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Parity Invariance For Strings In Twistor Space

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Abstract

Topological string theory with twistor space as the target makes visible some otherwise difficult to see properties of perturbative Yang-Mills theory. But left-right symmetry, which is obvious in the standard formalism, is highly unclear from this point of view. Here we prove that tree diagrams computed from connected $\boldsymbol{D}$-instanton configurations are parity-symmetric. The main point in the proof also works for loop diagrams.

1 Introduction

Perturbative Yang-Mills scattering amplitudes have many unexpected simplifications that have been found in very early studies [**1**], in more contemporary investigations of multi-gluon tree level scattering [**2**, **3**], and in

e-print archive:
http://lanl.arXiv.org/abs/hep-th/0403199

studies of loop diagrams [**4**, **5**].

A certain topological string theory [**6**] in which the target space is twistor space [**7**] gives a new approach to understanding some of these questions. (For an open string version of twistor string theory, see [**8**]. For an alternative proposal involving mirror symmetry, see [**9**]. See also further developments in [**10**].) However, while making some aspects of perturbative gauge theory more transparent, the twistor formalism obscures other properties such as the parity invariance or left-right symmetry of the model.

For example, tree amplitudes with all but two gluons having positive helicity are called maximal helicity violating or MHV amplitudes. They are described by a simple holomorphic function [**2**, **3**] that can be readily computed in twistor space [**6**]; the computation involves current algebra correlation functions, something which is natural in view of observations made some time ago [**11**]. Parity symmetry converts these amplitudes to amplitudes with all but two gluons having negative helicity; we call these the dual, opposite helicity, or googly MHV amplitudes. In the standard formalism, it is obvious that the parity conjugate amplitude is obtained by exchanging the two types of spinors (or, for real momenta, by simply complex-conjugating the amplitude). In the twistor formalism, this is far from apparent, but has been shown to be true at tree level if only connected D-instantons are considered [**12**], and also in another approach [**13**] based on tree diagrams with MHV amplitudes for vertices [**14**].

The purpose of the present paper is to make this result more transparent and to generalize it. We will consider amplitudes with p positive helicity gluons and q negative helicity gluons for arbitrary p, q. For any number of loops, we will show, in section 2, that the twistor amplitude can be expressed, after integrating out some of the variables and introducing some new ones, as an integral in which the integration region is symmetrical in p and q.

For tree diagrams, we go farther and prove in section 3 that the measure, as well as the integration region, is symmetric in p and q, and therefore that the amplitudes computed in twistor theory are parity-symmetric. We cannot extend this analysis to loop diagrams, because the proper definition of the integration measure for loop diagrams in twistor space (that is, for D-instanton configurations of positive genus) remains unclear.

The proof of parity invariance includes as a special case a new method of computing the tree level dual MHV amplitudes, that is the amplitudes with $(p, q) = (2, n-2)$, since the "ordinary" MHV amplitudes with $(p, q) = (n-2, 2)$ are readily computed from twistor space.

Another argument for parity symmetry in twistor string theory has been given in [**15**], based on a Fourier transform that presumably is a real analog of the Serre duality that we use in section 2.

As in [**12**], we consider only *connected* D-instantons. This poses something of a puzzle, because there is also evidence [**14**] that tree amplitudes can be computed from completely disconnected D-instanton configurations. It is not yet clear why there is apparently more than one way to compute Yang-Mills tree amplitudes from curves in twistor space.

2 Parity Invariance And Serre Duality

2.1 Wavefunctions

We consider in $U(N)$ gauge theory the scattering of external gluons that are described by spinors λ_i, $\widetilde{\lambda}_i$ and momenta $p_{i\,a\dot{a}} = \lambda_{i\,a}\widetilde{\lambda}_{i\,\dot{a}}$. A momentum eigenstate with momentum p_i is described in twistor space by a wavefunction that is roughly

$$\Psi_i(\lambda, \mu) = \overline{\delta}(\langle\lambda, \lambda_i\rangle)\exp(i[\mu, \widetilde{\lambda}_i]). \tag{1}$$

The idea is that the delta function describes a state with $\lambda = \lambda_i$, and the plane wave dependence on μ describes a state with $\widetilde{\lambda} = \widetilde{\lambda}_i$. Here essentially as in [**14**],[1] for any holomorphic function f, the symbol $\overline{\delta}(f)$ represents the closed $(0, 1)$-form $-id\overline{f}\,\delta^2(f)$, where the δ function is normalized so that $\int |dz\,d\overline{z}|\delta^2(z) = 1$. The normalization ensures that for any complex number b and any function $f(z)$, we have

$$\int dz\,\overline{\delta}(z - b)f(z) = f(b). \tag{2}$$

Actually, though the details will not be important for most purposes, the wavefunction (1) should be modified slightly to have the right homogeneity in all variables. Using the standard transformation law of the delta function, the object $\overline{\delta}(\langle\lambda, \lambda_i\rangle)$ is homogeneous of degree -1 in both λ and λ_i; when we want to make this explicit we write it as $\overline{\delta}_{(-1,-1)}(\langle\lambda, \lambda_i\rangle)$. On the support of the delta function, λ is a nonzero multiple of λ_i, so there is a well-defined and

[1]We will include a factor of $-i$ in the definition of $\overline{\delta}(f)$ relative to [**14**] to avoid having an unnatural factor in eqn. (2). What we here call $\delta^2(f)$ was called $\delta(f)$ in [**14**].

non-zero holomorphic function λ/λ_i. Hence we can define a more general delta function

$$\overline{\delta}_{(n-1,-n-1)}(\langle\lambda,\lambda_i\rangle) = (\lambda/\lambda_i)^n\overline{\delta}_{(-1,-1)}(\langle\lambda,\lambda_i\rangle). \tag{3}$$

The wavefunction for a positive helicity gauge boson of momentum $p_i = \lambda_i\widetilde{\lambda}_i$ is actually

$$\Psi_i^+(\lambda,\mu) = \overline{\delta}_{(0,-2)}(\langle\lambda,\lambda_i\rangle)\exp\Big(i[\mu,\widetilde{\lambda}_i](\lambda_i/\lambda)\Big). \tag{4}$$

The powers of λ/λ_i have been included to ensure that Ψ^+ is homogeneous of degree zero in λ,μ. It is also homogeneous of degree -2 under $(\lambda_i,\widetilde{\lambda}_i) \to (t\lambda_i, t^{-1}\widetilde{\lambda}_i)$, and this ensures, as expected (see section 2 of [**6**] for a review), that the scattering amplitude for a gluon of positive helicity scales under that transformation as t^{-2}. To write a twistor space wavefunction for a gluon of the same momentum and negative helicity, we must include the fermionic homogeneous coordinates ψ^A, $A = 1,\ldots,4$ of $\mathbb{CP}^{3|4}$. The wavefunction is

$$\Psi_i^-(\lambda,\mu,\psi) = \overline{\delta}_{(-4,2)}(\langle\lambda,\lambda_i\rangle)\psi^1\psi^2\psi^3\psi^4\exp\Big(i[\mu,\widetilde{\lambda}_i](\lambda_i/\lambda)\Big). \tag{5}$$

The weights are chosen so that the wavefunction is homogeneous of degree zero in overall scaling of λ,μ, and ψ. Under $(\lambda_i,\widetilde{\lambda}_i) \to (t\lambda_i, t^{-1}\widetilde{\lambda}_i)$, the wavefunction scales as t^2, and that therefore is the scaling of the scattering amplitude for a gluon of negative helicity. Again, this is the standard result.

To write a wavefunction for an external particle of helicity $h = 1 - k/2$, we write a similar formula with k factors of ψ. In addition, each external particle is also labeled by an element T_i of the Lie algebra of $U(N)$, which we have omitted in writing the wavefunctions.

2.2 Curves In Twistor Space

In twistor string theory, Yang-Mills scattering amplitudes are computed by integration over the moduli space of holomorphic curves in $\mathbb{CP}^{3|4}$. Such a curve can be described as follows.

Begin with an abstract Riemann surface C and a holomorphic line bundle $\mathcal{L}$ (which must have enough nonzero holomorphic sections or the following construction will be vacuous). A holomorphic map $\Phi : C \to \mathbb{CP}^{3|4}$ (which generically will be an embedding) is described by picking sections $P_a(x)$,

$Q_{\dot a}(x)$ and $\chi^A(x)$ of $\mathcal{L}$ (x denotes a point in C) and setting

$$\begin{aligned} \lambda_a &= P_a(x) \\ \mu_{\dot a} &= Q_{\dot a}(x) \\ \psi^A &= \chi^A(x). \end{aligned} \tag{6}$$

Geometrically, $\mathcal{L} = \Phi^*(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the usual line bundle over $\mathbb{CP}^{3|4}$ (whose sections are functions homogeneous of degree one in the homogeneous coordinates λ, μ, ψ of $\mathbb{CP}^{3|4}$), and hence every holomorphic curve in $\mathbb{CP}^{3|4}$ arises by this construction for some $\mathcal{L}$. Let $\mathcal{M}$ be the moduli space of such curves; the moduli are the moduli of C and $\mathcal{L}$ and the parameters that enter in picking the polynomials P_a, $Q_{\dot a}$, and χ^A. The parameters in the polynomials should be taken modulo an overall scaling, since a common scaling of P_a, $Q_{\dot a}$, and χ^A does not change C.

Each external gluon of momentum p_i and wavefunction Ψ_i couples to C via an interaction $W_i = \int_C \mathrm{Tr}\Psi_i \wedge V$, where V is a vertex operator (in the worldsheet theory of the $D1$-brane) that was described in [**6**]; the trace is taken in the Lie algebra of $U(N)$. To compute the scattering amplitudes for external gluons of momentum p_i and wavefunctions Ψ_i, we must evaluate the integral

$$\int_{\mathcal{M}} d\mu \left\langle \prod_i W_i \right\rangle \tag{7}$$

where $d\mu$ is a suitable holomorphic measure, and the integral really is taken over a suitable real cycle in $\mathcal{M}$.

There is no problem in integrating over the parameters in the polynomials P, Q, and χ. Indeed, each of P_a, $Q_{\dot a}$, and χ^A takes values in a common vector space $U = H^0(C, \mathcal{L})$. Picking an arbitrary basis u_σ, $\sigma = 1, \ldots, r$ for this vector space, we expand

$$\begin{aligned} P_a &= \sum_\sigma p_{\sigma\, a} u_\sigma \\ Q_{\dot a} &= \sum_\sigma q_{\sigma\, \dot a} u_\sigma \\ \chi^A &= \sum_\sigma \eta_\sigma^A\, u_\sigma. \end{aligned} \tag{8}$$

A natural measure for integrating over P, Q, and χ is then

$$\Omega_0 = \prod_{\sigma=1}^{r} \prod_{a,\,\dot a,\,A} dp_{\sigma\,a}\, dq_{\sigma\,\dot a}\, d\eta_\sigma^A. \tag{9}$$

This measure is independent of the choice of basis u_σ, since the number of bosonic and fermionic variables is the same for each σ. Since we really want to consider P, Q, and χ up to a common scaling, we want to integrate not over the space $\mathbb{C}^{4r|4r}$ of P, Q, and χ, but over the corresponding projective space $\mathbb{CP}^{4r-1|4r}$. Being $\mathbb{C}^*$-invariant, the natural measure Ω_0 descends to an equally natural measure Ω on $\mathbb{CP}^{4r-1|4r}$.

If C has genus zero, then C and $\mathcal{L}$ have no moduli, and hence the measure just described can serve as a measure on $\mathcal{M}$. For C of positive genus, C and $\mathcal{L}$ do have moduli. It is not at all clear what sort of measure should be used to integrate over those moduli, so at the moment there is not a clear framework for higher genus computations in twistor space. For this reason, in section 3, when we verify parity invariance of the scattering amplitudes in a precise fashion, we consider only the contributions from curves of genus zero. However, an important part of the calculation can be carried out for arbitrary genus, as we will now see.

2.3 Parity Symmetry And Duality

Because of the factor $\overline{\delta}(\langle\lambda,\lambda_i\rangle)$ in the wavefunction Ψ_i of the i^{th} external gluon, this gluon is actually attached to the curve C at a point x_i at which

$$\langle\lambda(x_i),\lambda_i\rangle = 0. \tag{10}$$

Equivalently, since $\lambda_a(x) = P_a(x)$ along C, the condition is

$$\langle P(x_i),\lambda_i\rangle = 0. \tag{11}$$

Twistor space is constructed in a way that breaks the parity symmetry between λ_i and $\widetilde{\lambda}_i$. $\widetilde{\lambda}_i$ enters the formalism quite differently. As we saw in the formulas of section 2.1 for the wavefunctions, the factor in Ψ_i that depends on $\widetilde{\lambda}_i$ is $\exp\Big(i[\mu,\widetilde{\lambda}_i](\lambda_i/\lambda)\Big)$. Since on C, $\mu_{\dot a} = Q_{\dot a}(x)$, this factor actually becomes $\exp\Big(i[Q(x_i),\widetilde{\lambda}_i](\lambda_i/\lambda)\Big)$. There is such a factor for each i, and these factors are the only factors in the integrand of (7) that depend on Q. So the integral over Q reads

$$\int dQ\, \prod_i \exp\Big(i[Q(x_i),\widetilde{\lambda}_i](\lambda_i/\lambda)\Big). \tag{12}$$

Concretely, the integral over Q is an integral $\int \prod_{\sigma\,\dot a} dq_{\sigma\,\dot a}$ over all of the coefficients when Q is expanded in the basis u_σ. As in [**12**], we assume that the integral should be taken as an integral over the real axis. This being so, the integral over Q simply gives a product of delta functions, asserting that the amplitude receives its contribution from curves C and sets of points x_i such that

$$\sum_i [Q(x_i), \widetilde{\lambda}_i](\lambda_i/\lambda(x_i)) = 0 \tag{13}$$

for every $Q \in H^0(C, \mathcal{L})$.

Now we can see why parity symmetry is a problem: the curves C and sets of points x_i that contribute to the scattering amplitude are constrained by the set of equations (11) and (13) in which λ_i and $\widetilde{\lambda}_i$ enter quite asymmetrically. Our goal is to express these conditions in a symmetrical fashion.

To do this, we let K denote the canonical line bundle of C, and for each point $x_i \in C$, we let $\mathcal{O}(x_i)$ denote the line bundle whose holomorphic sections are holomorphic functions on C that are allowed to have a simple pole at x_i. The line bundle $K(x_1, \dots, x_n) = K \otimes (\otimes_{i=1}^n \mathcal{O}(x_i))$ (which we also abbreviate as $K(x_i)$) has for its holomorphic sections the holomorphic differentials on C that may have simple poles at the x_i (and no other singularities). Such a differential ω has, at each point x_i where there may be a pole, a *residue*, a complex number $c_i = \mathrm{Res}_{x_i}\omega$.

Suppose instead that ω is a holomorphic section of $K(x_i) \otimes \mathcal{L}^{-1} = K \otimes \mathcal{L}^{-1} \otimes (\otimes_i \mathcal{O}(x_i))$. Thus, ω is now a holomorphic differential with values in $\mathcal{L}^{-1}$ that may have poles at the x_i. We can still define the residues $c_i = \mathrm{Res}_{x_i}\omega$, but now the c_i, instead of being complex numbers, take values in the vector spaces $\mathcal{L}^{-1}_{x_i}$, the fibers of $\mathcal{L}^{-1}$ at x_i.[2]

Suppose that, for $\dot a = 1, 2$, we can find $\widetilde{P}_{\dot a} \in H^0(C, K(x_i) \otimes \mathcal{L}^{-1})$ such that, for all i,

$$\widetilde{\lambda}_{i\,\dot a}(\lambda_i/\lambda(x_i)) = \mathrm{Res}_{x_i}\widetilde{P}_{\dot a}. \tag{14}$$

Notice that the left hand side of (14) takes values in $\mathcal{L}^{-1}_{x_i}$, because of the appearance of $\lambda(x_i)$ in the denominator. Thus, the left and right hand sides of (14) take values in the same vector space, and the equation makes sense.

[2]Locally, near x_i, we can trivialize $\mathcal{L}^{-1}$; once this is done, ω is an ordinary differential form with a possible pole at x_i, and its residue is a complex number. This number depends on how $\mathcal{L}^{-1}$ was trivialized; the intrinsic description is that c_i is a vector in the one-dimensional vector space $\mathcal{L}^{-1}_{x_i}$.

If so, let $\omega = Q^{\dot a}\widetilde{P}_{\dot a}$, for any $Q^{\dot a} \in H^0(C,\mathcal{L})$. As $Q^{\dot a}$, $\dot a = 1,2$, is a holomorphic section of $\mathcal{L}$, ω is a section of $K(x_i)$ and thus can be interpreted as an ordinary holomorphic differential on C with possible simple poles at the x_i. Its residues are therefore ordinary complex numbers, which simply equal $Q^{\dot a}\widetilde{\lambda}_{i\,\dot a}(\lambda_i/\lambda(x_i))$ (indeed, the residue of $\omega = Q\widetilde{P}$ at x_i is the value there of Q, which has no pole at x_i, times the residue of $\widetilde{P}$). The usual residue theorem asserts that the sum of these residues vanishes:

$$\sum_i Q^{\dot a}(x_i)\widetilde{\lambda}_{i,\dot a}(x_i)(\lambda_i/\lambda(x_i)) = 0. \tag{15}$$

But this is precisely the desired condition (13)!

Below, we will show that, conversely, (13) is satisfied only if there exists a differential $\widetilde{P}_{\dot a}$ obeying (14). Assuming this for a moment, we can now restate the basic conditions (11) and (13) to treat λ_i and $\widetilde{\lambda}_i$ symmetrically. We set $\widetilde{\mathcal{L}} = K(x_i)\otimes\mathcal{L}^{-1}$. Thus

$$\mathcal{L}\otimes\widetilde{\mathcal{L}} = K(x_i). \tag{16}$$

The right hand side of the last formula depends only on the choice of the curve C and the points $x_i \in C$; there is no asymmetry here between λ and $\widetilde{\lambda}$. The left hand side is symmetrical in $\mathcal{L}$ and $\widetilde{\mathcal{L}}$; when we exchange λ and $\widetilde{\lambda}$, we will also exchange $\mathcal{L}$ and $\widetilde{\mathcal{L}}$.

The basic equations obtained so far relating λ, λ_i, and $\widetilde{\lambda}_i$ to P_a and $\widetilde{P}_{\dot a}$ are as follows:

$$\begin{aligned} \lambda_a(x_i) &= P_a(x_i) \\ \langle\lambda(x_i),\lambda_i\rangle &= 0 \\ \widetilde{\lambda}_{i\,\dot a}(\lambda_i/\lambda(x_i)) &= \mathrm{Res}_{x_i}\widetilde{P}_{\dot a}. \end{aligned} \tag{17}$$

The second of these equations asserts that λ_i is a multiple of $\lambda(x_i)$, so (λ_i/λ) is a well-defined complex number, as is assumed in writing the third equation. Also, it follows from the first two equations that

$$\lambda_{i\,a} = w_i P_a(x_i) \tag{18}$$

for some w_i (which takes values in $\mathcal{L}^{-1}_{x_i}$). The third equation in (17) similarly implies that

$$\widetilde{\lambda}_{i\,\dot a} = \widetilde{w}_i\,\widetilde{P}_{\dot a}(x_i) \tag{19}$$

for some $\widetilde{w}_i$.

Eqn. (19) may require some elucidation. There are two ways to think about $\widetilde{P}_{\dot a}$. If it is viewed as a section of $K\otimes\mathcal{L}^{-1}$ that has a pole at x_i, then it has a residue $r_{i\,\dot a}$ at x_i which takes values in $\mathcal{L}^{-1}$. This is the point of view we used so far; it enabled us to invoke the residue theorem to derive (15). A different point of view is more helpful for understanding the symmetry between λ and $\widetilde{\lambda}$. If we view $\widetilde{P}_{\dot{a}}$ simply as a section of a line bundle $K(x_i)\otimes\mathcal{L}^{-1}$, then its "value" at x_i is an element which we call $\widetilde{P}_{\dot{a}}(x_i)$ of the fiber of this line bundle at x_i; this fiber is $(K(x_i)\otimes\mathcal{L}^{-1})_{x_i} = K(x_i)_{x_i}\otimes\mathcal{L}^{-1}_{x_i}$. The relation between the two points of view comes from the fact that, with $K(x_i)$ understood as the line bundle whose sections are holomorphic differentials with possible simple poles at the x_i, the fiber $K(x_i)_{x_i}$ is naturally trivial.[3] The trivialization is made by the residue map; if ω is a holomorphic differential with a simple pole at x_i, then the coefficient of the pole, which is the value of ω at x_i, is in a natural way a complex number, namely $\mathrm{Res}_{x_i}\omega$. Going back to our problem, what in one point of view is called $\mathrm{Res}_{x_i}\widetilde{P}_{\dot{a}}$, an element of $\mathcal{L}^{-1}_{x_i}$, is in the other point of view simply called $\widetilde{P}_{\dot{a}}(x_i)$, an element of $(K(x_i)\otimes\mathcal{L}^{-1})_{x_i}$. The two points of view are compatible because the latter space is isomorphic by the residue map to $\mathcal{L}^{-1}_{x_i}$.

This hopefully makes it clear that (19) is equivalent to the third equation in (17), with $\widetilde{w}_i = (\lambda(x_i)/\lambda_i)$. Since $\lambda_{i\,a} = w_iP_a = w_i\lambda_a(x_i)$, we similarly have $w_i = (\lambda_i/\lambda(x_i))$. Thus $w_i\widetilde{w}_i = 1$. We cannot expect to find any further constraints on w_i and $\widetilde{w}_i$, because we are free to rescale $\lambda_i\to t_i\lambda_i$, $\widetilde{\lambda}_i\to t_i^{-1}\widetilde{\lambda}_i$, for any $t_i\in\mathbb{C}^*$. Clearly all the conditions that we have described are completely symmetric in λ and $\widetilde{\lambda}$, establishing the parity invariance of the formalism.

For an alternative view of things, we combine the three equations in (17) to write

$$\lambda_{i\,a}\widetilde{\lambda}_{i\,\dot{a}} = P_a(x_i)\mathrm{Res}_{x_i}\widetilde{P}_{\dot{a}}. \tag{20}$$

Now let

$$\omega_{a\dot{a}} = P_a\widetilde{P}_{\dot{a}}. \tag{21}$$

Each component of $\omega_{a\dot{a}}$, $a,\dot{a}=1,2$, is a section of $K(x_i)$; that is, it is an ordinary holomorphic differential with possible simple poles at the x_i. The factorization (21) implies that

$$\omega_{a\dot{a}}\omega^{a\dot{a}} = 0. \tag{22}$$

[3]Our notation here is really too compressed. We recall that $K(x_i)$ is an abbreviation for $K(x_1,\dots,x_n) = K\otimes(\otimes_i\mathcal{O}(x_i))$. By $K(x_i)_{x_i}$ we mean $K(x_1,\dots,x_n)_{x_i}$, that is, the fiber of $K(x_1,\dots,x_n)$ at x_i. A partial justification of our overly compressed notation is that for analyzing the fiber of $K(x_1,\dots,x_n)$ at x_i, the existence and location of the x_j with $j\neq i$ are irrelevant. Thus the fiber of $K(x_1,\dots,x_n)$ at x_i is naturally isomorphic to the fiber of $K\otimes\mathcal{O}(x_i)$ at x_i; the latter is perhaps a better candidate for being called $K(x_i)_{x_i}$.

And finally, we have

$$\lambda_{i\,a}\widetilde{\lambda}_{i\,\dot a} = \mathrm{Res}_{x_i}\omega_{a\dot a}. \tag{23}$$

These conditions are actually closely related to the saddle point equation [**16**] that describes high energy, fixed angle scattering in string perturbation theory. The relation will be further explored elsewhere.

Clearly, equations (22) and (23) are symmetrical in λ and $\widetilde{\lambda}$. Moreover, all the structure described previously can be deduced from those equations. For example, (22) says that at any given $x \in C$, $\omega_{a\dot a}$ is a null vector and so has a factorization $\omega_{a\dot a} = P_a\widetilde{P}_{\dot a}$, unique up to $P \to tP$, $\widetilde{P} \to t^{-1}\widetilde{P}$. To make this factorization globally, we must interpret P and $\widetilde{P}$ as sections of suitable line bundles, reasoning as follows. We define a complex quadric $W \subset \mathbb{CP}^3$ that has homogeneous complex coordinates $Z_{a\dot a}$, $a, \dot a = 1, 2$ obeying $Z_{a\dot a}Z^{a\dot a} = 0$. As long as the $\omega_{a\dot a}$ have no common zeroes, which is true generically, we can define a map $\Phi : C \to W$ by setting $Z_{a\dot a} = \omega_{a\dot a}$; this maps C to W (and not just to $\mathbb{CP}^3$) since $\omega_{a\dot a}\omega^{a\dot a} = 0$. Since the $Z_{a\dot a}$ are homogeneous coordinates, Φ is well-defined even at poles of $\omega_{a\dot a}$ (if $\omega_{a\dot a}$ has a pole at $x = x_i$, one can define Φ near x_i by $Z_{a\dot a} = (x - x_i)\omega_{a\dot a}$). The quadric W is isomorphic as a complex manifold to $\mathbb{CP}^1 \times \mathbb{CP}^1$; there are two natural line bundles $\mathcal{O}(1)$ and $\mathcal{O}(1)'$ over it, and the $Z_{a\dot a}$ are elements of $H^0(W, \mathcal{O}(1) \otimes \mathcal{O}(1)')$. Moreover, there are sections $\lambda^W_a \in H^0(W, \mathcal{O}(1))$ and $\widetilde{\lambda}^W_{\dot a} \in H^0(W, \mathcal{O}(1)')$ with $Z_{a\dot a} = \lambda^W_a\widetilde{\lambda}^W_{\dot a}$.[4] Finally, to recover the formalism that we found above, we set $\mathcal{L} = \Phi^*(\mathcal{O}(1))$, $\widetilde{\mathcal{L}} = \Phi^*(\mathcal{O}(1)')$, $P_a = \Phi^*(\lambda^W_a)$, and $\widetilde{P}_{\dot a} = \Phi^*(\widetilde{\lambda}^W_{\dot a})$.

The Converse Statement

We still must prove the converse statement that (13) holds only if $\widetilde{\lambda}_{i\,\dot a}$ can be expressed as in (14) in terms of a suitable differential $\widetilde{P}_{\dot a}$.

We consider the exact sequence of sheaves

$$0 \to K \otimes \mathcal{L}^{-1} \xrightarrow{i} K(x_i) \otimes \mathcal{L}^{-1} \xrightarrow{r} \oplus_i \mathcal{L}^{-1}_{x_i} \to 0. \tag{24}$$

By $K \otimes \mathcal{L}^{-1}$ we mean the sheaf of sections of the line bundle $K \otimes \mathcal{L}^{-1}$, and similarly for $K(x_i) \otimes \mathcal{L}^{-1}$. Also, $\oplus_i \mathcal{L}^{-1}_{x_i}$ is the sheaf whose sections are families $\{\alpha_i\}$ with each α_i a vector in $\mathcal{L}^{-1}_{x_i}$. The map r is the "residue" map which maps a holomorphic section ϕ of $K(x_i) \otimes \mathcal{L}^{-1}$, which we recall

[4]The quickest way to prove these assertions is to start with $\mathbb{CP}^1 \times \mathbb{CP}^1$ and let λ^W_a and $\widetilde{\lambda}^W_{\dot a}$ denote the homogeneous coordinates of, respectively, the first and second factor (so they are the holomorphic sections of $\mathcal{O}(1)$ and $\mathcal{O}(1)'$, respectively). Then simply define a map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^3$ by $Z_{a\dot a} = \lambda^W_a\widetilde{\lambda}^W_{\dot a}$. Clearly this maps $\mathbb{CP}^1 \times \mathbb{CP}^1$ to the quadric $Z_{a\dot a}Z^{a\dot a} = 0$; it is easily seen to be an isomorphism.

is a holomorphic section of $K \otimes \mathcal{L}^{-1}$ with possible poles at the x_i, to the family $\{\alpha_i\}$ where $\alpha_i = \mathrm{Res}_{x_i}(\phi)$. The map i is the "inclusion" of sheaves, which maps a holomorphic section ρ of $K \otimes \mathcal{L}^{-1}$ to the "same" section of $K(x_i) \otimes \mathcal{L}^{-1}$. $i(\rho)$ does not have poles at x_i, and so has vanishing residues; hence $ri(\rho) = 0$. Conversely, if a section ϕ of $K(x_i) \otimes \mathcal{L}^{-1}$ is annihilated by r, that is, it has no poles at x_i, then it can be regarded as a section of $K \otimes \mathcal{L}^{-1}$, and hence is of the form $i(\rho)$ for some ρ. These assertions are part of the statement that the sequence (24) is exact. The remainder of the statement of exactness is the assertion that i is injective, which is obvious, and that r is surjective, which expresses the fact that locally the residues of a differential can be specified arbitrarily.

The short exact sequence of sheaves (24) leads to a long exact cohomology sequence which reads in part

$$\ldots H^0(C, K(x_i) \otimes \mathcal{L}^{-1}) \xrightarrow{r} \oplus_i \mathcal{L}^{-1}_{x_i} \xrightarrow{\delta} H^1(C, K \otimes \mathcal{L}^{-1}) \ldots . \tag{25}$$

In our problem, we have a family $\{\alpha_i\}$, with $\alpha_i = \widetilde{\lambda}_{i\,\dot{a}}(\lambda_i/\lambda)$ (in this discussion we regard $\dot{a}$ as a fixed number, 1 or 2), and we want to know if there is a global differential $\widetilde{P}$ such that $\alpha_i = \mathrm{Res}_{x_i}\widetilde{P}_i$. The exactness of the sequence (25) asserts that $\widetilde{P}$ exists if and only if $\delta(\{\alpha_i\}) = 0$. The definition of the map δ is that $\delta(\{\alpha_i\})$ is an element of $H^1(C, K \otimes \mathcal{L}^{-1})$ that can be represented by the $K \otimes \mathcal{L}^{-1}$-valued $(0,1)$-form $\zeta = \sum_i \alpha_i dx \overline{\delta}(x - x_i)$, where x is an arbitrary local holomorphic parameter near $x = x_i$ (and as explained in section 2.1, $\overline{\delta}(x - x_i) = id\overline{x}\delta^2(x - x_i)$). One can verify, using the transformation of the delta function under a change of coordinates, that ζ is independent of the choices of local coordinates.

ζ is nonzero as a differential form, but we need to know if it is nonzero as an element of $H^1(C, K \otimes \mathcal{L}^{-1})$. For this, we can use Serre duality, where asserts that $H^1(C, K \otimes \mathcal{L}^{-1})$ is the dual space to $H^0(C, \mathcal{L})$, and more precisely that an element $\zeta \in H^1(C, K \otimes \mathcal{L}^{-1})$ vanishes if and only if $\int_C \zeta Q = 0$ for every $Q \in H^0(C, \mathcal{L})$. In our case, because of the delta functions in the definition of ζ, the integral is trivially done: $\int_C \zeta Q = \sum_i \alpha_i Q(x_i)$. Putting it all together, we have established what we wanted to know: the family $\{\alpha_i\}$ can be written as $\mathrm{Res}_{x_i}\widetilde{P}$, for some global differential $\widetilde{P}$, if and only if $\sum_i \alpha_i Q(x_i) = 0$ for every $Q \in H^0(C, \mathcal{L})$.

This completes the demonstration that the twistor representation of the scattering amplitude can be expressed by integration over a left-right symmetric set of parameters – C, the x_i, $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, and P_a and $\widetilde{P}_{\dot{a}}$.

3 Explicit Evaluation For Genus Zero

In genus zero, we can make this much more explicit. Moreover, because the integration measure is known in genus zero, we can give a complete proof of parity invariance of the tree level scattering amplitudes, by showing that after the manipulation explained in section 2, the integration measure as well as the integration region becomes left-right symmetric.

To compute the scattering amplitude, we must integrate over the choice of curve C and line bundle $\mathcal{L}$, over the points $x_i \in C$ at which the external gluons are attached, and over the polynomials P_a, $Q_{\dot{a}}$, and χ^A. In genus zero, C and $\mathcal{L}$ have no moduli. The integration measure for the polynomials was explained in section 2. Finally, the integration measure over the x_i comes from the path integral of free fermions on the worldvolume of the $D1$-brane, as explained in [**6**]. We describe $C \cong \mathbb{CP}^1$ by homogeneous coordinates u^α, $\alpha = 1, 2$, and write u_i^α for the homogeneous coordinates of x_i. For a single trace subamplitude with external gluons attached in a definite cyclic order (which we take to be simply $123\ldots n$), the measure that comes from the worldvolume path integral is

$$\prod_i \int \langle u_i, du_i \rangle \frac{1}{\prod_k \langle u_k, u_{k+1} \rangle}. \tag{26}$$

The wave functions all contain a factor $\overline{\delta}(\langle P(u_i), \lambda_i \rangle)$, which is supported for u_i such that $P(u_i)$ is a multiple of λ_i. Which multiple it is does not matter, since (26) is homogeneous in u_i, for each i. It is convenient to take the multiple to be 1, which we do by using the fact that[5]

$$\begin{aligned} &\prod_i \int \langle u_i, du_i \rangle \frac{1}{\prod_k \langle u_k, u_{k+1} \rangle} \prod_m \overline{\delta}(\langle P(u_m), \lambda_m \rangle) \\ &= \prod_i \int d^2 u_i \frac{1}{\prod_k \langle u_k, u_{k+1} \rangle} \prod_m \overline{\delta}^2 (P_a(u_m) - \lambda_{m\,a}). \end{aligned} \tag{27}$$

One advantage of taking the multiple to be 1 is that we can drop all factors of (λ_i/λ) in the wavefunctions of section 2. Upon doing so, the wavefunctions look much more appealing.

[5]If the argument of the delta function on the left vanishes at $u_i = \alpha_i$, for some α_i, then by homogeneity it vanishes at $u_i = w\alpha_i$ for any w. However, on the left, u_i is a homogeneous variable and we can just set $w = 1$. Instead, on the right, as P is homogeneous of degree $q-1$, there are $q-1$ values of w for which the argument of the delta function vanishes; however, each contributes to the integral with a factor of $1/(q-1)$ that comes from the fact that if $f(x)$ vanishes at $x = x_0$, the contribution of this zero to $\int dx\, \delta(f(x))$ is $1/|f'(x_0)|$.

We suppose that p of the external gluons have positive helicity and q have negative helicity, with $p+q=n$. In this case, according to the rules in [**6**], the line bundle $\mathcal{L}$ is $\mathcal{L}=\mathcal{O}(q-1)$. The gluons with right-handed or positive helicity form a set R, and the gluons with left-handed or negative helicity form a set L. The polynomials P_a, $Q_{\dot a}$, and χ^A are of degree $q-1$ in the u^α. The space U of degree $q-1$ polynomials $f(u^\alpha)$ is q-dimensional. Taking advantage of the fact that there are precisely q points in L, we pick a basis for U consisting of basis elements f_i, $i\in L$, such that for all $i,j\in L$, $f_i(u_j)=\delta_{ij}$. (f_i is only defined for $i\in L$ and $f_i(u_j)$ is only constrained to equal δ_{ij} if in addition $j\in L$.) Explicitly (though we will never use this formula), this is accomplished by taking

$$f_i(u)=\prod_{j\in L,\,j\neq i}\frac{\langle u,u_j\rangle}{\langle u_i,u_j\rangle}. \tag{28}$$

Our general recipe for the integration measure says that we should expand (for example) $P_a=\sum_{i\in L}p_{i\,a}f_i$, whereupon the integration measure is

$$\prod_a dP_a=\prod_{i,a}dp_{i\,a}. \tag{29}$$

With our choice of basis, $p_{i\,a}=P_a(u_i)$ for $i\in L$, and we can alternatively write the integration measure as

$$\prod_{i\in L,a}dP_a(u_i). \tag{30}$$

The integration measures for the other polynomials Q and χ is precisely analogous and can be written as in (29) or (30).

The combined integration measure for P,Q, and χ is independent of the choice of basis, but picking a basis enables us to integrate over P, Q, or χ separately. This has several benefits. The first is that we can trivially integrate over χ and eliminate the fermions from the discussion. As we explained in section 2, for every $i\in L$, the external gluon wavefunction contains a factor $\prod_{A=1}^4\psi^A$. For the i^{th} external gluon, $\psi^A=\chi^A(u_i)$, so the dependence of the integrand on the χ^A is

$$\prod_{i\in L}\prod_{A=1}^{4}\chi^A(u_i). \tag{31}$$

On the other hand, with our choice of basis, the measure for integrating over χ is $\prod_{i,A}d\chi^A(u_i)$. The definition of fermion integration gives immediately

$$\int\prod_{i,A}d\chi^A(u_i)\prod_{j\in L}\prod_{B=1}^{4}\chi^B(u_j)=1, \tag{32}$$

so with our choice of basis, the fermion integral simply gives a factor of 1.

As described in section 2.2, the scattering amplitude also contains a factor

$$\int dQ_{\dot a} \exp\left(i\sum_j [Q(u_j),\widetilde\lambda_j]\right). \tag{33}$$

(As noted following eqn. (27), we can drop factors of λ/λ_j. This is done below without comment.) We saw in section 2.3 that the integral is supported on the locus on which u_j and $\widetilde\lambda_j$ are such that

$$\widetilde\lambda_{j\,\dot a} = \mathrm{Res}_{u_j}\widetilde P_{\dot a} \tag{34}$$

for some $\widetilde P_{\dot a} \in H^0(C, K(u_i)\otimes\mathcal{L}^{-1})$. We can write explicitly

$$\widetilde P_{\dot a} = \langle u, du\rangle \frac{T_{\dot a}(u)}{\prod_i \langle u_i, u\rangle}, \tag{35}$$

where $T_{\dot a}(u)$ is a polynomial homogeneous of degree $p-1$, or in other words is a section of $\widetilde{\mathcal{L}} = \mathcal{O}(p-1)$. This ensures that $\widetilde P_{\dot a}$ is a differential homogeneous of degree $1-q$ with possible simple poles at the u_i or in other words a section of $K(u_i)\otimes\mathcal{L}^{-1}$. Calculating the residues, we can rewrite (34) as

$$\widetilde\lambda_{j,\dot a} = \frac{T_{\dot a}(u_j)}{\prod_{k\neq j}\langle u_k, u_j\rangle}. \tag{36}$$

The scattering amplitude will be expressed as an integral over $T_{\dot a}$. We pick a basis $\widetilde f_\tau$ of $\widetilde U = H^0(C,\widetilde{\mathcal{L}})$ and expand $T_{\dot a}$ in this basis: $T_{\dot a} = \sum_\tau t_{\tau\,\dot a}\widetilde f_\tau$. The integration measure over $T_{\dot a}$ is then taken to be

$$\prod_{\tau,\,\dot a} dt_{\tau\,\dot a}. \tag{37}$$

Aiming for left-right symmetry, we define the basis of $\widetilde U$ in a "dual" fashion to the basis of U that was chosen earlier. As $\widetilde U$ is p-dimensional, we introduce one basis element $\widetilde f_i$ for each $i\in R$, normalized so that $\widetilde f_i(u_j) = \delta_{ij}$ for $j\in R$. Explicitly (though again we will not use this formula),

$$\widetilde f_i(u) = \prod_{j\in R,\, j\neq i} \frac{\langle u, u_j\rangle}{\langle u_i, u_j\rangle}. \tag{38}$$

The integration measure then becomes

$$\prod_{i\in R,\,\dot a} dT_{\dot a}(u_i), \tag{39}$$

in perfect parallel with the integration measure for P and Q as described earlier (see (30)).

We now expect that

$$\int dQ_{\dot a} \exp\left(i\sum_j [Q(u_j),\widetilde\lambda_j]\right) = g(u_1,\dots,u_n)\int dT_{\dot a}\prod_j \delta\left(\widetilde\lambda_{j\,\dot a} - \frac{T_{\dot a}(u_j)}{\prod_{k\neq j}\langle u_k,u_j\rangle}\right) \tag{40}$$

for some function $g(u_j)$. This expresses the fact that, as we know from section 2.3, the integral on the left has delta function support on the locus on which (36) is obeyed for some $T_{\dot a}$.

To determine $g(u_j)$, we act on both the left and right hand side with $\prod_{j\in L}\int d^2\widetilde\lambda_{j,\dot a}$. The integration contour is taken to be the real axis. On the left hand side of (40), the part of the exponent that involves $\widetilde\lambda_j$ for $j\in L$ is just $\sum_{j\in L}[q_j,\widetilde\lambda_j]$. (The q_j were defined so that $Q(u_j)=q_j$ for $j\in L$ and the measure in integrating over Q is just $\prod_{j\in L,\dot a} dq_{j\,\dot a}$.) This being so, we get

$$\prod_{j\in L}\int d^2\widetilde\lambda_{j,\dot a}\int dQ_{\dot a}\exp\left(i\sum_j[Q(u_j),\widetilde\lambda_j]\right) = (2\pi)^{2p}, \tag{41}$$

where each integral over $\widetilde\lambda_{j\,\dot a}$ gives a factor $2\pi\delta(q_{j\,\dot a})$, and the integral over Q is done with these delta functions. To apply $\prod_{j\in L}\int d^2\widetilde\lambda_{j,\dot a}$ to the right hand side of (40), we simply note that

$$\prod_{j\in L}\int d^2\widetilde\lambda_{j,\dot a}\int dT_{\dot a}\prod_m \delta\left(\widetilde\lambda_{m\,\dot a} - \frac{T_{\dot a}(u_m)}{\prod_{k\neq m}\langle u_k,u_m\rangle}\right) = \int dT_{\dot a}\prod_{m\in R}\delta\left(\widetilde\lambda_{m\,\dot a} - \frac{T_{\dot a}(u_m)}{\prod_{k\neq m}\langle u_k,u_m\rangle}\right), \tag{42}$$

where all we have done is to evaluate the integrals over the $\widetilde\lambda_j$ for $j\in L$ using the delta functions for $m\in L$. The delta functions that remain are therefore the ones for $m\in R$, and these, with our choice of basis, give simply

$$\prod_{k\in R}\int dt_{k\,\dot a}\prod_{m\in R}\delta\left(\widetilde\lambda_{m\,\dot a} - \frac{t_{m\,\dot a}}{\prod_{j\neq m}\langle u_j,u_m\rangle}\right) = \prod_{k\in R,\, m\neq k}\langle u_m,u_k\rangle^2. \tag{43}$$

On the right hand side, all factors are squared simply because each t_k has two components $t_{k\,\dot a}$, $\dot a=1,2$; k is restricted to R but m ranges over both L and R.

Comparing these formulas, we see that

$$g(u_1, \ldots, u_n) = (2\pi)^{2p} \prod_{k \in R,\, m \neq k} \langle u_m, u_k \rangle^{-2}. \tag{44}$$

Now, we assume that the underlying scattering amplitude is defined as

$$\frac{1}{(2\pi)^{2p}} \int dP\, dQ\, d\chi \left\langle \prod_i W_i \right\rangle \tag{45}$$

where as in (7), $\langle \prod_i W_i \rangle$ is the correlation function of vertex operators in the worldvolume theory of the D-instanton, and the integral over P, Q, and χ is the integral over the moduli space $\mathcal{M}$ of D-instantons, as described in [**6**] and in section 2. The only novelty we are adding here relative to what was explained in [**6**] is that a factor of $(2\pi)^{-2p}$ must be included in the definition in order to ensure parity symmetry. (This can be interpreted as a normalization factor in the wavefunctions or the gauge coupling constant.) We have seen that with our choice of basis, the integral over χ gives 1, and the integral over Q can be replaced by an integral over T with the extra factor $g(u_i)$. It also was shown in [**6**] that the evaluation of the correlation function gives $\prod_j \int \langle u_j, du_j \rangle \prod_m \langle u_m, u_{m+1} \rangle^{-1}$, and in eqn. (27) we explained how to convert $\int \langle u, du \rangle$ to $\int d^2 u$. (However, we will here take all integration variables, including u, to be real and replace $\overline{\delta}$ functions by ordinary delta functions.) So all told upon replacing the integral over Q by an integral over T via the above-described recipe, the formula for the scattering amplitude becomes

$$A = \int dP_a\, dT_{\dot a} \int \prod_j d^2 u_j \frac{1}{\prod_h \langle u_h, u_{h+1} \rangle} \frac{1}{\prod_{k \in R,\, m \neq k} \langle u_m, u_k \rangle^2} \prod_i \delta^2 \left(\lambda_{i\,a} - P_a(u_i) \right) \delta^2 \left(\widetilde{\lambda}_{i\,\dot a} - \frac{T_{\dot a}(u_i)}{\prod_{l \neq i} \langle u_l, u_i \rangle} \right). \tag{46}$$

We have not quite achieved manifest parity-invariance, but as will soon be clear, this can now be obtained by an elementary change of variables. Let $\phi_i = 1$ for $i \in L$ and 0 for $i \in R$, and let $\widetilde{\phi}_i = 1 - \phi_i$. Then as we will see, (46) is equivalent to the manifestly symmetric expression

$$A = \int dP_a\, dT_{\dot a} \int \prod_j d^2 u_j \frac{1}{\prod_m \langle u_m, u_{m+1} \rangle} \frac{1}{\prod_{l \neq k} \langle u_l, u_k \rangle^2} \prod_i \delta^2 \left(\lambda_{i\,a} - \frac{P_a(u_i)}{\prod_{s \neq i} \langle u_s, u_i \rangle^{\phi_i}} \right) \delta^2 \left(\widetilde{\lambda}_{i\,\dot a} - \frac{T_{\dot a}(u_i)}{\prod_{t \neq i} \langle u_t, u_i \rangle^{\widetilde{\phi}_i}} \right) \tag{47}$$

There are a total of $4n$ integration variables and $4n$ delta functions, so the integral really reduces to a sum over contributions of delta functions.

We make the change of variables

$$\begin{aligned} u_i &\to u_i \prod_{k\neq i}\langle u_k, u_i\rangle^{-1/(q-1)},\ i\in L \\ u_i &\to u_i,\ i\in R \\ P &\to P \\ T &\to \frac{T}{\prod_{j\in L,\, k\neq j}\langle u_k, u_j\rangle^{1/(q-1)}} \end{aligned} \tag{48}$$

We find that

$$\begin{aligned} P(u_i) &\to \frac{P(u_i)}{\prod_{k\neq i}\langle u_k, u_i\rangle},\ i\in L \\ P(u_i) &\to P(u_i),\ i\in R. \end{aligned} \tag{49}$$

These formulas show that the delta functions containing P in (45) transform into the delta functions containing P in (46). Also, since the measure for the P integration is $dP_a = \prod_{i\in L}\prod_a d^2P_a(u_i)$, the first formula in (48) shows that this measure transforms as

$$\prod_a dP_a \to \frac{\prod_a dP_a}{\prod_{i\in L,\, j\neq i}\langle u_j, u_i\rangle^2}. \tag{50}$$

(The denominator is squared because we must transform P_a for $a = 1, 2$.) We also have

$$\begin{aligned} \frac{T(u_i)}{\prod_{j\neq i}\langle u_j, u_i\rangle} &\to T(u_i),\ i\in L \\ \frac{T(u_i)}{\prod_{j\neq i}\langle u_j, u_i\rangle} &\to \frac{T(u_i)}{\prod_{j\neq i}\langle u_j, u_i\rangle},\ i\in R. \end{aligned} \tag{51}$$

These formulas imply that the delta functions containing T in (45) transform into the delta functions containing T in (46). Furthermore, since the measure for integrating over T is $dT_{\dot a} = \prod_{i\in R}\prod_{\dot a} dT_{\dot a}(u_i)$, the second formula shows that

$$\frac{\prod_{\dot a} dT_{\dot a}}{\prod_{i\in R,\, j\neq i}\langle u_j, u_i\rangle^2} \tag{52}$$

is invariant under the rescaling. Finally,

$$\frac{\prod_i d^2u_i}{\prod_j\langle u_j, u_{j+1}\rangle} \tag{53}$$

is invariant under any scaling of the u_i, and in particular under the transformation in (48). If we combine these assertions, we find that the given change of variables does indeed transform (46) into (47).

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