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Twisted r-spin potential and Givental's quantization

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Abstract

The universal curve $\pi\colon \overline{\mathcal{C}} \to \overline{\mathcal{M}}$ over the moduli space $\overline{\mathcal{M}}$ of stable r-spin maps to a target Kähler manifold X carries a universal spin bundle $\mathcal{L} \to \overline{\mathcal{C}}$. Therefore, the moduli space $\overline{\mathcal{M}}$ itself carries a natural K-theory class $R\pi_*\mathcal{L}$.

We introduce a *twisted* r-spin Gromov–Witten potential of X enriched with Chern characters of $R\pi_*\mathcal{L}$. We show that the twisted potential can be reconstructed from the ordinary r-spin Gromov–Witten potential of X via an operator that assumes a particularly simple form in Givental's quantization formalism.

1 Introduction

In [23] Mumford used the Grothendieck–Riemann–Roch formula to express the Chern characters of the Hodge bundle over the moduli space of stable curves via other tautological classes.

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Faber and Pandharipande [12] generalized his result to spaces of stable maps. They showed that the resulting formula allows one to express the so-called *twisted* Gromov–Witten potential (enriched with Chern characters of the Hodge bundle) of any target Kähler manifold via the usual Gromov–Witten potential.

Givental [15] noted that the above result admitted a strikingly concise formulation in the framework of his quantization formalism. This allows many explicit calculations of twisted Gromov–Witten potentials starting from known "untwisted" potentials (see [10, 11, 27] and references therein).

In the present paper, we generalize all these steps to the spaces of r-spin structures and maps, Theorem 1.1. Witten's r-spin conjecture, proved in [13], determines the r-spin untwisted Gromov–Witten potential of the point. Theorem 1.1 can be regarded as the natural tool to calculate r-spin Gromov–Witten potentials beyond the untwisted cases, see Remark 1.2.

1.1 Mumford's formula and Givental's formalism

1.1.1 Moduli spaces

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of genus-g stable curves with n marked points, and let $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. In one-to-one correspondence with the marked points, natural cohomology classes $\psi_1, \dots, \psi_n$ in $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ are defined on the moduli space: the cotangent lines at the ith markings form a line bundle $L_i \to \overline{\mathcal{M}}_{g,n}$, and ψ_i equals $c_1(L_i)$. Moreover, if $p\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map, we define $\kappa_d = p_*(\psi_{n+1}^{d+1})$ in $H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, for $d \geq 1$. (We refer to [16] for an overview of these and other geometrically defined cohomology classes on $\overline{\mathcal{M}}_{g,n}$ and their properties.)

1.1.2 Boundary

Set

$$\mathcal{D} = \overline{\mathcal{M}}_{g-1,n+2} \sqcup \bigsqcup_{\substack{l+l'=g \\ I \sqcup I' = \{1,\dots,n\}}} \overline{\mathcal{M}}_{l, I\cup\{n+1\}} \times \overline{\mathcal{M}}_{l', I'\cup\{n+2\}}.$$

Then there is a natural map $j\colon \mathcal{D} \to \overline{\mathcal{M}}_{g,n}$, obtained by gluing together the marked points number $n+1$ and $n+2$. The image of j is the boundary of $\overline{\mathcal{M}}_{g,n}$.

1.1.3 Mumford's formula

In [23], Mumford applied the Grothendieck–Riemann–Roch formula to express the Chern characters $\mathrm{ch}_d(\Lambda)$ of the Hodge bundle $\Lambda \to \overline{\mathcal{M}}_{g,n}$. His formula reads, for $d \geq 1$,

$$\mathrm{ch}_d(\Lambda) = \frac{B_{d+1}}{(d+1)!}\left[\kappa_d - \sum_{i=1}^{n} \psi_i^d + \frac{1}{2} j_* \left(\sum_{a=0}^{d-1} \psi_{n+1}^a (-\psi_{n+2})^{d-1-a}\right)\right], \quad (1.1)$$

where B_{d+1} is the $(d+1)$th Bernoulli number. In particular, $\mathrm{ch}_d(\Lambda) = 0$ for even d, because odd Bernoulli numbers B_{d+1} vanish.

1.1.4 Differential operators

Let X be a Kähler manifold and $(h_\mu)_{\mu \in B}$ a basis of the cohomology space $H = H^*(X, \mathbb{Q})$ such that $h_1 = 1$. For simplicity, we assume that X has only even cohomology. Let $g_{\mu\mu'}$ be the matrix of the Poincaré duality form on H in the basis (h_μ). Denote by $X_{g,n,D}$ the space of degree D stable maps with target X, of genus g, with n marked points. Here D is an effective cycle in $H_2(X, \mathbb{Z})$. Let $[X_{g,n,D}]^{\mathrm{v}}$ denote its virtual fundamental class.

Introduce the following generating series in the variables Q, s_d, $d \geq 1$, and t_a^μ, $a \geq 0$, $\mu \in B$:

$$F_g(\boldsymbol{t}, \boldsymbol{s}) = \sum_{n \geq 1} \sum_{D} Q^D \sum_{\substack{a_1, \ldots, a_n \\ \mu_1, \ldots, \mu_n}} \int_{[X_{g,n,D}]^v} \exp\Big(\sum_{d \geq 1} s_d \, \mathrm{ch}_d(\Lambda)\Big)$$
$$\times \prod_{i=1}^{n} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \cdot \frac{t_{a_1}^{\mu_1} \ldots t_{a_n}^{\mu_n}}{n!}.$$

The series $F = \sum_{g \geq 0} \hbar^{g-1} F_g$ is called the *twisted Gromov–Witten potential of* X. Let $Z = \exp F$.

Denote by $g^{\mu\mu'}$ the inverse matrix of $g_{\mu\mu'}$. From the expression of $\mathrm{ch}_d(\Lambda)$, Faber and Pandharipande [12] deduced the following claim: *we have*

$$\frac{\partial Z}{\partial s_d} = L_d Z,$$

where L_d is the linear differential operator depending only on the variables t_a^μ:

$$L_d = \frac{B_{d+1}}{(d+1)!}\left[\frac{\partial}{\partial t^1_{d+1}} - \sum_{a\geq 0,\mu} t_a^\mu \frac{\partial}{\partial t^\mu_{a+d}} + \frac{\hbar}{2}\sum_{\substack{a+a'=d-1\\ \mu,\mu'}} (-1)^{a'} g^{\mu\mu'} \frac{\partial^2}{\partial t_a^\mu \partial t_{a'}^{\mu'}}\right].$$

1.1.5 Givental's quantization

Let $\mathcal{H} = H((z^{-1}))$ be the space of H-valued Laurent series in z, finite in the positive direction and possibly infinite in the negative direction. This space bears a natural symplectic form

$$\omega(f_1, f_2) = \mathrm{Res}_{z=0} \sum_{\mu,\mu'} g_{\mu\mu'} f_1^\mu(-z) f_2^{\mu'}(z).$$

If we write a Laurent series in the form

$$f(z) = \sum_{\substack{a\geq 0\\ \mu\in B}} q_a^\mu z^a h_\mu + \sum_{\substack{a\geq 0\\ \mu,\mu'\in B}} p_{a,\mu} g^{\mu\mu'} (-1/z)^{a+1} h_{\mu'}$$

(the first sum is finite, while the second one can be infinite), then $p_{a,\mu}, q_a^\mu$ form a set of Darboux coordinates on $\mathcal{H}$. The coordinates q_a^μ are identified with the variables t_a^μ above via

$$\begin{aligned} q_1^1 &= t_1^1 - 1, \\ q_a^\mu &= t_a^\mu \qquad \text{for other } a \text{ and } \mu. \end{aligned}$$

The strange-looking shift for $a = 1, \mu = 1$ is called the *dilaton shift*.

Functions (or *Hamiltonians*) on the space $\mathcal{H}$ are quantized according to Weyl's rules: q_a^μ is transformed into the operator of multiplication by $q_a^\mu/\sqrt{\hbar}$, while $p_{a,\mu}$ is transformed into the operator $\sqrt{\hbar}\,\partial/\partial q_a^\mu$. By convention, the derivations are applied before the multiplications. Using these rules, we obtain the following result.

The operator L_d is the Weyl quantization of the Hamiltonian

$$P_d = \frac{B_{d+1}}{(d+1)!}\left[-\sum_{\substack{a\geq 0\\ \mu\in B}} q_a^\mu p_{a+d,\mu} + \frac{1}{2}\sum_{\substack{a+a'=d-1\\ \mu,\mu'\in B}} (-1)^{a'} g^{\mu\mu'} p_{a,\mu} p_{a',\mu'}\right].$$

Every Hamiltonian P determines a vector field on (or an infinitesimal symplectic transformation of) $\mathcal{H}$, given by $\omega^{-1}(dP)$. Since our Hamiltonian is of pure degree 2, the corresponding vector field is linear. Following [15], we obtain that *the vector field on $\mathcal{H}$ determined by P_d is given by the multiplication by*

$$\frac{B_{d+1}}{(d+1)!} z^d. \tag{1.2}$$

The Hamiltonian can, of course, be recovered from the vector field (up to an additive constant). Thus the not-so-simple Mumford formula (1.1) turns out to be encoded in the strikingly simple expression (1.2).

1.1.6 Characteristic classes

Moduli spaces of r-spin curves $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ and of r-spin maps $X_{g,n,D}^{r,\boldsymbol{m}}$ will be introduced in Section 2. Essentially, they classify stable curves or maps enriched with an r-spin structure in the sense of Witten: in the case of a smooth curve with no markings, an r-spin structure is a line bundle $\mathcal{L}$, which is an rth tensor root of the canonical line bundle on the curve.

These moduli spaces come with universal curves $\pi : \overline{\mathcal{C}}_{g,n}^{r,\boldsymbol{m}} \to \overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ and $\pi : \overline{\mathcal{C}}_{g,n,D}^{r,\boldsymbol{m}}(X) \to X_{g,n,D}^{r,\boldsymbol{m}}$. The universal curve has n sections $s_1, \dots s_n$ specifying the marked points and carries a universal r-spin structure $\mathcal{L}_{g,n}^{r,\boldsymbol{m}}$ (which is either a line bundle or a sheaf, depending on the construction, see Section 2). Let ω be the relative dualizing sheaf of the universal curve and $\omega_{\log}$ the relative dualizing sheaf twisted by the divisor of the sections $\sum [s_i]$.

On the moduli space $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ or $X_{g,n,D}^{r,\boldsymbol{m}}$ we consider the following cohomology classes:

$$\begin{aligned}
\psi_i &= c_1(s_i^* \omega), \\
\kappa_d &= \pi_*(c_1(\omega_{\log})^{d+1}), \\
\mathrm{ch}_d &= \mathrm{ch}_d(R\pi_* \mathcal{L}_{g,n}^{r,\boldsymbol{m}}),
\end{aligned}$$

where ch_d denotes the term in degree d of the Chern character.

Remark 1.1. The classes ψ_i and κ_d are the pullbacks of the analogous classes in the Chow ring of the stack $\overline{\mathcal{M}}_{g,n}$ of stable curves (respectively, the stack $X_{g,n,D}$ of stable maps) via the natural morphism $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}} \to \overline{\mathcal{M}}_{g,n}$ (respectively $X_{g,n,D}^{r,\boldsymbol{m}} \to X_{g,n,D}$).

1.2 The main result

Let X be a target Kähler manifold and (h_μ) a basis in $H^*(X,\mathbb{Q})$ (as before, we assume that X has only even cohomology).

Denote by $\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v$ the virtual r-spin class of $X^{r,\boldsymbol{m}}_{g,n,D}$ (see Section 3 for more details). Let $\mathrm{ev}_i\colon X^{r,\boldsymbol{m}}_{g,n,D} \to X$ be the evaluation maps.

Let $B_d(x)$ be the Bernoulli polynomials. They are defined by

$$\frac{e^{tx}\,t}{e^t-1} = \sum \frac{B_d(x)}{d!} t^d.$$

Definition 1.1. The power series $F_{g,r}(\boldsymbol{t},\boldsymbol{s})$

$$\sum_D Q^D \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{m_1,\ldots,m_n\\ a_1,\ldots,a_n\\ \mu_1,\ldots,\mu_n}} \frac{1}{r^{g-1}} \int\limits_{\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v} \exp\Bigl(\sum_{d\geq 1} s_d \,\mathrm{ch}_d\Bigr) \prod_{i=1}^n \psi_i^{a_i}\, \mathrm{ev}_i^*(h_{\mu_i}) t_{a_i}^{\mu_i\otimes m_i}.$$

is called the *genus-g r-spin twisted Gromov–Witten potential* of X.

Let $F_r(\boldsymbol{t},\boldsymbol{s}) = \sum_{g\geq 0} \hbar^{g-1} F_{g,r}(\boldsymbol{t},\boldsymbol{s})$ and $Z_r = \exp(F_r)$.

Consider the vector space $\mathbb{Q}^{r-1}$ with basis $e_1,\ldots,e_{r-1}$ endowed with the quadratic form g with coefficients $g_{ab} = \delta_{a+b,r}$. A diagonal matrix in basis $e_1,\ldots,e_{r-1}$ will be denoted by $\mathrm{diag}[u_1,\ldots,u_{r-1}]$. Denote by H the space $H = H^*(X,\mathbb{Q})\otimes\mathbb{Q}^{r-1}$. The metric on H is the tensor product of the Poincaré paring on $H^*(X,\mathbb{Q})$ and the metric g_{ab} on $\mathbb{Q}^{r-1}$.

Let $\mathcal{H} = H\left(\left(\frac{1}{z}\right)\right)$ as in Section 1.1.5.

Theorem 1.1. *We have*

$$\frac{\partial Z_r}{\partial s_d} = L_d Z_r,$$

where L_d is a differential operator in the variables $\boldsymbol{t}$ obtained by Weyl quantization of the infinitesimal symplectic transformation of $\mathcal{H}$ given by

$$\frac{z^d}{(d+1)!}\cdot \mathrm{Id}\otimes \mathrm{diag}\left[B_{d+1}\left(\frac{1}{r}\right),\ldots,B_{d+1}\left(\frac{r-1}{r}\right)\right].$$

Remark 1.2. This theorem actually gives a way of computing the twisted r-spin potential of X starting from the ordinary r-spin potential of X, not twisted by the classes ch_d.

More precisely, let $f_r = F_r|_{s_1=s_2=\cdots=0}$ be the ordinary r-spin Gromov–Witten potential of X. Then we have

$$\exp(F_r) = \exp\Bigl(\sum s_d L_d\Bigr) \exp(f_r).$$

1.3 Plan of the paper

In Section 2 we give an overview of the existing constructions of the natural compactification of spaces of r-spin curves. In Section 3 we explain in more detail the properties of the moduli spaces of r-stable spin maps and the virtual fundamental classes. In Section 4 we recall and slightly generalize the results obtained in [9] by applying the Grothendieck–Riemann–Roch formula to the spin bundle on the so-called universal r-bubbled curve. In Section 5 we put these results in the framework of Givental's quantization.

2 Moduli of r-spin curves: an overview

Here we describe the construction of the moduli spaces of r-stable spin curves and maps. The integer $r \geq 2$ is fixed once and for all. We denote by $\mathbb{Z}_r$ the group of rth roots of unity and set $\xi_r = \exp(2\pi \mathrm{i}/r)$.

All schemes and stacks throughout this paper are defined over $\mathbb{C}$.

2.1 Smooth curves

Let $(C; s_1, \ldots, s_n)$ be a smooth genus-g curve with $n \geq 1$ marked points. Choose n integers $m_1, \ldots, m_n \in \mathbb{Z}$ such that $2g - 2 + n - \sum m_i$ is divisible by r. An *r-spin structure of type $\boldsymbol{m} = (m_1, \ldots, m_n)$* on C is a line bundle $\mathcal{L}$ over C together with an identification

$$\varphi\colon \mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_{\log}\left(-\sum_{i=1}^{n} m_i[s_i]\right),$$

where ω is the cotangent line bundle of C and $\omega_{\log} = \omega\left(\sum [s_i]\right)$.

The number m_i is called the *index* of $\mathcal{L}$ at x_i.

There are exactly r^{2g} nonisomorphic r-spin structures of type $\boldsymbol{m}$ on every smooth curve. Each of them has r "trivial" automorphisms $\mathcal{L} \to \mathcal{L}$ given by the multiplications by rth roots of unity along its fibers. A curve endowed with an r-spin structure is called a *smooth r-spin curve.*

The moduli space $\mathcal{M}_{g,n}^{r,\boldsymbol{m}}$ of smooth r-spin curves is an r^{2g}-sheeted unramified covering of the moduli space of smooth curves $\mathcal{M}_{g,n}$ in the sense that its fibre is constant and consists of r^{2g} copies of the same zero-dimensional stack. However, since each point of the fibre is equipped with the r "trivial" automorphisms mentioned above, the forgetful map $\mathcal{M}_{g,n}^{r,\boldsymbol{m}} \to \mathcal{M}_{g,n}$ is of degree r^{2g-1}.

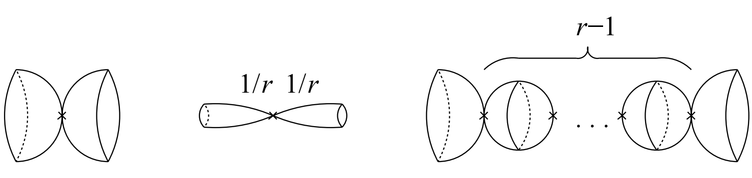
Remark 2.1. Let $\boldsymbol{m} = (m_1, \dots, m_n)$ and $\boldsymbol{m'} = (m_1, \dots, m_i + r, \dots, m_n)$. There is a natural isomorphism between $\mathcal{M}_{g,n}^{r,\boldsymbol{m}}$ and $\mathcal{M}_{g,n}^{r,\boldsymbol{m'}}$. Therefore, from now on, we will always choose $m_1, \dots, m_n$ in $\{1, \dots, r\}$.

2.2 Curves with nodes

There exists a natural compactification $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ of the space $\mathcal{M}_{g,n}^{r,\boldsymbol{m}}$. This compactification can be constructed in three different (but equivalent) ways, and there are, accordingly, three different versions of the universal curve over the compactified moduli space (see figure 1). We are going to describe the behavior of the universal curve at the neighborhood of a node in all three versions.

In the universal curve $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ over the space of stable curves, from a local point of view, there is a unique type of node. The local picture of π at the neighborhood of a node is given by $(x, y) \mapsto t = xy$. However, in the universal curve over the space of r-stable curves, there are r different types of nodes: they are distinguished by assigning to the branches of the curve at the node two integers $a, b \in \{1, \dots, r\}$ such that either $a = b = r$ or $a + b = r$, see below.

(1) *Coarse r-spin curves.* Coarse (or scheme-theoretic) r-spin curves were introduced by Jarvis [17, 18] using relatively torsion-free sheaves.



(A) (B) (C)

Figure 1: (A) a stable curve, (B) an r-stable curve, and (C) an r-bubbled curve.

The local picture of the universal coarse r-spin curve at the neighborhood of a node is given by the equation $xy = t^r$, where t is a local coordinate on the moduli space. Thus the universal curve has an A_{r-1} singularity at $x = y = t = 0$.

If $a = b = r$, then locally, at the neighborhood of the node, $\mathcal{L}$ is a line bundle endowed with an isomorphism

$$\mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_{\log}.$$

(There is no need to twist $\omega_{\log}$ by a divisor of marked points since we are working at the neighborhood of the node.)

If $a + b = r$, then $\mathcal{L}$ is no longer a line bundle, but a (rank-1 torsion-free) sheaf generated by two elements η and ξ, modulo the relations $y\xi = t^a\eta$, $x\eta = t^b\xi$. There is a map from $\mathcal{L}^{\otimes r}$ to the sheaf of sections of $\omega_{\log}$ given by

$$\xi^i\eta^j \mapsto t^{jb}x^{a-j}\frac{dx}{x} = -t^{ia}y^{b-i}\frac{dy}{y},$$

for $i + j = r$. This map is, of course, not an isomorphism.

A stable curve endowed with a sheaf $\mathcal{L}$ like that will be called a *stable coarse r-spin curve.* We often need to generalize this to stable maps from C to a Kähler manifold X. In this case the source curve is a prestable curve, which is not necessarily stable. When C is equipped with a sheaf $\mathcal{L}$ as above, we say that C is a *prestable coarse r-spin curve.*

(2) *r-bubbled spin curves.* The previous construction was improved in [6]. The idea is to transform every stable curve with at least one node into a semistable curve. One obtains that, on the semistable curves, it is enough to consider locally free sheaves.

We provide a precise description as follows. Consider the resolution of the A_{r-1} singularity at the origin of $\{xy = t^r\}$ by $\lfloor r/2 \rfloor$ iterated blowups. One obtains a smooth universal curve, where every node is replaced by a chain of $r - 1$ complex projective lines.

On this universal curve, $\mathcal{L}$ is a line bundle, and we have a map

$$\mathcal{L}^{\otimes r} \to \omega_{\log}\left(-\sum_{i=1}^{n} m_i[s_i]\right).$$

This map is *not* an isomorphism. Indeed, its vanishing divisor is a weighted sum of the projective lines that form the $(r-1)$-chain, with coefficients

$$a, \quad 2a, \quad \ldots, \quad (b-1)a, \quad ba, \quad b(a-1), \quad \ldots, \quad 2b, \quad b,$$

for $a, b \in \{1, \ldots, r-1\}$ and $a+b=r$.

In this paper, a curve like that will be called an *r-bubbled curve*. When endowed with the line bundle $\mathcal{L}$ as above, it will be called an *r-bubbled spin curve*.

(3) *r-stable spin curves.* Another way to improve the initial construction was given in [2, 8]. A stack structure at the neighborhood of every node of every singular curve is introduced. Note that the surface $xy = t^r$ is the quotient of the smooth surface $XY = t$ by the action of the group $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ via $X \mapsto \xi_r X$, $Y \mapsto \xi_r^{-1} Y$. In other words, the surface $xy = t^r$ is actually the coarse space of a smooth stack. Over this smooth stack, $\mathcal{L}$ is a line bundle and

$$\mathcal{L}^{\otimes r} \to \omega_{\log}\left(-\sum_{i=1}^{n} m_i [s_i]\right)$$

is a global isomorphism. These stacks are special cases of Abramovich and Vistoli's stack-theoretic curves: stacks with stabilizers of arbitrary order at the nodes and at the markings (they are called "twisted curves" in [3]).

The fibers of this version of the universal curve are stable curves endowed with a nontrivial stack structure at the nodes. They are called *r-stable curves* and the *r-prestable curves* are defined analogously. The neighborhood of a node in an r-stable curve is isomorphic to an ordinary node $XY = 0$ endowed with the group action of $\mathbb{Z}_r$ via $X \mapsto \xi_r X$, $Y \mapsto \xi_r^{-1} Y$. When endowed with the line bundle $\mathcal{L}$ as above, the curve is called an *r-(pre)stable spin curve.*

An r-stable curve with k nodes has exactly r^k times as many automorphisms as the associated coarse stable curve [1, Theorem. 7.1.1].

In this picture, the numbers a and b result from the $\mathbb{Z}_r$-action involved in the local picture of $\mathcal{L}$ at the two branches. More precisely, locally on the $\mathbb{Z}_r$-space

$$\{XY = 0\} \text{ with } \mathbb{Z}_r \text{ - action}(X, Y) \mapsto (\xi_r X, \xi_r^{-1} Y),$$

the total space of $\mathcal{L}$ is the $\mathbb{Z}_r$-space

$$\{(X, Y, T) \mid XY = 0\} \text{ with } \mathbb{Z}_r \text{ - action}(X, Y, T) \mapsto (\xi_r X, \xi_r^{-1} Y, \xi_r^a T).$$

The index b in $\{1,\ldots,r\}$ is determined by $a+b\equiv 0 \mod r$ or by interchanging X with Y in the local picture.

On the three constructions: Among the three constructions, the third one (involving r-stable curves) is best fit for constructing the compactification. Indeed, $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ is just the solution of the corresponding moduli problem. On the other hand, the objects defined in the first two constructions, carry less information. In these definitions, it can happen that two r-spin curves lying over two distinct points of the boundary of $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ are isomorphic. Thus in these constructions, the moduli functor yields a singular stack, which needs to be normalized if we want to obtain $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$. (The authors in [6, 17] addressed this issue: the moduli functor of r-spin structures should be refined in an appropriate way [6, 18, p. 26; (a,b)], but we do not discuss this here.)

On the other hand, if we wish to apply the Grothendieck–Riemann–Roch (GRR) formula, it turns out that it is more straightforward to work with the second construction involving r-bubbled curves. This happens because the GRR formula applies without modifications to morphisms between stacks only if the fibers of the morphisms are schemes, which is not true for r-stable curves.

In this paper we will talk about spin structures on r-(pre)stable curves when describing moduli spaces and morphisms between them, but we will switch to r-bubbled curves when we want to apply GRR and calculate the Chern character of the K-theoretical direct image of the r-spin structure. The following remark explains why this direct image coincides with the direct image via the r-(pre)stable curve.

Remark 2.2. There are the following morphisms between universal curves:

$$p'\colon \overline{\mathcal{C}}^{\text{bubble}} \to \overline{\mathcal{C}}^{\text{coarse}} \qquad \text{and} \qquad p''\colon \overline{\mathcal{C}}^{\text{stacky}} \to \overline{\mathcal{C}}^{\text{coarse}}.$$

Moreover, we have $\mathcal{L}^{\text{coarse}} = p'_*\mathcal{L}^{\text{bubble}} = p''_*\mathcal{L}^{\text{stacky}}$.

2.3 Boundary

The structure of the boundary of $\overline{\mathcal{M}}_{g,n}^{r,\boldsymbol{m}}$ is similar to that of $\overline{\mathcal{M}}_{g,n}$, but there is a new subtlety.

Notation. *By* $[n]$ *we denote the set* $\{1,\ldots,n\}$*. Let* $I\subset[n]$*. For any multiindex* $\boldsymbol{m}=(m_1,\ldots,m_n)$*, we denote by* $\boldsymbol{m}_I$ *the multiindex* $(m_i)_{i\in I}$*.*

For any nonnegative integers n and g we define the following involutions.

(1) Given $l \in \{0, \dots, g\}$, we write l' for $g - l$.

(2) Given a subset $I \subset [n]$ we write I' for $[n] \setminus I$.

(3) Given $q \in \{1, \dots, r\}$ we write q' for the integer in $\{1, \dots, r\}$ satisfying $q + q' \in r\mathbb{Z}$.

The boundary $\partial\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$ of $\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$ is the moduli space of *singular* r-stable spin curves. The normalization $N(\partial\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n})$ of the boundary is the moduli space of pairs $(C, \text{node of } C)$, where C is a singular r-stable spin curve. Finally, we consider a double cover $\mathcal{D}$ of the normalization, namely, the moduli space of triples (C, node of C, branch of C at the node).

While in the case of moduli spaces of stable curves, the space $\mathcal{D}$ turned out to be a disjoint union of several smaller moduli spaces, the picture here is more complicated: the space $\mathcal{D}$ is not isomorphic, but can be projected to a disjoint union of smaller moduli spaces.

The stack $\mathcal{D}$ is naturally equipped with two line bundles whose fibers are the cotangent lines to the chosen branch of the *coarse* stable curve and to the other branch. We write

$$\psi, \psi' \in H^2(\mathcal{D}, \mathbb{Q})$$

for their respective first Chern classes. Note that in this notation, we privilege the coarse curve, because in this way the classes ψ and ψ' are more easily related to the classes ψ_i introduced in Section 1.1.6.

Recall that the spin bundle $\mathcal{L}$ determines local indices a and b in $\{1, \dots, r\}$ satisfying $a + b \equiv 0 \mod r$ in one-to-one correspondence with the branches of the node. Therefore, to each point (C, node of C, branch of C at the node) we associate an index $q \in \{1, \dots, r\}$ by setting either $q = a$ or $q = b$ depending on the branch.

We can decompose $\mathcal{D}$ according to the topological type of the node and the index of the spin bundle $\mathcal{L}$ at the chosen branch:

$$\mathcal{D} = \bigsqcup_{\substack{0 \le l \le g \\ I \subseteq [n]}} \mathcal{D}_{l,I} \sqcup \bigsqcup_{1 \le q \le r} \mathcal{D}^q_{\mathrm{irr}}. \tag{2.1}$$

A point of $\mathcal{D}_{l,I}$ corresponds to a curve with a node that divides it into a component of genus l with marking set I and a component of genus l' with marking set I'. A point of $\mathcal{D}^q_{\mathrm{irr}}$ corresponds to a curve with a nonseparating node, the index of the spin bundle at the chosen branch being q. We have the natural morphisms $j_{l,I}\colon \mathcal{D}_{l,I} \longrightarrow \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$ and $j^q_{\mathrm{irr}}\colon \mathcal{D}^q_{\mathrm{irr}} \longrightarrow \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$.

Remark 2.3. Over $\mathcal{D}_{l,I}$ the multiplicity index $q \in \{1,\dots,r\}$ is constant and satisfies

$$2l-1-\sum_I (m_i-1) \equiv q \mod r. \tag{2.2}$$

We denote this index by $q(l,I)$. On the other hand, on $\mathcal{D}^q_{\mathrm{irr}}$ the index q is constant by definition. Let us set $\mathcal{D}_{\mathrm{irr}} = \bigsqcup_q \mathcal{D}^q_{\mathrm{irr}}$.

There exist natural morphisms from $\mathcal{D}_{l,I}$ and $\mathcal{D}^q_{\mathrm{irr}}$ to the moduli of r-stable spin curves of smaller dimension (see [19]).

$$\begin{tikzcd}
& \mathcal{D}_{l,I} \arrow[dl, "\mu_{l,I}"'] \arrow[dr, "j_{l,I}"] & l\in\{0,\dots,g\},\ I\subseteq[n],\ q=q(l,I) \\
\overline{\mathcal{M}}^{r,(\boldsymbol{m}_I,q)}_{l,|I|+1} \times \overline{\mathcal{M}}^{r,(\boldsymbol{m}_{I'},q')}_{l',|I'|+1} & & \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{D}^q_{\mathrm{irr}} \arrow[dl, "\mu^q_{\mathrm{irr}}"'] \arrow[dr, "j^q_{\mathrm{irr}}"] & q\in\{1,\dots,r\} \\
\overline{\mathcal{M}}^{r,(\boldsymbol{m},q,q')}_{g-1,n+2} & & \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}
\end{tikzcd}$$

These morphisms are obtained by

(1) normalizing the curve at the node and taking the pullback of the spin bundle to the normalization,
(2) "forgetting" the orbifold structure at the two new marked points (this is the same as passing to the coarse space, but only locally), and
(3) replacing the spin bundle $\mathcal{L}$ by the sheaf of its invariant sections (this sheaf turns out to be locally free at the two new marked points).

Remark 2.4. When $q = r$, we actually obtain two r-spin structures of types $(\boldsymbol{m}_I, 0)$ and $(\boldsymbol{m}_{I'}, 0)$. Indeed, $q = r$ is the case where, locally at the node, the line bundles are pullbacks of rth roots of $\omega_{\log}$ on the coarse space. If we want the indices m_i to lie in the set $\{1,\dots,r\}$ as usual, we must further compose the functor $\mu_{l,I}$ with the canonical isomorphisms recalled in Remark 2.1 shifting by r the two multiindices at the two new markings.

We set $\mu_{\mathrm{irr}} = \bigsqcup_q \mu_{\mathrm{irr}}^q$.

Let us prove that the degree of $\mu_{l,I}$ and μ_{irr} equals 1. Assume, for simplicity that we are in the case where the generic curve has trivial automorphism group. Set $d = \mathrm{GCD}(q, r)$. Then we claim the following:

A generic geometric point in the image of $\mu_{l,I}$ has a stabilizer of order r^2. It has one geometric preimage with stabilizer of order r^2. The degree of $\mu_{l,I}$ equals 1.

A generic geometric point in the image of μ_{irr} has a stabilizer of order r. It has d geometric preimages with stabilizers of order rd. The degree of μ_{irr} equals 1.

$$\overset{\frac{1}{r^2}}{\bullet} \xrightarrow{\ \mu_{l,I}\ } \overset{\frac{1}{r^2}}{\bullet} \qquad\qquad \underbrace{\overset{\frac{1}{rd}}{\bullet} \quad \cdots \quad \overset{\frac{1}{rd}}{\bullet}}_{d} \xrightarrow{\ \mu_{\mathrm{irr}}\ } \overset{\frac{1}{r}}{\bullet}$$

For both morphisms, we consider a generic point in the image and work out the fiber over it.

For $\mu_{l,I}$, the point in the image is a pair formed by a smooth genus-l $(|I|+1)$-marked curve C_1 and a smooth genus-l' $(|I'|+1)$-marked curve C_2, each with trivial automorphism group, and each equipped with spin bundles L_1 and L_2. Then, due to the "trivial" automorphisms acting by multiplication along the fibers of L_1 and — independently — along the fibers of L_2, the point has a stabilizer of order r^2. The geometric points of the fiber are "gluings" of $L_1 \to C_1$ and $L_2 \to C_2$; that is to say: r-stable spin curves yielding $L_1 \to C_1$ and $L_2 \to C_2$ when we apply procedures (1–3) listed above. There is only one coarse curve obtained from identifying the $(|I|+1)$th point of C_1 to the $(|I'|+1)$th of C_2 and, more importantly, there is only one r-stable curve C over it (see for example [24, Lemma. 5.3]). Furthermore, the fact that the node is separating implies that there is only one line bundle L up to isomorphism gluing L_1 and L_2. Therefore the fiber of $\mu_{l,I}$ contains a single point. We now show that its stabilizer has order r^2 as desired. As mentioned above, the order of $\mathrm{Aut}(C, \text{markings})$ equals r, because of the presence of a node [1, Theorem 7.1.1]. By pulling back the spin bundle L via any of these automorphisms, we get another spin bundle gluing L_1 and L_2. We just showed that, up to isomorphism, there exists only one such gluing; therefore, we conclude that each automorphism of the r-stable curve lifts to an automorphism of the r-stable spin curve. Furthermore, the r-stable spin curve has exactly r times as many automorphism as the r-stable curve, because the morphisms lifting the identity of C are exactly r "trivial" automorphisms acting by multiplication on the fibers of L.

For μ^q_{irr} a generic point in the image is a spin bundle L_0 over a smooth genus-$(g-1)$ $(n+2)$-marked r-stable curve C_0 satisfying $\mathrm{Aut}(C_0,$ markings$) = 1$. Its automorphism group has order r: it consists of the "trivial" automorphisms acting by multiplication on the fibers of L_0. The geometric points of the fibers are "gluings": r-stable spin curves yielding C_0 and L_0 through procedures (1–3). Again, and for the same reasons as above, there is only one r-stable curve C yielding C_0 after normalization and passage to the coarse space. However, since the node is nonseparating, there are exactly r spin bundles gluing L_0 on the nodal curve C. Let us first show that on C there are exactly r gluings of the trivial bundle on C_0: these are the sheaves $T(0), T(1), \dots, T(r-1)$ of regular functions $f\colon C_0 \to \mathbb{C}$ satisfying a compatibility conditions $f(s_{n+1}) = \xi_r^i f(s_{n+2})$ at the $(n+1)$th marking s_{n+1} and at the $(n+2)$nd marking s_{n+2} of C_0. In fact, $\{T(0), T(1), \dots, T(r-1)\}$ is a cyclic group generated by $T(1)$ and acting freely and transitively on the gluings of $L_0 \longrightarrow C_0$ on C; therefore, once one of such gluings L is fixed, we can write them all as $\{L \otimes T(0), L \otimes T(1), \dots, L \otimes T(r-1)\}$. These line bundles are pairwise nonisomorphic over C, i.e., an isomorphism

$$\begin{tikzcd} L \otimes T(i) \arrow[rr, "\sim"] \arrow[dr] & & L \otimes T(j) \arrow[dl] \\ & C & \end{tikzcd}$$

exists only if $i = j$.

The automorphisms of $(C, \text{markings})$, form a cyclic group of order r, because of the presence of a node [1, Theorem 7.1.1]. Denote by α a generator of this group. Let $d = GCD(r, q)$. In [8, Proposition 2.5.3], the first author shows that there exists an isomorphism

$$\begin{tikzcd} L \otimes T(i) \arrow[r, "\sim"] \arrow[d] & L \otimes T(j) \arrow[d] \\ C \arrow[r, "\sim", "\alpha"'] & C \end{tikzcd}$$

if and only if $j = i + d$.

In this way these automorphisms identify r/d by r/d the spin bundles

$$\{L \otimes T(0), L \otimes T(1), \dots, L \otimes T(r-1)\}.$$

Furthermore, only the d automorphisms $\alpha^0, \alpha^{\frac{r}{d}}, \dots, \alpha^{(d-1)\frac{r}{d}}$ lift to automorphisms of the r-stable spin curve. As above, due to the "trivial" automorphism acting by multiplication on the fibers, we observe that the automorphisms of the r-stable spin curve are r times as many as the automorphisms lifting from the r-stable curve. In this way, the fiber consists of d distinct geometric points with stabilizers of order rd.

Remark 2.5. We point out that $\mu_{l,I}$ and μ^q_{irr} are not isomorphisms in general. The discussion above makes it evident for μ^q_{irr}: in general, the generic fiber contains more than one geometric point. A more detailed analysis shows that also $\mu_{l,I}$ is not an isomorphism in general.[1]

The discussion above can be extended to the case of stable maps, as we will see in the next section.

3 Spaces of spin maps, virtual classes, and twisted potentials

The space $X^{r,\boldsymbol{m}}_{g,n,D} = \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n,D}(X)$ is the *moduli space of r-stable spin maps of degree D*: these are certain types of maps from r-prestable spin curves to the Kähler manifold X whose image cycle is rationally equivalent to D. The stack $X^{r,\boldsymbol{m}}_{g,n,D}$ is equipped with a universal r-prestable curve, whose coarse space is the universal coarse prestable curve, in which A_{r-1} singularities appear. The desingularization of this coarse curve is the universal r-bubbled curve, which is well suited for GRR calculations.

3.1 Moduli of r-stable spin maps

Let X be a Kähler manifold. Fix $r \geq 2$, two positive integers g and n, and an effective cycle D in $H_2(X, \mathbb{Z})$. By $X^{r,\boldsymbol{m}}_{g,n,D}$, we denote the category of r-stable

[1]We consider for simplicity the case $q(l, I) = r$, but the discussion extends whenever $q = q(l, I)$ is not prime to r. We claim that $\mu_{l,I}$ is not fully faithful. Take an automorphism acting by multiplication by two different factors on the fibers of L_1 and L_2 defined above. Such an automorphism is not in the image of the functor $\mu_{l,I}$. After applying $\mu_{l,I}$ by following procedures (1–3), we find that the r^2 automorphisms of the r-stable spin curve $L \to C$ only yield automorphisms acting by multiplication by the same factor on L_1 and L_2. Indeed, the case $q(l, I) = r$ implies that L is a pullback from the coarse space; since each automorphism $\alpha \in \mathrm{Aut}(C, \text{markings})$ fix the coarse space, the line bundle $\alpha^* L$ is equal — and not only isomorphic — to L. In this way, all automorphisms lifting α to $L \to C$ yield the identity on $L_0 \to C_0$.

spin maps. An object is given by the data

$$
\begin{tikzcd}
\mathcal{L} \arrow[dr] & & \\
& C \arrow[r] \arrow[d, "\pi"] & X \\
& B \arrow[u, bend left=60, "s_1"] \arrow[u, bend left=30, "s_n"'] &
\end{tikzcd}
\qquad \varphi\colon \mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_{\log}(-\textstyle\sum_{i=1}^{n} m_i[s_i(B)]),
$$

satisfying the following conditions:

(1) $C \to B$ is an r-prestable curve. More explicitly C is a stack of relative dimension one over a base scheme B, its smooth locus is represented by a scheme, and its singularities are nodes with cyclic stabilizers of order r. The local picture at the nodes is given by $[\{XY = t\}/\mathbb{Z}_r]$, where t is a local parameter on the base scheme and the group $\mathbb{Z}_r$ acts by $(X, Y, t) \mapsto (\xi_r X, \xi_r^{-1} Y, t)$.
(2) The coarse schemes and morphisms between schemes corresponding to $C \to B$, $C \to X$, and $s_1, \dots, s_n\colon B \to C$ form a stable map of genus g over B, of degree D, and marked at n distinct smooth points.
(3) $\mathcal{L}$ is a line bundle on C and φ sets an isomorphism between $\mathcal{L}^{\otimes r}$ and $\omega_{\log}(-\sum_{i=1}^{n} m_i[s_i(B)])$.

Morphisms are defined in the natural way. Since C is a stack, in order to obtain a category, we need to consider one-morphisms up to two-isomorphisms (this yields a two-category equivalent to a category as detailed in [3]).

It follows immediately from [3, 8, 24] that $X^{r,\boldsymbol{m}}_{g,n,d}$ is a proper Deligne–Mumford stack. Beside Kontsevich's construction of the stack of stable maps, the key fact is the existence of an algebraic stack of r-prestable curves (a straightforward consequence of Olsson's description [24] of the stack of all Abramovich and Vistoli's stack-theoretic curves). We detail this as follows:

(i) The functor retaining only the coarse schemes and morphisms between schemes corresponding to $C \to B$, $C \to X$, and $s_1, \dots, s_n\colon B \to C$ lands on Kontsevich's proper stack $X_{g,n,D}$:

$$p\colon X^{r,\boldsymbol{m}}_{g,n,D} \to X_{g,n,D}. \tag{3.1}$$

(ii) Note that $X_{g,n,D}$ naturally maps to $\mathfrak{M}_{g,n}$, the stack of genus-g n-pointed prestable curves. Morphism (3.1) is the base change via

$p\colon X_{g,n,D} \to \mathfrak{M}_{g,n}$ of the proper morphism

$$\mathfrak{M}^{r,\boldsymbol{m}}_{g,n} \to \mathfrak{M}_{g,n} \tag{3.2}$$

sending r-prestable curves equipped with an r-spin structure $\mathcal{L}^{\otimes r} \cong \omega_{\log}(-\sum_{i=1}^n m_i[s_i(B)])$ to the coarse prestable curves. Morphism (3.2) is proper and represented by Deligne–Mumford stacks. This can be regarded as a consequence of [24], showing that the stack of r-prestable curves is proper over the stack of prestable curves, and of [8], showing that the functor of rth roots is proper over the stack of r-prestable curves.

Remark 3.1. If X is a point, we recover the stack $\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$ and (3.1) is the morphism $\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

Remark 3.2. Moduli of stable maps equipped with r-spin structures were introduced in [20] by means of Jarvis's notion of coarse stable r-spin curve. As mentioned above, with this technique several technical points concerning the singularities appearing in the moduli stack and the stabilization morphisms need to be addressed. The use of Abramovich and Vistoli's stack-theoretic curves simplifies the treatment of the moduli stack and avoids dealing with singularities. This approach to Gromov–Witten r-spin theory via stack-theoretic curves was alluded to in [19, Section 2.3, 7]. After Olsson [24], this treatment becomes straightforward. From the point of view of enumerative geometry, the two approaches of [20] or via stack-theoretic curves are equivalent; this follows immediately from the case treated in [2], which can be regarded as the case $X = \mathrm{pt}$ (see also [9, Section 4.3]).

An irreducible component of $X^{r,\boldsymbol{m}}_{g,n,D}$ whose generic points correspond to singular curves is, in general, nonreduced. Indeed, such a component is projected onto a ramification locus of morphism (3.2). (Another explanation: following [2] one can realize $X^{r,\boldsymbol{m}}_{g,n,D}$ as the moduli stack of stable maps to a *target stack*. Such moduli stacks may well be nonreduced.)

3.2 The genus-g r-spin twisted Gromov–Witten potential of X

The stack $X^{r,\boldsymbol{m}}_{g,n,D}$ is equipped with a virtual r-spin class, a homology class playing the role of the virtual fundamental class in the standard Gromov–Witten theory of stable maps. This is the class used in the definition of intersection numbers in Gromov–Witten r-spin theory. We recall its definition as follows (we refer to Definitions 4.7 and 5.4 in [20]):

$$\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v = c_W(\boldsymbol{m}) \cap p^*[X_{g,n,D}]^v,$$

where $c_W(\boldsymbol{m})$ is the so called Witten's top Chern class, a cohomology class, whose degree is opposite to the Euler characteristic of the universal spin structure. (The explicit expression for the degree is $((r-2)(g-1)-n+\sum_i m_i)/r$, by Riemann–Roch.) The class c_W has several compatible constructions [7, 22, 25], which extend naturally from $\overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$ to $X^{r,\boldsymbol{m}}_{g,n,D}$. Furthermore, this class can be understood as the virtual fundamental homology class of the space of $1/r$-differentials on stable maps, but we do not develop this point here.

The definitions of the tautological classes $\psi_1,\ldots,\psi_n$, κ_d, and ch_d introduced in Section 1.1.6 naturally extend over $X^{r,\boldsymbol{m}}_{g,n,D}$.

The coefficients appearing in the r-spin Gromov–Witten potential are intersection number of the tautological classes against the virtual r-spin class $\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v$. We recall from Section 1.1 the definition of the genus-g r-spin twisted Gromov–Witten potential of X, $F_{g,r}(\boldsymbol{t},\boldsymbol{s})$

$$\sum_D Q^D \sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{m_1,\ldots,m_n\\ a_1,\ldots,a_n\\ \mu_1,\ldots,\mu_n}}\frac{1}{r^{g-1}}\int_{\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v}\exp\Big(\sum_{d\geq 1}s_d\,\mathrm{ch}_d\Big)\prod_{i=1}^n \psi_i^{a_i}\,\mathrm{ev}_i^*(h_{\mu_i})t_{a_i}^{\mu_i\otimes m_i}.$$

The total r-spin twisted Gromov–Witten potential of X is given by $F_r(\boldsymbol{t},\boldsymbol{s})=\sum_{g\geq 0}\hbar^{g-1}F_{g,r}(\boldsymbol{t},\boldsymbol{s})$.

3.3 Properties of the virtual r-spin class

We state the properties of the virtual r-spin class $\left[X^{r,\boldsymbol{m}}_{g,n,d}\right]^v$ needed in the rest of the paper. The main result is Proposition 3.1, which can be regarded as the generalization to r-spin maps of the main factorization properties of the virtual fundamental class $[X_{g,n,D}]^v$. The notation follows closely Faber and Pandharipande's treatment [12, Section 1.2].

Remark 3.3. In [19, Section 5], the authors provide a statement of the Gromov–Witten r-spin factorization properties after pushforward via the forgetful morphism $p\colon X^{r,\boldsymbol{m}}_{g,n,D}\to X_{g,n,D}$. These factorization properties are sufficient if we wish to intersect $\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v$ only with pullbacks from $X_{g,n,D}$. However, in the r-spin twisted Gromov–Witten potential above, we consider intersections of classes such as ch_d that are not pullbacks from $X_{g,n,D}$.

To begin with, at (3.1–3.3), we set up natural morphisms π, j, and μ needed in the statements. First, recall the morphism forgetting the $(n+1)$th point:

$$\pi\colon X^{r,(\boldsymbol{m},1)}_{g,n+1,D} \longrightarrow X^{r,\boldsymbol{m}}_{g,n,D}. \tag{3.3}$$

Second, we extend the discussion Section 2.3 of the boundary locus to the substack $\partial X^{r,\boldsymbol{m}}_{g,n,D} \hookrightarrow X^{r,\boldsymbol{m}}_{g,n,D}$ of singular objects: i.e., r-spin structures over singular r-prestable curves mapping to X. By the same definitions of Section 2.3, we get the locus $\mathcal{D}$ classifying triples (C, node of C, branch of C at the node), which admits the natural decomposition

$$\mathcal{D} = \bigsqcup_{\substack{0\leq l\leq g\\ I\subseteq[n]}} \mathcal{D}_{l,I} \sqcup \bigsqcup_{1\leq q\leq r} \mathcal{D}^q_{\mathrm{irr}}$$

analogue to (2.1). The images of the morphisms $j_{l,I}\colon \mathcal{D}_{l,I} \longrightarrow X^{r,\boldsymbol{m}}_{g,n,D}$ and $j^q_{\mathrm{irr}}\colon \mathcal{D}^q_{\mathrm{irr}} \longrightarrow X^{r,\boldsymbol{m}}_{g,n,D}$ describe the entire boundary locus. As in Section 2.3, the stack $\mathcal{D}$ can be projected to certain moduli stacks, which we denote by Δ_ξ, where ξ labels the topological type of the splitting of the curve and of the map at the node.

First, we introduce the set Ω of such splittings ξ

$$\Omega = \Omega_{\mathrm{irr}} \sqcup \bigsqcup_{\substack{0\leq l\leq g\\ I\subseteq[n]}} \Omega_{l,I},$$

with $\Omega_{l,I} = \{(l, I, A) \mid A \in H_2(X,\mathbb{Z})\}$, and $\Omega_{\mathrm{irr}} = \{(\mathrm{irr}, q) \mid q \in \{1,\dots,r\}\}$. To each $\xi \in \Omega$, we attach a stack Δ_ξ as follows. For $\xi = (l, I, A) \in \Omega_{l,I}$, we denote by Δ_ξ the stack fitting in the fiber diagram

$$\begin{tikzcd} \Delta_\xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{r,(\boldsymbol{m}_{I'},q')}_{l',|I'|+1,A'} \arrow[d, "\mathrm{ev}'"] \\ X^{r,(\boldsymbol{m}_I,q)}_{l,|I|+1,A} \arrow[r, "\mathrm{ev}"'] & X, \end{tikzcd}$$

where $A' = D - A \in H_2(X,\mathbb{Z})$, q equals the index $q(l, I)$ defined by equation (2.2), and the fibered product is taken with respect to the morphisms ev and ev$'$ evaluating the $(|I| + 1)$th and the $(|I'| + 1)$th point. Equivalently, we say

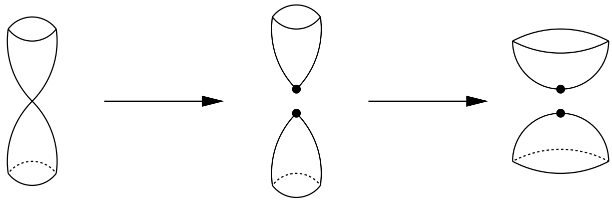


Figure 2: The morphisms μ.

that Δ_ξ fits in the fiber diagram

$$\begin{tikzcd} \Delta_\xi \arrow[rr] \arrow[d] & & X \arrow[d, "\delta=(\mathrm{id},\mathrm{id})"] \\ X^{r,(\boldsymbol{m}_I,q)}_{l,|I|+1,A} \times X^{r,(\boldsymbol{m}_{I'},q')}_{l',|I'|+1,A'} \arrow[rr, "\mathrm{ev}\times\mathrm{ev}'"'] \arrow[urr, phantom, "\square"] & & X \times X, \end{tikzcd}$$

where $\delta\colon X \to X \times X$ is the diagonal morphism.

Similarly, for $\xi = (\mathrm{irr}, q) \in \Omega_{\mathrm{irr}}$, the stack Δ_ξ fits in the fiber diagram

$$\begin{tikzcd} \Delta_\xi \arrow[rr] \arrow[d] & & X \arrow[d, "\delta=(\mathrm{id},\mathrm{id})"] \\ X^{r,(\boldsymbol{m},q,q')}_{g-1,n+2,D} \arrow[rr, "(\mathrm{ev},\mathrm{ev}')"'] \arrow[urr, phantom, "\square"] & & X \times X, \end{tikzcd}$$

where the morphisms ev and ev′ evaluating the $(n + 1)$st and the $(n + 2)$nd point.

By applying to $\mathcal{D}_{l,I}$ and $\mathcal{D}^q_{\mathrm{irr}}$ the morphism defined in Section 2.3 (see figure 2) we obtain the diagrams

$$\begin{tikzcd} & \mathcal{D}_{l,I} \arrow[dl, "\mu_{l,I}"'] \arrow[dr, "j_{l,I}"] & \\ \bigsqcup_{\xi\in\Omega_{l,I}} \Delta_\xi & & X^{r,\boldsymbol{m}}_{g,n,D} \end{tikzcd} \qquad l \in \{0, \dots, g\},\ I \subseteq [n] \tag{3.4}$$

$$\begin{tikzcd} & \mathcal{D}^q_{\mathrm{irr}} \arrow[dl, "\mu^q_{\mathrm{irr}}"'] \arrow[dr, "j^q_{\mathrm{irr}}"] & \\ \Delta_{(\mathrm{irr},q)} & & X^{r,\boldsymbol{m}}_{g,n,D} \end{tikzcd} \qquad q \in \{1, \dots, r\} \tag{3.5}$$

Remark 3.4. We notice that certain stacks Δ_ξ may well be empty. This is indeed the case if A or $D-A$ is not an effective cycle in $H^2(X,\mathbb{Z})$.

The stacks Δ_ξ are moduli functors classifying r-stable spin maps and, therefore, are equipped with virtual r-spin classes $[\Delta_\xi]^v$ obtained, as above, by intersecting the virtual fundamental class of the corresponding moduli stack of stable maps with Witten's class c_W. Such virtual r-spin classes $[\Delta_\xi]^v$ can be explicitly obtained as follows (Axiom 4 of [5], see also [12, Section 1.2]). For $\xi=(l,I,A)\in\Omega_{l,I}$, we have

$$\left[\Delta_{(l,I,A)}\right]^v = \left[X^{r,(\boldsymbol{m}_I,q)}_{l,|I|+1,A}\right]^v \times \left[X^{r,(\boldsymbol{m}_{I'},q')}_{l',|I'|+1,A'}\right]^v \cap (\mathrm{ev}\times\mathrm{ev}')^{-1}(\delta), \tag{3.6}$$

where δ is the diagonal cycle in $X\times X$. For $\xi=(\mathrm{irr},q)\in\Omega_{\mathrm{irr}}$, we have

$$[\Delta_{(\mathrm{irr},q)}]^v = \left[X^{r,(\boldsymbol{m},q,q')}_{g-1,n+2,D}\right]^v \cap (\mathrm{ev},\mathrm{ev}')^{-1}(\delta). \tag{3.7}$$

The factorization property relates the virtual r-spin class of Δ_ξ and $X^{r,\boldsymbol{m}}_{g,n,D}$. Note that it is delicate to restrict the virtual fundamental class of $X^{r,\boldsymbol{m}}_{g,n,D}$ to the boundary locus, because $X^{r,\boldsymbol{m}}_{g,n,D}$ may be singular or simply not of the expected dimension. This problem already arises for moduli of stable maps, where calculations involve the natural morphism from $X_{g,n,D}$ to the nonsingular algebraic stack $\mathfrak{M}_{g,n}$ of prestable n-pointed genus-g curves. Similarly, we need to regard $X^{r,\boldsymbol{m}}_{g,n,D}$ alongside with the natural morphism to the nonsingular algebraic stack $\mathfrak{M}^{r,\boldsymbol{m}}_{g,n}$ of r-prestable curves equipped with an r-spin structure of type $\boldsymbol{m}$:

$$X^{r,\boldsymbol{m}}_{g,n,D} \longrightarrow \mathfrak{M}^{r,\boldsymbol{m}}_{g,n}.$$

Then, we take refined pullbacks $(\mathfrak{j}_{l,I})^!$ and $(\mathfrak{j}^q_{\mathrm{irr}})^!$ (see [14]) via the natural morphisms

$$\mathfrak{j}_{l,I}\colon \mathfrak{D}_{l,I}\to\mathfrak{M}^{r,\boldsymbol{m}}_{g,n} \qquad\text{and}\qquad \mathfrak{j}^q_{\mathrm{irr}}\colon \mathfrak{D}^q_{\mathrm{irr}}\to\mathfrak{M}^{r,\boldsymbol{m}}_{g,n}$$

induced by the usual decomposition of the locus $\mathfrak{D}$ mapping to the boundary locus of $\mathfrak{M}^{r,\boldsymbol{m}}_{g,n}$ representing r-spin structures over singular r-prestable curves.

Clearly, the morphisms $\mathrm{j}_{l,I}$, $\mathrm{j}^q_{\mathrm{irr}}$, $j_{l,I}$, and j^q_{irr} fit in the fiber diagrams

$$\begin{tikzcd} \mathcal{D}_{l,I} \arrow[r, "j_{l,I}"] \arrow[d] \arrow[dr, phantom, "\square"] & X^{r,\boldsymbol{m}}_{g,n,D} \arrow[d] \\ \mathfrak{D}_{l,I} \arrow[r, "\mathrm{j}_{l,I}"'] & \mathfrak{M}^{r,\boldsymbol{m}}_{g,n}. \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}^q_{\mathrm{irr}} \arrow[r, "j^q_{\mathrm{irr}}"] \arrow[d] \arrow[dr, phantom, "\square"] & X^{r,\boldsymbol{m}}_{g,n,D} \arrow[d] \\ \mathfrak{D}^q_{\mathrm{irr}} \arrow[r, "\mathrm{j}^q_{\mathrm{irr}}"'] & \mathfrak{M}^{r,\boldsymbol{m}}_{g,n}. \end{tikzcd}$$

where $\mathfrak{D}_{l,I}$ and $\mathfrak{D}^q_{\mathrm{irr}}$ are the terms of the natural decomposition

$$\mathfrak{D} = \bigsqcup_{\substack{0\leq l\leq g\\ I\subseteq[n]}} \mathfrak{D}_{l,I} \sqcup \bigsqcup_{1\leq q\leq r} \mathfrak{D}^q_{\mathrm{irr}}.$$

Proposition 3.1. *The virtual r-spin class $\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v$ satisfies the following properties.*

Factorization property. *For any $l \in \{0,\dots,g\}$ and $I \subseteq [n]$, we have*

$$(\mu_{l,I})_*(\mathrm{j}_{l,I})^!\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v = \sum_{\xi\in\Omega_{l,I}} [\Delta_\xi]^v,$$

and for any $q \in \{1,\dots,r\}$, we have

$$(\mu^q_{\mathrm{irr}})_*(\mathrm{j}^q_{\mathrm{irr}})^!\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v = [\Delta_{(\mathrm{irr},q)}]^v,$$

Forgetting property: *We have*

$$[X^{r,(\boldsymbol{m},1)}_{g,n+1,D}]^v = \pi^*[X^{r,\boldsymbol{m}}_{g,n,D}]^v,$$

where π is the morphism $X^{r,(\boldsymbol{m},1)}_{g,n+1,D} \longrightarrow X^{r,\boldsymbol{m}}_{g,n,D}$ forgetting the $(n+1)$th point.

Proof. The equations above are based on the properties of the virtual fundamental class $[X_{g,n,D}]^v$ proved in [4,5,21] and on the properties of Witten's class $c_W(\boldsymbol{m})$ proved in [19,25,26].

The forgetting property is an immediate consequence of the analogous properties for $[X_{g,n,D}]^v$ and $c_W(\boldsymbol{m})$.

On the other hand, the factorization property requires the Isogeny property of [4,5]. This condition claims that for any $l \in \{0,\dots,g\}$ and $I \subseteq [n]$, we have

$$(\mathrm{j}_{l,I})^! p^*[X_{g,n,D}]^v = (\mu_{l,I})^*(p^*[X_{l,|I|+1,A}]^v \times p^*[X_{l',|I'|+1,A'}]^v \\ \cap (\mathrm{ev} \times \mathrm{ev}')^{-1}(\delta)),$$

where, on the right-hand side, δ is the diagonal in $X \times X$ and we implicitly sum over the parameter A ranging over $H_2(X, \mathbb{Z})$. Furthermore, for $q \in \{1, \dots, r\}$, we have

$$(\mathrm{j}_{\mathrm{irr}}^q)^! p^*[X_{g,n,D}]^v = (\mu_{\mathrm{irr}}^q)^* \left(p^*[X_{g-1,n+2,D}]^v \cap (\mathrm{ev} \times \mathrm{ev}')^{-1}(\delta) \right).$$

Witten's cohomology class $c_W(\boldsymbol{m})$ satisfies the following factorization properties proved in [19, 25, 26]. For any $l \in \{0, \dots, g\}$ and for any $I \subseteq [n]$, we have

$$(\mu_{l,I})_*(j_{l,I})^*(c_W(\boldsymbol{m})) = c_W(\boldsymbol{m}_I, q) \times c_W(\boldsymbol{m}_{I'}, q').$$

For any $q \in \{1, \dots, r\}$, we have

$$(\mu_{\mathrm{irr}}^q)_*(j_{\mathrm{irr}}^q)^*(c_W(\boldsymbol{m})) = c_W(\boldsymbol{m}, q, q').$$

The projection formula for $\mu_{l,I}$ and μ_{irr}^q yields immediately the desired factorization property. □

Remark 3.5. We briefly recall why we restrict the range of the indices m_i to $\{1, \dots, r\}$. Indeed, it makes sense to define the moduli of r-stable spin curves for any multiindex $\boldsymbol{m} \in \mathbb{Z}^n$. On the other hand, the construction of Witten's class $c_W(\boldsymbol{m})$ requires that the entries m_i are positive. Furthermore, this extended Witten's class $c_W(\boldsymbol{m})$ satisfies the so-called descending and the vanishing properties. The descending property claims that for all $i \in \{1, \dots, n\}$, we have

$$c_W(\boldsymbol{m} + r\boldsymbol{\delta}_i) = -\frac{m_i}{r} \ \psi_i \ c_W(\boldsymbol{m}).$$

The vanishing property claims that, if $m_i \in r\mathbb{Z}$ for some $1 \leq i \leq n$, then we have

$$c_W(\boldsymbol{m}) = 0.$$

Automatically, the virtual r-spin class $\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^{\mathrm{v}}$ shares analogous recursive relations. Because of these properties, it makes sense to restrict ourselves to the values $m_i \in \{1, \ldots, r-1\}$.

4 Applying the Grothendieck–Riemann–Roch formula

In [9, Theorem 1.1.2], the first author applied the Grothendieck–Riemann–Roch formula to the r-spin structure $\mathcal{L}$ over the universal curve. These methods can be easily generalized to any family of r-bubbled spin curves. In Section 4.1, we describe how the GRR formula works in this case. In Section 4.2 we intersect the GRR formula with the virtual r-spin class of $X^{r,\boldsymbol{m}}_{g,n,D}$. Finally in Section 4.3 we use this version of the GRR formula to deduce the differential equation satisfied by the twisted r-spin potential. The final result, Proposition 4.3, is the crucial ingredient allowing us to generalize Givental's quantization.

4.1 The GRR formula for r-bubbled spin curves

In [9] the GRR formula was applied to the universal curve $\pi\colon \overline{\mathcal{C}}^{r,\boldsymbol{m}}_{g,n} \to \overline{\mathcal{M}}^{r,\boldsymbol{m}}_{g,n}$. In this section, we briefly describe these computations to show that they actually work for any family of r-spin curves with maximal variation.

Definition 4.1. A family of n-pointed, r-prestable curves $\pi : C \to B$ is a family *with maximal variation* if for any $b \in B$ the Kodaira–Spencer homomorphism $T_bB \to \mathrm{Ext}^1(\Omega_{C_b}, \mathcal{O}_{C_b})$ is surjective.

It follows from the definition, that in a family $C \to B$ with maximal variation, the boundary $\partial B = \{b \in B \mid C_b \text{ is singular}\}$ is a normal crossings divisor in B. As in Section 2.3, we construct a smooth scheme D whose points are triples $(b \in \partial B, \text{node of } C_b, \text{branch at the node})$. The scheme D has a decomposition $D = \bigsqcup_{l,I} D_{l,I} \sqcup \bigsqcup_q D^q_{\mathrm{irr}}$ endowed with morphisms $j_{l,I}\colon D_{l,I} \to B$ and $j^q_{\mathrm{irr}}\colon D^q_{\mathrm{irr}} \to B$.

The schemes B and D are equipped with the tautological cohomology classes $\kappa_d, \psi_1, \ldots, \psi_n \in H^*(B, \mathbb{Z})$ and $\psi, \psi' \in H^2(D, \mathbb{Z})$ as in Section 1.1.6.

The following result is a slight generalization of the main theorem of [9].

Proposition 4.1 [9]**.** *Consider a family of r-prestable curves $C \to B$ with maximal variation over a nonsingular scheme B, equipped with an r-spin* *structure* $\mathcal{L}$

$$\begin{tikzcd}
\mathcal{L} \arrow[dr] & \\
& C \arrow[d, "\pi"] \\
& B \arrow[u, bend left=60, "s_1"] \arrow[u, bend left=30, "s_n"']
\end{tikzcd}
\qquad \varphi\colon \mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_{\log}(-\textstyle\sum_{i=1}^n m_i[s_i(B)]).$$

Then, we have

$$\begin{aligned}
\mathrm{ch}_d(R\pi_*\mathcal{L}) = {}& \frac{B_{d+1}(\frac{1}{r})}{(d+1)!}\kappa_d - \sum_{i=1}^{n} \frac{B_{d+1}(\frac{m_i}{r})}{(d+1)!}\psi_i^d \\
&+ \frac{r}{2}\sum_{q=1}^{r} \frac{B_{d+1}(\frac{q}{r})}{(d+1)!}\,(j_{\mathrm{irr}}^q)_*\left(\sum_{a+a'=d-1}(\psi)^a(-\psi')^{a'}\right) \\
&+ \frac{r}{2}\sum_{\substack{0\le l\le g\\ I\subseteq[n]}} \frac{B_{d+1}(\frac{q(l,I)}{r})}{(d+1)!}\,(j_{l,I})_*\left(\sum_{a+a'=d-1}(\psi)^a(-\psi')^{a'}\right).
\end{aligned}$$

Sketch of a proof. We want to calculate the Chern character of the direct image $R\pi_*\mathcal{L}$. The first step of the proof is to replace the r-stable curve $\pi\colon C \to B$ by the corresponding r-bubbled curve $\widetilde{\pi}\colon \widetilde{C} \to B$. Replacing π_* with $\widetilde{\pi}_*$ does not change the higher direct image because of the identities of Remark 2.2. The GRR formula applied to this situation reads

$$\mathrm{ch}(R\widetilde{\pi}_*\mathcal{L}) = \pi_*\big(\mathrm{td}(\widetilde{\pi})\mathrm{ch}(\mathcal{L})\big).$$

Now the aim is to compute the right-hand side.

The maximal variation condition guarantees that the bubbles form a normal crossings divisor in $\widetilde{C}$, while the nodes of the singular fibers of $\widetilde{C}$ form a smooth subscheme of pure codimension 2.

We can construct $r-1$ families $P_i \to D$, $1 \le i \le r-1$, of projective lines over D that map to the bubbles of $\widetilde{C}$ (starting from the closest bubble to the chosen branch). In the sequel a *bubble* will be the image of one of the P_i in $\widetilde{C}$. Each family P_i has two disjoint sections, where the bubble intersects neighboring bubbles or branches. The normal sheaves relative to these sections are line bundles over D. The first Chern classes of these line bundles will be called *bubble classes.* They are important for two reasons:

(1) For a family with smooth fibers, the class $\mathrm{td}(\widetilde{\pi})$ is just the Todd class of the relative tangent vector bundle; but in our case not all fibers are smooth, and the nodes give a contribution to $\mathrm{td}(\widetilde{\pi})$. This contribution is expressed in terms of the bubble classes.

(2) The rth tensor power $\mathcal{L}^{\otimes r}$ is isomorphic to the relative dualizing line bundle $\omega_{\log}$ to $\widetilde{C}$ twisted by the divisors formed by the bubbles (see Section 2.2). Therefore, to evaluate $\mathrm{ch}(\mathcal{L})$ we need to know the intersections between the classes represented by the bubbles. The intersection of two different bubbles is simply their geometric intersection (this is guaranteed by the maximal variation condition). However, the selfintersection of a bubble is more complicated and can be described in terms of the bubble classes.

It is explained in [9] that the bubble classes can be expressed via ψ and ψ' over every component $D_{l,I}$ and D^q_{irr}. Once we know this, a computation (though not a simple one) leads to the result stated in the proposition. $\square$

4.2 The GRR formula and the virtual r-spin class

Proposition 4.1 allows us to express the homology class $\mathrm{ch}_d \cap \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v$.

Proposition 4.2. *We have*

$$
\begin{aligned}
&\mathrm{ch}_d \cap \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v \\
&\quad= \frac{B_{d+1}(\frac{1}{r})}{(d+1)!}\kappa_d \cap \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v - \sum_{i=1}^{n} \frac{B_{d+1}(\frac{m_i}{r})}{(d+1)!}\psi_i^d \cap \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v \\
&\qquad+ \frac{r}{2}\sum_{q=1}^{r} \frac{B_{d+1}(\frac{q}{r})}{(d+1)!}\,(j^q_{\mathrm{irr}})_* \left(\sum_{a+a'=d-1} (\psi)^a(-\psi')^{a'} \cap (\mathrm{j}^q_{\mathrm{irr}})^! \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v\right) \\
&\qquad+ \frac{r}{2}\sum_{\substack{0\le l\le g\\ I\subseteq[n]}} \frac{B_{d+1}(\frac{q(l,I)}{r})}{(d+1)!}\,(j_{l,I})_* \left(\sum_{a+a'=d-1} (\psi)^a(-\psi')^{a'} \cap (\mathrm{j}_{l,I})^! \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v\right).
\end{aligned}
$$

Proof. In order to apply Proposition 4.1, we need to show that $X^{r,\boldsymbol{m}}_{g,n,D}$ can be embedded into a nonsingular Deligne–Mumford stack $\widetilde{B}$ equipped with an n-pointed r-prestable curve $\widetilde{\pi}\colon \widetilde{C} \to \widetilde{B}$ of maximal variation and with an

r-spin structure $\widetilde{\mathcal{L}}$ fitting in the following fiber diagram:

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{L}} \arrow[d] \\
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{C} \arrow[d] \\
X_{g,n,D}^{r,\boldsymbol{m}} \arrow[r] & \widetilde{B},
\end{tikzcd}$$

where $\mathcal{L} \to C$ is the universal r-prestable spin curve over $X_{g,n,D}^{r,\boldsymbol{m}}$. This is actually the crucial step of the proof: once $\widetilde{B}$ is constructed, since Proposition 4.1 holds for $R\widetilde{\pi}_*\widetilde{\mathcal{L}}$ over $\widetilde{B}$, the desired equation follows from the projection formula for refined intersections [14, Proposition 8.1.1(c)].

First, by Faber and Pandharipande [12, Section 1.2, Proposition 1], there exists an embedding of $X_{g,n,D}$ into a nonsingular Deligne–Mumford stack $\widetilde{B}'$ and a prestable curve $\widetilde{C}' \to \widetilde{B}'$, whose variation is maximal, and which extends the universal curve of $X_{g,n,D}$. The markings also extend, and indeed we can regard $\widetilde{C}' \to \widetilde{B}'$ as a morphism $\widetilde{B}' \to \mathfrak{M}_{g,n}$ extending $X_{g,n,D} \to \mathfrak{M}_{g,n}$. Consider the fibered product $\widetilde{B} = \widetilde{B}' \times_{\mathfrak{M}_{g,n}} \mathfrak{M}_{g,n}^{r,\boldsymbol{m}}$. By construction (see (i) and (ii)), the stack $X_{g,n,D}^{r,\boldsymbol{m}}$ is the fiber over $\widetilde{B}'$ of $\widetilde{B} \to \widetilde{B}'$; therefore, it is embedded in $\widetilde{B}$. Furthermore, $\widetilde{B}$ is naturally equipped with an r-prestable curve and an r-spin structure, which extends the universal r-spin structure of $X_{g,n,D}^{r,\boldsymbol{m}}$.

In this construction, it is crucial to notice that $\widetilde{B}$ is a nonsingular Deligne–Mumford stack. This follows from Olsson's description of the stack of r-prestable curves. Indeed, in [24], the morphism $\widetilde{B} \to \widetilde{B}'$ at a point $x\colon \operatorname{Spec} k \to \widetilde{B}'$ is locally represented by the flat, finite morphism of nonsingular Deligne–Mumford stacks

$$[(\operatorname{Spec} \widetilde{I})/(\boldsymbol{\mu}_r^m)] \to \operatorname{Spec} I, \tag{4.1}$$

where the notation is chosen as follows. The scheme $\operatorname{Spec} I$ is the versal deformation space of x at $\widetilde{B}'$. The index m equals the number of nodes of the curve $\widetilde{C}'_x$ over $x \in \widetilde{B}'$. The ring $\widetilde{I}$ equals $I[z_1, \dots, z_m]/(z_i^r - t_i, \forall i)$, where $t_1, \dots, t_m \in I$ are chosen so that $\{t_i = 0\} \subset \operatorname{Spec} I$ is the locus where the ith node persists. Finally, the group $(\boldsymbol{\mu}_r)^m$ acts by multiplication on the coordinates $(z_1, \dots, z_m)$. $\square$

4.3 The differential operator

Consider the genus-g r-spin twisted Gromov–Witten potential $F_g(\boldsymbol{s}, \boldsymbol{t})$ of X. Set

$$F_r(\boldsymbol{t}, \boldsymbol{s}) = \sum_{g \geq 0} \hbar^{g-1} F_{g,r}(\boldsymbol{t}, \boldsymbol{s}), \qquad \text{and} \qquad Z_r = \exp(F_r).$$

Proposition 4.3. *We have*

$$\frac{\partial}{\partial s_d} Z_r = L_d Z_r,$$

where L_d is the operator

$$L_d = \frac{B_{d+1}(\frac{1}{r})}{(d+1)!} \frac{\partial}{\partial t^{1\otimes 1}_{d+1}} - \sum_{\substack{a\geq 0 \\ \mu\otimes m}} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} t_a^{\mu\otimes m} \frac{\partial}{\partial t^{\mu\otimes m}_{a+d}}$$
$$+ \frac{\hbar}{2} \sum_{\substack{a+a'=d-1 \\ \mu,\mu' \\ m,m'}} (-1)^{a'} g^{\mu\otimes m, \mu'\otimes m'} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} \frac{\partial^2}{\partial t_a^{\mu\otimes m} \partial t_{a'}^{\mu'\otimes m'}},$$

where all summations are taken over the range $a \geq 0$ and $1 \leq m \leq r$.

Proof. We can write the statement in terms of F_r using $Z_r = \exp F_r$; we get

$$\frac{\partial F_r}{\partial s_d} = \frac{B_{d+1}(\frac{1}{r})}{(d+1)!} \frac{\partial F_r}{\partial t^{1\otimes 1}_{d+1}} - \sum_{\substack{a\geq 0 \\ \mu\otimes m}} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} t_a^{\mu\otimes m} \frac{\partial F_r}{\partial t^{\mu\otimes m}_{a+d}}$$
$$+ \frac{\hbar}{2} \sum_{\substack{a+a'=d-1 \\ \mu,\mu' \\ m,m'}} (-1)^{a'} g^{\mu\otimes m, \mu'\otimes m'} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} \frac{\partial^2 F}{\partial t_a^{\mu\otimes m} \partial t_{a'}^{\mu'\otimes m'}}$$
$$+ \frac{\hbar}{2} \sum_{\substack{a+a'=d-1 \\ \mu,\mu' \\ m,m'}} (-1)^{a'} g^{m\otimes \mu, m'\otimes \mu'} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} \frac{\partial F_r}{\partial t_a^{\mu\otimes m}} \frac{\partial F_r}{\partial t_{a'}^{\mu'\otimes m'}}. \tag{4.2}$$

Write the right-hand side as $R_1 + R_2 + R_3 + R_4$. Using Proposition 4.2, we decompose also $\partial F_r / \partial s_d$ as the sum of four terms involving κ_d, ψ_i^d, classes in the image of $\bigsqcup_q j^q_{\mathrm{irr}}$, and classes in the image of $\bigsqcup_{l,I} j_{l,I}$. In each the following four steps we identify these summands.

Step 1. intersection numbers involving κ_d. The class κ_d can be regarded as the pushforward of ψ_{n+1}^{d+1} via $\pi\colon X^{r,(\boldsymbol{m},1)}_{g,n+1,D} \to X^{r,\boldsymbol{m}}_{g,n,D}$. Then, by the forgetting property, the projection formula yields

$$\begin{aligned}
&\int_{\left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v} \kappa_d \prod_{i=1}^{n} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^{k} \mathrm{ch}_{d_j} \\
&\quad = \int_{\left[X^{r,(\boldsymbol{m},1)}_{g,n+1,D}\right]^v} \psi_{n+1}^{d+1} \prod_{i=1}^{n} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^{k} \mathrm{ch}_{d_j},
\end{aligned}$$

where we used the equation $\psi_{n+1}^{d+1}\pi^*\psi_i = \psi_{n+1}^{d+1}\psi_i$ for all $d \geq 0$, and $1 \leq i \leq n$. In this way, we get R_1.

Step 2. intersection numbers involving ψ_i^d. These intersections are already in the desired form and yield immediately R_2.

Step 3. intersection numbers involving classes in the image of $j_{\mathrm{irr},q}$. For all $1 \leq q \leq r$, we intersect $\left(\frac{\hbar}{r}\right)^{g-1} \prod_{i=1}^{n} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^{k} \mathrm{ch}_{d_j}$ with the homology class

$$\frac{r}{2}(j^q_{\mathrm{irr}})_* \left(\sum_{a+a'=d-1} (\psi)^a (-\psi')^{a'} \cap (\mathrm{j}^q_{\mathrm{irr}})^! \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v \right)$$

and we multiply by $B_{d+1}(q/r)/(d+1)!$. Taking aside this last factor, we carry out the intersection on $\mathcal{D}^q_{\mathrm{irr}}$, and we get

$$\begin{aligned}
\frac{\hbar^{g-1}}{2r^{g-2}} &\left[\sum_{a+a'=d-1} (\mu^q_{\mathrm{irr}})^* \left((\psi_{n+1})^a (-\psi_{n+2})^{a'} \right) \cdot (j^q_{\mathrm{irr}})^* \right. \\
&\quad \left. \times \left(\prod_{i=1}^{n} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^{k} \mathrm{ch}_{d_j} \right) \right] \cap (\mathrm{j}^q_{\mathrm{irr}})^! \left[X^{r,\boldsymbol{m}}_{g,n,D}\right]^v
\end{aligned}$$

Now instead of integrating via j^q_{irr} we can integrate via μ^q_{irr}. Notice the identity

$$(j^q_{\mathrm{irr}})^* \left(\prod_{i=1}^{n} \psi_i^{a_i}\ \mathrm{ev}_i^*(h_{\mu_i}) \right) = (\mu^q_{\mathrm{irr}})^* \left(\prod_{i=1}^{n} \psi_i^{a_i}\ \mathrm{ev}_i^*(h_{\mu_i}) \right).$$

As in [19], we also notice that the terms of the Chern character in degree $d \geq 1$, satisfy the identity

$$(j^q_{\mathrm{irr}})^* \mathrm{ch}_d = (\mu^q_{\mathrm{irr}})^* (\mathrm{ch}_d).$$

Therefore, the intersection number is

$$\frac{\hbar^{g-1}}{2r^{g-2}} \sum_{a+a'=d-1} (\mu^q_{\mathrm{irr}})^* \left((\psi_{n+1})^a (-\psi_{n+2})^{a'} \prod_{i=1}^n \psi_i^{a_i} \ \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^k \mathrm{ch}_{d_j} \right) \\ \cap \ (\mathrm{j}^q_{\mathrm{irr}})^! \left(\left[X^{r,\boldsymbol{m}}_{g,n,D} \right]^v \right).$$

Using the projection formula for μ^q_{irr} and the factorization property we get

$$\frac{\hbar^{g-1}}{2r^{g-2}} \sum_{a+a'=d-1} (-1)^{a'} \int_{\left[\Delta_{(\mathrm{irr},q)}\right]^v} (\psi_{n+1})^a (\psi_{n+2})^{a'} \prod_{i=1}^n \psi_i^{a_i} \ \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^k \mathrm{ch}_{d_j}.$$

Recall that by the Künneth formula the Poincaré dual class of the diagonal $[\delta]$ in $X \times X$ can be written as $\sum_{\mu,\mu'} g^{\mu,\mu'} h_\mu \times h_{\mu'}$, where $g^{\mu,\mu'}$ is the inverse matrix of the Poincaré pairing matrix $g_{\mu,\mu'}$ on $H^*(X,\mathbb{Q})$. In this way, we obtain the intersection numbers of $X^{r,(\boldsymbol{m},q,q')}_{g-1,n+2,D}$ with an extra $\hbar/2$ factor

$$\frac{\hbar}{2} \left(\frac{\hbar^{g-2}}{r^{g-2}} \sum_{\substack{a+a'=d-1 \\ \mu,\mu'}} (-1)^{a'} g^{\mu,\mu'} \int_{\left[X^{r,(\boldsymbol{m},q,q')}_{g-1,n+2,D}\right]^v} (\psi_{n+1})^a \mathrm{ev}_i^*(h_\mu) (\psi_{n+2})^{a'} \mathrm{ev}_i^*(h_{\mu'}) \right. \\ \left. \times \prod_{i=1}^n \psi_i^{a_i} \ \mathrm{ev}_i^*(h_{\mu_i}) \prod_{j=0}^k \mathrm{ch}_{d_j} \right),$$

which, if we sum over $q \in \{1, \ldots, r\}$ and take the factor $B_{d+1}(q/r)/(d+1)!$ into account, agrees with R_3.

Step 4. intersection numbers involving classes in the image of $j_{l,I}$. Set $I \subseteq [n]$ and $l \in \{0, \ldots, g\}$, and write q for $q(l,I)$. We show that the term involving $Q^D t^{\mu_1 \otimes m_1}_{a_1} \ldots t^{\mu_n \otimes m_n}_{a_n} \hbar^{g-1}$ in R_4 equals

$$Q^D t^{\mu_1 \otimes m_1}_{a_1} \ldots t^{\mu_n \otimes m_n}_{a_n} \left(\frac{\hbar}{r} \right)^{g-1} \frac{1}{n!} \prod_{i \in [n]} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \exp\Big(\sum_h s_h \mathrm{ch}_h \Big) \\ \cap \frac{r}{2} \frac{B_{d+1}\left(\frac{q}{r}\right)}{(d+1)!} (j_{l,I})_* \left(\sum_{a+a'=d-1} \psi^a (-\psi')^{a'} \cap \mathrm{j}^!_{l,I} \left[X^{r,\boldsymbol{m}}_{g,n,D} \right] \right).$$

As in the previous step we put aside the factor $B_{d+1}(q/r)/(d+1)!$, and we carry out the intersection on $\mathcal{D}_{l,I}$. We get

$$\sum_{a+a'=d-1} (-1)^{a'} Q^D \frac{t_{a_1}^{\mu_1 \otimes m_1} \dots t_{a_n}^{\mu_n \otimes m_n}}{n!} \frac{\hbar^{g-1}}{2r^{g-2}}$$
$$\times (j_{l,I})^* \left(\prod_{i\in[n]} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \exp\big(\sum_h s_h \mathrm{ch}_h\big) \right)$$
$$\times (\mu_{l,I})^* \left((\psi_{|I|+1})^a \times (\psi_{|I'|+1})^{a'} \right) \cap (\mathrm{j}_{\mathrm{irr}}^q)^! \left[X_{g,n,D}^{r,\boldsymbol{m}} \right]^v, \tag{4.3}$$

where we identified the classes ψ and ψ' on $\mathcal{D}_{l,I}$ with pullbacks via $\mu_{l,I}$.

Notice that the classes ψ_i, $\mathrm{ev}_i^*(h_{\mu_i})$, and ch_d satisfy the relations

$$(j_{l,I})^* \left(\prod_{i=1}^n \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \right) = (\mu_{l,I})^* \left(\prod_I \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \times \prod_{I'} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \right),$$
$$(j_{l,I})^* \mathrm{ch}_d = (\mu_{l,I})^* ((\mathrm{ch}_d \times 1) + (1 \times \mathrm{ch}_d)), \quad \text{for } d \geq 1.$$

Recall the factorization property of the virtual class from Proposition 3.1

$$(\mu_{l,I})_* (\mathrm{j}_{l,I})^! \left[X_{g,n,D}^{r,\boldsymbol{m}} \right]^v = \sum_{(l,I,A)\in\Omega_{l,I}} \left[\Delta_{(l,I,A)} \right]^v$$
$$= \left[X_{l,b,A}^{r,(\boldsymbol{m}_I,q)} \right]^v \left[X_{l',b',A'}^{r,(\boldsymbol{m}_{I'},q')} \right]^v \cap (\mathrm{ev} \times \mathrm{ev}')^{-1}(\delta),$$

where $b = |I| + 1$ and $b' = |I'| + 1$. These relations, together with formal properties of the exponent function, allow us to rewrite each summand appearing in the alternate sum (4.3) in terms of intersections on the fiberd product $X_{l,b,A}^{r,(\boldsymbol{m}_I,q)} \times_X X_{l',b',A'}^{r,(\boldsymbol{m}_{I'},q')}$:

$$\frac{\hbar}{2} \sum_{A+A'=D} \Bigg[Q^A \left(\frac{\hbar}{r}\right)^{l-1} \prod_{i\in I} \frac{t_{a_i}^{\mu_i \otimes m_i}}{n!}$$
$$\times \left(\psi_b^a \mathrm{ev}_b^*(h_\mu) \prod_{i\in I} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \exp\left(\sum_h s_h \mathrm{ch}_h \right) \right) \cap \left[X_{l,b,A}^{r,\boldsymbol{m}_I,q} \right]^v \Bigg]$$
$$\Bigg[Q^{A'} \left(\frac{\hbar}{r}\right)^{l'-1} \prod_{i\in I'} \frac{t_{a_i}^{\mu_i \otimes m_i}}{n!}$$
$$\times \left(\psi_{b'}^{a'} \mathrm{ev}_{b'}^*(h_{\mu'}) \prod_{i\in I'} \psi_i^{a_i} \mathrm{ev}_i^*(h_{\mu_i}) \exp\left(\sum_h s_h \mathrm{ch}_h \right) \right) \cap \left[X_{l',b',A'}^{r,\boldsymbol{m}_{I'},q'} \right]^v \Bigg],$$

where the Künneth formula $[\delta] = \sum_{\mu,\mu'} g^{\mu,\mu'} h_\mu \times h_{\mu'}$ has been used. Taking into account the factor $B_{d+1}(q/r)/(d+1)!$, this yields the monomial of R_4 involving $Q^D t_{a_1}^{\mu_1 \otimes m_1} \dots t_{a_n}^{\mu_n \otimes m_n} \hbar^{g-1}$. ☐

5 Givental's quantization

Now we can prove our main Theorem 1.1.

We use the notation from Section 1.2. In particular, $H = H^*(X, \mathbb{Q}) \otimes \mathbb{Q}^{r-1}$ is a vector space with a quadratic form g and $\mathcal{H} = H((z^{-1}))$ is the corresponding infinite-dimensional symplectic space.

Proposition 5.1. *The operator L_d from Proposition 4.3 is obtained by the Weyl quantization rules from the hamiltonian*

$$P_d = -\sum_{a=0}^{\infty} \sum_{\mu,m} \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} q_a^{\mu\otimes m} p_{a+d,\mu\otimes m} + \frac{1}{2} \sum_{\substack{a+a'=d-1 \\ \mu,\mu',m,m'}} (-1)^d \frac{B_{d+1}(\frac{m}{r})}{(d+1)!} g^{\mu\otimes m,\mu'\otimes m'} p_{a,\mu\otimes m} p_{a',\mu'\otimes m'}.$$

on $\mathcal{H}$.

Proposition 5.2. *The vector field, or the infinitesimal symplectic transformation, induced by this hamiltonian is the multiplication by*

$$\frac{z^d}{(d+1)!} \operatorname{id} \otimes \operatorname{diag} \left[B_{d+1}\left(\frac{1}{r}\right), \dots, B_{d+1}\left(\frac{r-1}{r}\right) \right].$$

Both propositions are proved by a simple computation.

Theorem 1.1 follows. ☐

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