 0$ for $\|v\| \to \infty$, then $\chi(\boldsymbol{v})$ reaches the lower bound $1 - \sqrt{1/2}$. There are remedies for such an undesired feature (see e.g. Schlather 2002; Davison and Gholamrezaee 2012).

We realized that an alternative solution can also be obtained by working with a spatial partition model similar to Page and Quintana (2016, Proposition 3.3). More precisely, let $\{S_1, \ldots, S_{k_n}\}$ denote a partition of the set $\{1, \ldots, n\}$ into k_n subsets, with $S_h \subset \{1, \ldots, n\}$. The cluster membership is denoted by $C(\boldsymbol{s}_1), \ldots, C(\boldsymbol{s}_n)$ where $C(\boldsymbol{s}_i) = h$ implies that $i \in S_h$. Define $\boldsymbol{Y}_h = \{Y(\boldsymbol{s}_i),\, i \in S_h\}$, $\mathbf{C}_h = \{C(\boldsymbol{s}_i),\, i \in S_h\}$, $h = 1, \ldots, k_n$, and suppose that $\boldsymbol{Y}_h | \boldsymbol{\psi}, \mathbf{C}_h, \sim EG(\boldsymbol{\psi}_h)$, where EG denotes the finite dimensional distribution of an extremal-Gaussian max-stable random field, $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{k_n})$ and $\boldsymbol{\psi}_h$ is a cluster specific vector of parameters. We assume also that $\boldsymbol{Y}_h | \boldsymbol{\psi}, \mathbf{C}_h$ and $\boldsymbol{Y}_l | \boldsymbol{\psi}, \mathbf{C}_l$ are conditionally independent for $h \neq l$.

The coefficient of upper tail dependence between $Y(\boldsymbol{s}+\boldsymbol{v})$ and $Y(\boldsymbol{s})$ turns out to be

$$\lim_{u \to 1} \mathbb{P}[F_{\boldsymbol{s}+\boldsymbol{v}}\{Y(\boldsymbol{s}+\boldsymbol{v})\} > u | F_{\boldsymbol{s}}\{Y(\boldsymbol{s})\} > u] = \chi(\boldsymbol{v}) \mathbb{P}\{C(\boldsymbol{s}+\boldsymbol{v}) = C(\boldsymbol{s})\}, \tag{8}$$

i.e. it coincides with the right-hand side of (7) times the probability of the sites belonging to the same cluster. Note that the resulting model is not stationary.

As an illustrative example, the left panel of Figure 2 displays the plot of (7) versus distances with a black solid line, obtained using the correlation (3), with $\tau^2 = 1$ and $\phi = 0.5$. A solid blue line displays the corresponding plot for (8) when a spatial partition model is included; specifically, we consider the model in Page and Quintana (2016, Table 1) with $\alpha = 1.77$ and $M = 10$. As a consequence, with the spatial partition model an improvement is gained in making the coefficient of upper tail dependence approach the asymptotically independent case for large distances, as desired.

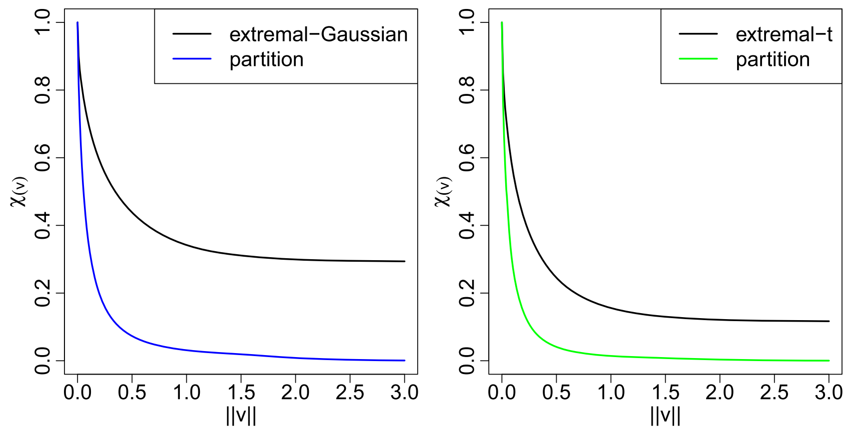

Figure 2: Coefficients of tail dependence.

It is worth noting that the result (8) actually holds for every instance of max-stable random field, provided that $\chi(\boldsymbol{v})$ indicates the coefficient of upper tail dependence between the components of the conditioned vector $\boldsymbol{Y}_h|\boldsymbol{\psi}, \mathbf{C}_h$.

In the right panel of Figure 2 we consider another important example of a max-stable random field: the extremal-t. This type of model has gained popularity due to its capability of fitting the data in different applications (see e.g. Davison et al. 2012). Although the coefficient of upper tail dependence can reach the asymptotically independent case, it is worth noting that with some parameters' values it may no longer hold true.

The right panel displays the plot of the coefficient of upper tail dependence for the extremal-t max-stable random field. A solid blue line depicts $\chi(\boldsymbol{v})$ with 3 degrees of freedom and the same specifications as before. The green solid line also includes the spatial partition model. In the former instance the marginal variables are dependent also at large distances, whereas in the latter instance this is not longer the case.

An, in our view interesting, open problem is the full development of the spatial partition model approach in this context.

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