

MODERATE DEVIATIONS AND NONPARAMETRIC INFERENCE FOR MONOTONE FUNCTIONS

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This paper considers self-normalized limits and moderate deviations of nonparametric maximum likelihood estimators for monotone functions. We obtain their self-normalized Cramér-type moderate deviations and limit distribution theorems for the nonparametric maximum likelihood estimator in the current status model and the Grenander-type estimator. As applications of the results, we present a new procedure to construct asymptotical confidence intervals and asymptotical rejection regions of hypothesis testing for monotone functions. The theoretical results can guarantee that the new test has the probability of type II error tending to 0 exponentially. Simulation studies also show that the new nonparametric test works well for the most commonly used parametric survival functions such as exponential and Weibull survival distributions.

1. Introduction. The maximum likelihood estimator (MLE) of a monotone density function is the Grenander estimator [see Grenander (1956), Prakasa Rao (1969), and Groeneboom (1985)]. The asymptotic distribution of the Grenander estimator was obtained in Prakasa Rao (1969). Groeneboom and Wellner (1992) derived the limit distribution of the nonparametric maximum likelihood estimator (NPMLE) for current status data. Banerjee and Wellner (2001) proposed a likelihood ratio (LR) test for monotone functions to construct pointwise confidence intervals for the monotone functions and obtained its limit distribution. The method was studied further in Banerjee and Wellner (2005a, 2005b) and Banerjee (2007). Banerjee and Wellner (2001, 2005a, 2005b) investigated the power behavior of the LR test, and demonstrated the superior performance of the LR test over several existing methods in the current status model. Recently,

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Groeneboom, Jongbloed and Witte (2010) proposed two estimators: the maximum smoothed likelihood and the smoothed maximum likelihood estimators and established their asymptotic normality, but the asymptotic variances depend on the derivative of the density function. Moreover, Groeneboom (2014) studied the maximum smoothed likelihood estimator for the interval censoring model, Groeneboom and Jongbloed (2014) discussed these nonparametric estimators, and Groeneboom and Jongbloed (2015) provided an alternative way to deduce the distribution of the likelihood ratio and extended it to the Grenander estimator.

Asymptotic behavior for Grenander's estimator and its extensions have been studied widely in the last 30 years. The asymptotic normality of the L_1 -distance between the decreasing density and its Grenander estimator was obtained by Groeneboom, Hooghiemstra and Lopuhaä (1999), and a similar result for monotone regression can be found in Durot (2002). Huang and Wellner (1995) studied the limit distribution of a Grenander-type estimator for a monotone hazard rate in a right-censoring model. Durot (2007) gave the limit distribution for the L_p -error of a Grenander-type estimator. Recently, the limiting distribution of the supremum distance was studied by Durot, Kulikov and Lopuhaä (2012), and a law of the iterated logarithm for the Grenander's estimator was established by Dümbgen, Wellner and Wolff (2016).

Recently, self-normalized Cramér-type moderate deviations in statistics have attracted wide attention; see, for example, Liu and Shao (2010, 2013), Chang, Shao and Zhou (2016). In these papers, the statistics can be represented analytically by samples. The NPMLE of current status data and Grenander-type estimator are inverse statistical problems, and their weak limits are the maximum point of a Brownian motion minus a parabola. In this paper, we study self-normalized limit distributions and Cramér-type moderate deviations of the NPMLE for monotone functions. The self-normalized method is based on the self-normalized Cramér-type moderate deviations, and the self-normalized limit distributions for the NPMLE in the current status model and Grenander-type estimator are obtained. As applications, we use these results to construct asymptotical confidence intervals and asymptotical rejection regions of hypothesis testing for monotone functions. The new procedure for statistical inference is different from the LR method in Banerjee and Wellner (2001) and Groeneboom and Jongbloed (2015) and the smoothed maximum likelihood method in Groeneboom, Jongbloed and Witte (2010). Theoretically, the proposed test has the probability of type II error tending to 0 with exponential decay by moderate deviations. Such theoretical properties of the type 2 error are not generally well studied for other competing methods.

The rest of the paper is organized as follows. Section 2 presents the main results including the self-normalized Cramér-type moderate deviations, and the self-normalized limit distributions for the NPMLE in the current status model and Grenander-type estimator. In Section 3, the main results in the current status model are proved using a general Cramér-type moderate deviation theorem and a general

limit distribution theorem. The proofs of these general results are postponed to Sections 5. Our proofs are based on the strong approximation, Talagrand deviation inequality and small ball estimates. In Section 4, we use the theoretical results to construct statistical hypothesis tests and pointwise confidence intervals. In addition, we conduct simulation studies to evaluate the finite-sample performance of the proposed method and compare it with the LR test in Section 4. The proofs of some technical lemmas and the results for Grenander-type estimator are given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)], and some useful known results are presented in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)] for the convenience of the reader.

2. Main results. We consider two types of problems: the NPMLE in the current status model and the Grenander-type estimator.

2.1. *Current status data.* Consider a study that concerns the time to occurrence of a certain event of interest such as failure or onset of a disease. In many cases, the event time cannot be observed completely since it is often too expensive or even impossible to observe the exact event occurrence time. However, the current status of the subjects at a random examination time can be much more easily observed. This means that for each subject, the event occurrence time is unknown, but it is known whether the event has occurred before the examination time. Such data are called current status data or case 1 interval censored data in survival analysis [Groeneboom and Wellner (1992), Huang and Wellner (1995), Groeneboom (1996), Sun (2006), Groeneboom, Jongbloed and Witte (2010)].

Let X_1, \dots, X_n and T_1, \dots, T_n be independent i.i.d. samples from unknown distribution functions F_0 and G on the nonnegative half-line, respectively. Set $\delta_i = I_{\{X_i \leq T_i\}}$. Here, X_i is interpreted as the (unobserved) time of onset of a disease; T_i is a check-up time at which patient i is observed to be ill or not: $\delta_i = 1$ or 0 . The observations consist of the pairs $(\delta_1, T_1), \dots, (\delta_n, T_n)$ in the current status model. The NPMLE \hat{F}_n for F_0 is defined to be the one that maximizes the log-likelihood function

$$F \mapsto \sum_{i=1}^n (\delta_i \log F(T_i) + (1 - \delta_i) \log(1 - F(T_i))),$$

where F is a right-continuous distribution function. Since the likelihood function depends on F only through the values $F(T_i)$, the NPMLE is not unique. To avoid this ambiguity, \hat{F}_n is assumed to be a constant on each interval $[T_{(i-1)}, T_{(i)})$, where $T_{(i)}$ is the i th order statistic of T_1, \dots, T_n . A basic assumption is as follows.

ASSUMPTION (A1). Let $t_0 > 0$ be a fixed point such that $0 < F_0(t_0), G(t_0) < 1$. Let F_0 and G be continuous on $[0, \infty)$ and differentiable on a closed neighborhood $[t_0 - \epsilon_0, t_0 + \epsilon_0]$ of the point t_0 , with strictly positive derivatives $f_0(t)$ and $g(t)$, $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$, respectively, where $0 < \epsilon_0 < t_0$.

Throughout the paper, the notation $\xi_n \xrightarrow{d} \xi$ means that the random sequence ξ_n converges to a random variable ξ in distribution. The notation $\operatorname{argmin}_{t \in U} f(t)$ [resp., $\operatorname{argmax}_{t \in U} f(t)$] stands for the point $t \in U$ at which $f(t)$ is minimized (resp., maximized), where the largest value is chosen when multiple minimizers (resp., maximizers) exist.

Groeneboom and Wellner (1992) established the strong consistency of the NPMLE, and derived the limit distribution of the NPMLE $\hat{F}_n(t_0)$ for a fixed point t_0 as follows: under Assumption (A1) stated above,

$$\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \xrightarrow{d} \tau,$$

where

$$\sigma_F^2(t_0) := \frac{4F_0(t_0)(1 - F_0(t_0))f_0(t_0)}{g(t_0)}, \quad \tau := \operatorname{argmin}_{t \in \mathbb{R}} \{W(t) + t^2\},$$

and $\{W(t), t \in \mathbb{R}\}$ is a two-sided Wiener process with $W(0) = 0$.

By Corollaries 3.3 and 3.4 in Groeneboom (1989), the density function f_τ of τ is symmetric and bounded continuous, and has the tail behavior

$$(2.1) \quad f_\tau(x) \sim \frac{2}{\operatorname{Ai}'(a_1)} 4^{1/3} |x| \exp\left\{-\frac{2}{3}|x|^3 + 2^{1/3} a_1 |x|\right\}, \quad |x| \rightarrow \infty$$

where $a_1 \approx -2.3381$ is the largest zero of the Airy function Ai and $\operatorname{Ai}'(a_1) \approx 0.7002$. By Lemma 2.1 in Hooghiemstra and Lopuhaä (1998), the distribution function F_τ of τ satisfies

$$(2.2) \quad 1 - F_\tau(x) \sim \frac{1}{\operatorname{Ai}'(a_1)} 4^{1/3} \frac{1}{x} \exp\left\{-\frac{2}{3}x^3 + 2^{1/3} a_1 x\right\}, \quad x \rightarrow +\infty.$$

In order to use Groeneboom and Wellner’s (1992) result for statistical inference, we need to construct a consistent estimator of f_0 or to obtain a self-normalized limit distribution of the NPMLE. This is one of our motivations. Another motivation of the paper is to study the tail probabilities of $\hat{F}_n(t_0) - F_0(t_0)$ and to construct confidence intervals and rejection regions for hypothesis testing for $F_0(t_0)$. We consider the following asymptotic behavior of $\hat{F}_n(t_0) - F_0(t_0)$: Cramér-type moderate deviation and moderate deviation principle (MDP).

The Cramér-type moderate deviation is to find $b_n \rightarrow \infty$ such that

$$\sup_{x \in [0, b_n]} \left| \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x\right) - 1 \right| \rightarrow 0.$$

The moderate deviation principle (MDP) is to find two suitable sequences of positive numbers $b_n \rightarrow \infty$ and $\beta_n \rightarrow \infty$ with $\frac{n^{1/3}}{b_n} \rightarrow \infty$ such that

$$P\left(\left|\frac{1}{b_n} \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}}\right| > x\right) \approx e^{-I(x)\beta_n},$$

where $I(x)$ is a suitable *rate function* and β_n is called the *speed*. If one replaces $\frac{n^{1/3}}{b_n} \rightarrow \infty$ by $\frac{n^{1/3}}{b_n} \rightarrow c > 0$, this result is called the large deviation principle (LDP). Generally, the rate function I of the MDP can be easily calculated, and provides a useful method for statistical inference. Some notion and properties in large deviation are given in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)].

Generally, the bigger b_n is, the faster the convergence rate is. In the i.i.d. case, $b_n = O(n^{1/6})$ is the best for the Cramér-type moderate deviation [see Petrov (1975), de la Peña, Lai and Shao (2009)]. For the NPMLE, due to a technical reason, we assume that b_n satisfies, as $n \rightarrow \infty$,

$$(2.3) \quad b_n \rightarrow \infty \quad \text{and} \quad \frac{b_n^{10} \log n}{n^{1/3}} \rightarrow 0.$$

That is, $b_n \rightarrow \infty$ and $b_n = o(n^{1/30}/(\log n)^{1/10})$.

To study moderate deviations of the NPMLE, we need the following assumption.

ASSUMPTION (A2). The functions $f_0(t)$ and $g(t)$ are Lipschitz continuous on $[t_0 - \epsilon_0, t_0 + \epsilon_0]$.

Let \hat{g}_n be a classical kernel density estimator of g_0 defined by

$$(2.4) \quad \hat{g}_n(t) = \frac{1}{nh_n} \sum_{k=1}^n I_{[0,1]} \left(\frac{t - T_k}{h_n} \right),$$

where $h_n \rightarrow 0$ and $\frac{nh_n}{b_n^3} \rightarrow \infty$. Then by the large deviation principle for the classical kernel density estimator [see the proof of Proposition 3.1 in Gao (2003)], for any $r > 0$,

$$(2.5) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P(|\hat{g}_n(t_0) - g_0(t_0)| > r) = -\infty,$$

since $\lim_{n \rightarrow \infty} \frac{nh_n}{b_n^3} = \infty$. In order to obtain a faster speed, one can take $h_n = \frac{1}{\log n}$. We can also replace $\log n$ by a sequence r_n with $\log n \leq r_n = o(n^{1/3}/b_n)$ to obtain a faster convergence speed.

Let us introduce two estimators of f_0 as follows:

$$(2.6) \quad \hat{f}_n(t) = \frac{n^{1/3}}{b_n \log n} (\hat{F}_n(t + b_n \log n/n^{1/3}) - \hat{F}_n(t))$$

and

$$(2.7) \quad \tilde{f}_n(t) = \frac{n^{1/3}}{\log n} (\hat{F}_n(t + \log n/n^{1/3}) - \hat{F}_n(t)).$$

For the estimators $\hat{f}_n(t)$ and $\tilde{f}_n(t)$, we also have the following exponential convergence.

PROPOSITION 2.1. (i) Assume that Assumptions (A1) and (A2) hold. Then for any $r > 0$,

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P(|\hat{F}_n(t_0) - F_0(t_0)| > r) = -\infty$$

and

$$(2.9) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P(|\hat{f}_n(t_0) - f_0(t_0)| > r) = -\infty.$$

(ii) Assume that Assumption (A1) holds. Then for any $r > 0$,

$$(2.10) \quad \lim_{n \rightarrow \infty} P(|\tilde{f}_n(t_0) - f_0(t_0)| > r) = 0.$$

Now, we state the self-normalized Cramér-type moderate deviations and the self-normalized limit distribution theorem for the NPMLE in the current status model.

THEOREM 2.1. Assume that Assumptions (A1) and (A2) hold. Then for each $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,

$$(2.11) \quad \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x\right) \rightarrow 1,$$

that is,

$$(2.12) \quad \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x\right) \rightarrow 1$$

and

$$(2.13) \quad \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \leq -x\right) \rightarrow 1,$$

where $\hat{\sigma}_n^2(t_0) := 4\hat{F}_n(t_0)(1 - \hat{F}_n(t_0))\hat{f}_n(t_0)/\hat{g}_n(t_0)$.

In particular, the sequence $\{\frac{n^{1/3} \hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}}, n \geq 1\}$ satisfies the MDP in \mathbb{R} with speed b_n^3 and rate function $I(x) = \frac{2}{3}|x|^3$, that is, for any open subset A of \mathbb{R} ,

$$\liminf_{n \rightarrow \infty} \frac{1}{b_n^3} \log P\left(\frac{n^{1/3} \hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}} \in A\right) \geq - \inf_{x \in A} I(x),$$

and for any closed subset B of \mathbb{R} ,

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^3} \log P\left(\frac{n^{1/3} \hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}} \in B\right) \leq - \inf_{x \in B} I(x).$$

THEOREM 2.2. *Assume that Assumption (A1) holds. Then*

$$(2.14) \quad \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\tilde{\sigma}_n^2(t_0))^{1/3}} \xrightarrow{d} \tau,$$

where $\tilde{\sigma}_n^2(t_0) := 4\hat{F}_n(t_0)(1 - \hat{F}_n(t_0))\tilde{f}_n(t_0)/\hat{g}_n(t_0)$.

2.2. Grenander-type estimator. Let λ be a decreasing (or increasing) function on $[0, 1]$ and $\Lambda(t) = \int_0^t \lambda(u) du, t \in [0, 1]$. Let $\Lambda_n(t)$ be a cadlag (right continuous with finite left-hand limits at every point) step process and let $\hat{\Lambda}_n$ be the least concave majorant (resp., greatest convex majorant) of Λ_n . The Grenander-type estimator $\hat{\lambda}_n$ of λ is defined as the left derivative of $\hat{\Lambda}_n$. Then $\hat{\lambda}_n$ is monotone, left-continuous and piecewise constant.

We introduce the following assumptions.

(GA1) The function λ is monotone and differentiable on $[0, 1]$,

$$\inf_{t \in [0,1]} |\lambda'(t)| > 0 \quad \text{and} \quad \sup_{t \in [0,1]} |\lambda'(t)| < \infty.$$

(GA2) Let W be either a Brownian bridge or a Brownian motion. There exist positive absolute constants $C, K, \kappa, L : [0, 1] \rightarrow \mathbb{R}$ and versions of Λ_n and W such that for all $r \in [0, n]$,

$$(2.15) \quad \begin{aligned} P\left(\sup_{u \in [0,1]} |n(\Lambda_n(u) - \Lambda(u)) - n^{1/2}W(L(u))| \geq C \log n + r\right) \\ \leq K \exp\{-\kappa r\}. \end{aligned}$$

Moreover, L is increasing and twice differentiable in $(0, 1)$, and $L'(s) > 0$ for any $s \in (0, 1)$.

We define an estimator of λ' as follows:

$$(2.16) \quad \hat{\lambda}_n^{(1)}(t) = \frac{n^{1/3}}{b_n \log n} (\hat{\lambda}_n(t + b_n \log n / n^{1/3}) - \hat{\lambda}_n(t)).$$

The estimators $\hat{\lambda}_n(t)$ and $\hat{\lambda}_n^{(1)}(t)$ also have exponential convergence.

PROPOSITION 2.2. *Assume that (GA1) and (GA2) hold.*

(i) *For any $r > 0$, we have*

$$(2.17) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log \max\{P(|\hat{\lambda}_n(t) - \lambda(t)| > r), P(|\hat{\lambda}_n^{(1)}(t) - \lambda'(t)| > r)\} = -\infty.$$

(ii) *If $\liminf_{n \rightarrow \infty} \frac{b_n}{(\log n)^3} > 1$, then*

$$(2.18) \quad P\left(\lim_{n \rightarrow \infty} \hat{\lambda}_n(t) = \lambda(t)\right) = 1 \quad \text{and} \quad P\left(\lim_{n \rightarrow \infty} \hat{\lambda}_n^{(1)}(t) = \lambda'(t)\right) = 1.$$

The following result is the self-normalized Cramér-type moderate deviations for the Grenander-type estimator.

THEOREM 2.3. *Assume that (GA1) and (GA2) hold. Then for $t \in (0, 1)$ and $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,*

$$(2.19) \quad \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))}{(4L'(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \geq x\right) \rightarrow 1.$$

In particular, the sequence $\{ \frac{n^{1/3}}{b_n} \frac{\hat{\lambda}_n(t) - \lambda(t)}{(4L'(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}, n \geq 1 \}$ satisfies the MDP in \mathbb{R} with speed b_n^3 and rate function $I(x) = \frac{2}{3}|x|^3$.

The next subsections give several corollaries of Theorem 2.3.

2.2.1. The monotone density function. Let X_1, \dots, X_n be independent random variables taking values in $[0, 1]$ with the common distribution function F and the monotone density function f . Then the monotone estimator based on the empirical distribution function of X_1, \dots, X_n is the Grenander estimator \hat{f}_n .

COROLLARY 2.1. *Assume that (GA1) holds for f . Then for $t \in (0, 1)$ and $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,*

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{f}_n(t) - f(t))}{(4\hat{f}_n(t)|\hat{f}_n^{(1)}(t)|)^{1/3}} \geq x\right) \rightarrow 1.$$

2.2.2. Monotone hazard function with right-censored data. Let $(X_1, \delta_1), \dots, (X_n, \delta_n)$ be a right-censored sample, where $X_i = \min\{T_i, Y_i\}$, $\delta_i = I_{\{T_i \leq Y_i\}}$, $T_i, i = 1, \dots, n$ are nonnegative i.i.d. failure times, and $Y_i, i = 1, \dots, n$ are i.i.d. censoring times independent of $T_i, i = 1, \dots, n$. Assume that the common distribution function F of $T_i, i = 1, \dots, n$ is absolutely continuous with density function f . Assume that F has a monotone hazard rate $\lambda = f/(1 - F)$. Let N_n be the Nelson–Aalen estimator, defined as follows: if $t_1 < \dots < t_k$ are the distinct uncensored times and n_j is the number of X_i 's that are greater than or equal to t_j ($j = 1, \dots, k$), then

$$N_n(t) = \sum_{i=1}^{k-1} I_{[t_i, t_{i+1})}(t) \sum_{j \leq i} \frac{1}{n_j} + I_{[t_k, 1)}(t) \sum_{j \leq k} \frac{1}{n_j}.$$

Let Λ_n be the restriction of N_n to $[0, 1]$ and let G be the common distribution function of $Y_i, i = 1, \dots, n$. Let $\hat{\lambda}_n$ be the monotone estimator based on Λ_n [see [Huang and Wellner \(1995\)](#)]. Let F_n and G_n be the Kaplan–Meier estimators of F and G , respectively.

COROLLARY 2.2. *Suppose that (GA1) holds, $F(1) < 1$, $\lim_{t \rightarrow 1} G(t) < 1$, and G has a bounded continuous first derivative on $(0, 1)$. Then for $t \in (0, 1)$ and $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,*

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))(1 - F_n(t))(1 - G_n(t))}{(4\hat{\lambda}_n(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \geq x\right) \rightarrow 1.$$

2.2.3. *The monotone regression model.* Consider the model

$$Y_{i,n} = \lambda(i/n) + \varepsilon_{i,n}, \quad i = 1, \dots, n,$$

where $\varepsilon_{i,n}, i = 1, \dots, n$, are independent random variables with mean zero. Let

$$\Lambda_n(t) = \frac{1}{n} \sum_{i \leq nt} Y_{i,n}, \quad t \in [0, 1].$$

Then the monotone estimator $\hat{\lambda}_n$ based on Λ_n is a slight modification of the Brunk estimator.

COROLLARY 2.3. *Suppose that (GA1) holds and there exists $\delta > 0$ such that $\sup_{i,n} E(\exp\{\delta|\varepsilon_{i,n}|\}) < \infty$. Assume, moreover, that $\text{var}(\varepsilon_{i,n}) = \sigma^2(i/n)$ for some $\sigma^2 : [0, 1] \rightarrow \mathbb{R}_+$. If σ^2 has a bounded first derivative and satisfies $\inf_t \sigma^2(t) > 0$, then for $t \in (0, 1)$ and $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,*

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))}{(4\sigma^2(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \geq x\right) \rightarrow 1.$$

3. Proofs of the main results. In this section, we only prove Proposition 2.1, and Theorems 2.1 and 2.2. The proofs of the results on the Grenander-type estimator are similar and will be given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)]. The following two results, Theorems 3.1 and 3.2, play an important role in our proofs. Their proofs will be given in Section 5.

THEOREM 3.1. *Assume that Assumptions (A1) and (A2) hold. Let χ_n be a sequence of nonnegative numbers such that $b_n^{4.5} \chi_n \rightarrow 0$ as $n \rightarrow \infty$. Then for each $\rho > 0$, uniformly for $0 \leq x \leq \rho b_n$,*

$$(3.1) \quad \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \geq x\right) \rightarrow 1.$$

In particular, the sequence $\{ \frac{n^{1/3} \hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)}{(\sigma_F^2(t_0))^{1/3}}, n \geq 1 \}$ satisfies the MDP in \mathbb{R} with speed b_n^3 and rate function $I(x) = \frac{2}{3}|x|^3$.

The next result is an extension of Groeneboom and Wellner’s limit distribution result.

THEOREM 3.2. *Assume that Assumption (A1) holds. Let χ_n be a sequence of nonnegative numbers such that $\chi_n \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$\frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \xrightarrow{d} \tau.$$

PROOF OF PROPOSITION 2.1. Equation (2.8) is obvious. When n is large enough,

$$\begin{aligned} |\hat{f}_n(t_0) - f_0(t_0)| &\leq \frac{1}{\log n} \frac{n^{1/3}}{b_n} \{ |\hat{F}_n(t_0 + b_n \log n/n^{1/3}) - F_0(t_0 + b_n \log n/n^{1/3})| \\ &\quad + |\hat{F}_n(t_0) - F_0(t_0)| \} \\ &\quad + \left| \frac{F(t_0 + b_n \log n/n^{1/3}) - F(t_0)}{b_n \log n/n^{1/3}} - f_0(t_0) \right|. \end{aligned}$$

Since

$$\left| \frac{F(t_0 + b_n \log n/n^{1/3}) - F(t_0)}{b_n \log n/n^{1/3}} - f_0(t_0) \right| \rightarrow 0,$$

then by Theorems 3.1, (2.9) holds. The proof of (2.10) is similar to that of (2.9). □

PROOF OF THEOREM 2.1. By Proposition 2.1(i) and the Delta method in large deviations [Gao and Zhao (2011)], for any $\epsilon > 0$,

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P \left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1 \right| > \epsilon \right) = -\infty.$$

For each $\rho > 0$ and $0 \leq x \leq \rho b_n$, we have

$$\begin{aligned} &\frac{1}{1 - F_\tau(x)} P \left(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x \right) \\ &\leq \frac{1}{P(\tau \geq x)} P \left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1 \right| > 1 \right) \\ &\quad + \frac{P(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x(1 - \frac{b_n \log n}{n^{1/3}}))}{P(\tau \geq x(1 - \frac{b_n \log n}{n^{1/3}}))} \frac{P(\tau \geq x(1 - \frac{b_n \log n}{n^{1/3}}))}{1 - F_\tau(x)} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x\right) \\ & \geq -\frac{1}{P(\tau \geq x)} P\left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1 \right| > 1\right) \\ & \quad + \frac{P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x\left(1 + \frac{b_n \log n}{n^{1/3}}\right)\right) P(\tau \geq x\left(1 + \frac{b_n \log n}{n^{1/3}}\right))}{P(\tau \geq x\left(1 + \frac{b_n \log n}{n^{1/3}}\right)) (1 - F_\tau(x))}. \end{aligned}$$

Note that by (2.2) and (3.2),

$$\sup_{x \in [0, \rho b_n]} \frac{1}{P(\tau \geq x)} P\left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1 \right| > 1\right) \rightarrow 0$$

and

$$\sup_{0 \leq x \leq \rho b_n} \left| \frac{1}{1 - F_\tau(x)} P\left(\tau \geq x\left(1 + \frac{b_n \log n}{n^{1/3}}\right)\right) - 1 \right| \rightarrow 0.$$

Then (2.12) follows from (3.1) with $\chi_n = 0$. \square

PROOF OF THEOREM 2.2. By Proposition 2.1(ii) and the continuous mapping theorem [van der Vaart and Wellner (1996), page 20], for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left(\left| \frac{(\sigma_F^2(t_0))^{1/3}}{(\tilde{\sigma}_n^2(t_0))^{1/3}} - 1 \right| > \epsilon\right) = 0.$$

Again, the conclusion follows from Theorem 3.1 with $\chi_n = 0$. \square

4. Applications. In this section, we focus on applications of the self-normalized Cramér-type MDP to statistical inference for the current status model. We also will present some simulation results to compare the finite-sample performance of the new test constructed by using the self-normalized Cramér-type MDP and the LR test [Banerjee and Wellner (2001, 2005a, 2005b), Groeneboom and Jongbloed (2015)].

4.1. *Hypothesis testing.* In this subsection, we apply the results in Section 2 to hypothesis testing problems for unknown distribution function F in the current status model. Let F_0 and F_1 be two distributions on $[0, \infty)$ and let $t_0 \in (0, \infty)$. Assume that $F_0(t_0) \neq F_1(t_0)$. Consider testing

$$H_0 : F(t_0) = F_0(t_0) \quad \text{versus} \quad H_1 : F(t_0) = F_1(t_0).$$

We can use Theorem 2.1 to construct a rejection region and to prove the probability of type II error tending to 0 with exponential decay. Consider the test statistic

$$T_n := \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{\hat{\sigma}_n},$$

where $\hat{\sigma}_n = (\hat{\sigma}_n^2(t_0))^{1/3}$.

Given $0 < \alpha < 1$, by Theorem 2.1, under H_0 ,

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \rho b_n]} \left| \frac{P(|T_n| \geq x)}{2(1 - F_\tau(x))} - 1 \right| = 0.$$

Take the rejection region as $\{|T_n| \geq c(\alpha)\}$, where $c(\alpha)$ is a positive constant such that $\alpha/2 = 1 - F_\tau(c(\alpha))$. Then the probability β_n of type II error is

$$\beta_n = P(|T_n| < c(\alpha) | F_1(t_0)).$$

Next, we establish the exponential convergence of β_n .

THEOREM 4.1. *Suppose that Assumptions (A1) and (A2) hold for F_0 and F_1 . Then the probability of type II error tends to 0 with exponential decay speed $\exp\{-rb_n^3\}$ for all $r > 0$, that is,*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log \beta_n = -\infty.$$

PROOF. The probability β_n of type II error is

$$\begin{aligned} \beta_n &= P(|T_n| < c(\alpha) | F_1(t_0)) \\ &\leq P\left(\left| \frac{n^{1/3}(\hat{F}_n(t_0) - F_1(t_0))}{b_n \hat{\sigma}_n} \right| \geq \frac{n^{1/3}}{b_n \hat{\sigma}_n} |F_0(t_0) - F_1(t_0)| - \frac{c(\alpha)}{b_n} \mid F_1(t_0)\right). \end{aligned}$$

By Proposition 2.1(i) and the Delta method in large deviations [Gao and Zhao (2011)], for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P\left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_{F_1}^2(t_0))^{1/3}}{\hat{\sigma}_n} - 1 \right| > \epsilon \mid F_1(t_0)\right) = -\infty.$$

Thus, by Theorem 2.1, we have $\lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log \beta_n = -\infty$. \square

4.2. Pointwise confidence interval. For given confidence level $1 - \alpha$, let $\tau_{\alpha/2}$ be a critical value such that $P(\tau > \tau_{\alpha/2}) = \alpha/2$.

A direct application of Theorem 2.1 yields an approximate $(1 - \alpha)$ confidence interval for $F_0(t_0)$ as

$$\left(\hat{F}_n(t_0) - \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2}, \hat{F}_n(t_0) + \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2} \right).$$

4.3. *Simulation study.* We conducted a simulation study to assess the finite-sample performance of the proposed test by rejecting $H_0 : F(t_0) = F_0(t_0)$ if

$$|\hat{F}_n(t_0) - F_0(t_0)| \geq \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2},$$

where $\hat{F}_n(t_0)$ is the NPMLE of $F_0(t_0)$. For $\alpha = 0.05$, one can find $\tau_{0.025} = 0.9982$ from Table 2 in Groeneboom and Wellner (2001) or Table I in Abrevaya and Huang (2005).

For the testing problem, other available methods include the LR, MLE-based, score and Wald test statistics [see Banerjee and Wellner (2001, 2005a, 2005b)]. Based on simulations, Banerjee and Wellner (2005a) concluded that the LR-based method is better than the MLE-based methods for the situations considered, while Banerjee and Wellner (2005b) showed that the LR and score statistics are comparable and are more powerful than the Wald statistic. Note that these so-called score test statistics are based on the difference between the unrestricted and the restricted MLEs, which can be considered as extensions of the LR test statistic. Here, we focus on comparison of the new test and the LR test. To compute the LR test statistic, we also need to find the unrestricted MLE. For this, we used the new method presented in Section 2 of Groeneboom and Jongbloed (2015).

To generate current status data, suppose that failure time X follows an exponential distribution or a Weibull distribution, and examination time T follows a uniform distribution over an interval $[0, T_0]$. Let $\{x_i, i = 1, \dots, n\}$ and $\{t_i, i = 1, \dots, n\}$ be independent samples from the distributions of X and T , respectively. The generated current status data consist of $\{(t_i, \delta_i), i = 1, \dots, n\}$ with $\delta_i = I_{(x_i \leq t_i)}$.

Consider two cases: (i) $F_0(t) = 1 - \exp(-0.5t)$ and $F_1(t) = 1 - \exp(\lambda t)$ with $\lambda \neq 0.5$; and (ii) $F_0(t) = 1 - \exp(-(0.5t)^{0.5})$ and $F_1(t) = 1 - \exp(-(\lambda t)^{0.5})$ with $\lambda \neq 0.5$. We set $T_0 = 2$ under H_0 and 3 or 5 under H_1 .

To apply the proposed test, we need to choose b_n . Since b_n is assumed to satisfy the following condition:

$$b_n \rightarrow \infty \quad \text{and} \quad b_n = o(n^{1/30}/(\log n)^{1/10}),$$

we can simply take $b_n = n^{1/(30+\varepsilon)}$ with $\varepsilon > 0$. According to the general principle “the bigger b_n is, the faster the convergence rate is”, we only considered values of b_n close to $n^{1/30}$ in our simulations. In addition, we need to choose h_n for estimation of g_0 . As pointed out in Section 2.1, we prefer to take $h_n = 1/r_n$ where $r_n = n^{1/(3+\delta)}/b_n$ with $\delta > 0$. Here, we took values of r_n close to $n^{1/3}/b_n$ to obtain a faster convergence speed. All the simulation results reported here are based on 1000 Monte Carlo replications using the R software, while the corresponding R code is available via the website <http://www.mypolyuweb.hk/~mazhao>.

Tables 1 and 2 report the estimated sizes and powers of the proposed test with $b_n = n^{1/30.1}$ and $r_n = n^{1/(3.1)}/b_n$ and LR test at nominal level 0.05 based on simulated current status data for two cases with different values of λ . In each setting,

TABLE 1
Estimated sizes and powers of the proposed and LR tests at nominal level 0.05 for the case of exponential distribution

λ	T_0	$n = 100$		$n = 200$		$n = 300$	
		Proposed	LR	Proposed	LR	Proposed	LR
0.5	2	0.058	0.049	0.058	0.063	0.048	0.055
0.7	3	0.261	0.157	0.309	0.227	0.408	0.324
0.9		0.513	0.427	0.681	0.645	0.737	0.732
0.3		0.517	0.386	0.641	0.558	0.727	0.682
0.1	5	1.000	1.000	1.000	1.000	1.000	1.000
0.7		0.262	0.124	0.278	0.190	0.320	0.232
0.9		0.446	0.339	0.520	0.446	0.636	0.592
0.3		0.543	0.331	0.568	0.434	0.637	0.542
0.1		1.000	0.997	1.000	1.000	1.000	1.000

we take $t_0 = 1$ and considered three sample sizes $n = 100, 200, 300$. It can be seen from the tables that the estimated sizes using both tests are close to nominal level 0.05, and the estimated powers of the proposed test are higher than those of the LR test for the situations considered here. Very similar results were obtained for the proposed test with other values of b_n close to $n^{1/30}$ and other values of r_n close to $n^{1/3}/b_n$ as well.

In addition, we conducted simulations to compare the power behavior of the proposed test and the LR test for the contiguous alternatives with the exponential and Weibull distributions, respectively. Let f_0 be the density function of F_0 .

TABLE 2
Estimated sizes and powers of the proposed and LR tests at nominal level 0.05 for the case of Weibull distribution

λ	T_0	$n = 100$		$n = 200$		$n = 300$	
		Proposed	LR	Proposed	LR	Proposed	LR
0.5	2	0.064	0.038	0.054	0.046	0.041	0.062
0.7	3	0.273	0.079	0.263	0.116	0.270	0.138
0.9		0.405	0.215	0.431	0.290	0.498	0.389
0.3		0.390	0.239	0.419	0.320	0.480	0.390
0.1	5	0.980	0.964	1.000	0.999	0.999	0.999
0.7		0.302	0.065	0.262	0.083	0.284	0.113
0.9		0.376	0.133	0.366	0.194	0.412	0.270
0.3		0.454	0.216	0.432	0.251	0.446	0.309
0.1		0.965	0.916	0.994	0.992	1.000	0.999

Following Banerjee and Wellner (2005b), we define local alternatives $\{F_n\}$ as

$$F_n(t) = F_0(t) + n^{-1/3} B(n^{1/3}(t - t_0)),$$

where $B(t) = \eta f_0(t_0)(c - |t|)1_{[-c,c]}(t)$, and t_0, c, η are fixed positive numbers. Take $\eta = 0.9$. The failure times can be generated by using the method given in Banerjee and Wellner (2005b). Again suppose examination time T follows a uniform distribution over interval $[0, 3]$. The simulation results are shown in Tables 3 and 4. The tables include the estimated powers (the rejection percentages of the null hypothesis over 1000 Monte Carlo replicates) of the proposed test and the LR test for the contiguous alternatives $H_1 : F(t_0) = F_n(t_0)$ with different values of t_0 in two cases of F_0 , respectively. The results indicate that the proposed test outperforms the LR test for the contiguous alternatives considered here.

We also carried out a simulation study using the *Simulation Setting 3* designed by Banerjee and Wellner (2005a) with t_0 as chosen there. The results of estimated confidence intervals indicate that our method does not perform as well as the LR method for this case where the function F_0 behaves capriciously.

TABLE 3
 Estimated powers of the proposed and LR tests at nominal level 0.05 for the contiguous alternatives with the exponential distribution

t_0	c	$n = 100$		$n = 200$		$n = 300$	
		Proposed	LR	Proposed	LR	Proposed	LR
0.5	1	0.225	0.060	0.167	0.071	0.157	0.059
	2	0.284	0.110	0.230	0.125	0.232	0.137
	3	0.506	0.361	0.427	0.325	0.438	0.342
	4	0.734	0.633	0.705	0.635	0.689	0.618
	5	0.888	0.837	0.884	0.870	0.893	0.870
1.0	1	0.152	0.052	0.156	0.053	0.141	0.075
	2	0.225	0.073	0.191	0.090	0.177	0.091
	3	0.356	0.192	0.354	0.189	0.350	0.188
	4	0.559	0.372	0.532	0.351	0.546	0.377
	5	0.762	0.606	0.754	0.580	0.747	0.600
1.5	1	0.148	0.048	0.139	0.054	0.133	0.055
	2	0.225	0.093	0.195	0.067	0.173	0.074
	3	0.323	0.121	0.299	0.132	0.267	0.124
	4	0.491	0.253	0.486	0.253	0.463	0.273
	5	0.660	0.432	0.662	0.466	0.637	0.435
2.5	1	0.218	0.062	0.139	0.059	0.102	0.056
	2	0.288	0.085	0.197	0.083	0.158	0.074
	3	0.351	0.142	0.274	0.141	0.254	0.128
	4	0.441	0.215	0.337	0.208	0.299	0.178
	5	0.537	0.353	0.472	0.327	0.417	0.328

TABLE 4

Estimated powers of the proposed and LR tests at nominal level 0.05 for the contiguous alternatives with Weibull distribution

t_0	c	$n = 100$		$n = 200$		$n = 300$	
		Proposed	LR	Proposed	LR	Proposed	LR
0.5	1	0.208	0.048	0.182	0.049	0.171	0.042
	2	0.286	0.068	0.258	0.075	0.246	0.075
	3	0.448	0.136	0.382	0.105	0.369	0.142
	4	0.624	0.324	0.581	0.275	0.552	0.245
	5	0.752	0.535	0.759	0.546	0.756	0.521
1.0	1	0.212	0.060	0.212	0.055	0.184	0.059
	2	0.262	0.058	0.231	0.057	0.188	0.054
	3	0.343	0.085	0.291	0.076	0.283	0.074
	4	0.432	0.112	0.392	0.125	0.387	0.138
	5	0.536	0.180	0.541	0.215	0.523	0.227
1.5	1	0.198	0.054	0.192	0.046	0.187	0.059
	2	0.256	0.061	0.233	0.040	0.218	0.050
	3	0.294	0.058	0.258	0.062	0.247	0.064
	4	0.351	0.091	0.348	0.094	0.331	0.113
	5	0.445	0.155	0.440	0.150	0.422	0.153
2.5	1	0.223	0.070	0.142	0.066	0.100	0.060
	2	0.269	0.082	0.168	0.079	0.127	0.063
	3	0.272	0.109	0.210	0.097	0.183	0.101
	4	0.301	0.124	0.214	0.122	0.187	0.109
	5	0.310	0.157	0.245	0.169	0.208	0.160

4.4. *Conclusions.* We have proposed a new test for current status data using the self-normalized Cramér-type MDP. When the size is fixed, the new test has the probability of a type II error tending to 0 with exponential decay, and thus it has good power. Our simulation study has demonstrated that the null distribution of the proposed test statistic has a reasonable approximation and that the new test is more powerful than the LR test for most of the settings considered here. However, when the true distribution function behaves capriciously, the proposed approach does not work as well as the LR method. Compared with the LR test, the new test enjoys the following advantages:

- (i) The probability of type II error tends to 0 with exponential decay, which implies that the power behavior of the new test is very good.
- (ii) The rejection region of the proposed test has a simple analytical expression, and thus is easy to implement. In contrast, the rejection region of the LR test must be obtained numerically.
- (iii) Computationally, conducting the proposed test procedure is quite simple, but performing the LR test procedure is time-consuming.

5. Proofs of Theorems 3.1 and 3.2. In this section, we propose a new approach to study the moderate deviation problem. Let us first give an outline of the proof. In Section 5.1, we convert the moderate deviation estimates of $\hat{F}_n(t_0 + \chi_n)$ to those for the locations τ_n of minimums of a sequence of processes $Z_n(t), n \geq 1$. Then, in Section 5.2, we show that the processes $\{Z_n(t), n \geq 1\}$ can be strongly approximated by the processes $\{\hat{Z}_n(t), n \geq 1\}$, where $\hat{Z}_n(t)$ can be represented by a two-sided Wiener process plus a parabolic drift. After that, in Section 5.3, we show that τ_n is exponentially equivalent to the locations $\hat{\tau}_n$ of minimums of the processes $\hat{Z}_n(t)$ by the Talagrand deviation inequality and small ball estimates. The proofs of Theorems 3.1 and 3.2 are given respectively in Sections 5.4 and 5.5 using the asymptotic properties of $\hat{\tau}_n$.

5.1. *An equivalent representation of the moderate deviations.* In this subsection, we convert the moderate deviation estimates of $\hat{F}_n(t_0)$ to those of the minima of a sequence of stochastic processes.

Let \mathbb{P}_n be the empirical measure of $(X_1, T_1), \dots, (X_n, T_n)$, that is,

$$\mathbb{P}_n(A) = \mathbb{P}_n I_A = \frac{1}{n} \sum_{i=1}^n I_A(X_i, T_i), \quad A \in \mathcal{B}(\mathbb{R}_+^2).$$

Set $A = \{(x, t) : x \leq t\}$ and define

$$V_n(t) = \frac{1}{n} \sum_{i=1}^n \delta_i I_{[0,t]}(T_i) = \mathbb{P}_n I_{A \cap (\mathbb{R} \times [0,t])},$$

$$G_n(t) = \frac{1}{n} \sum_{i=1}^n I_{[0,t]}(T_i) = \mathbb{P}_n I_{\mathbb{R} \times [0,t]}.$$

The function $s \mapsto V_n \circ G_n^{-1}(s)$ equals the cumulative sum diagram [van der Vaart and Wellner (1996)], and hence, for every observation point t_i ,

$$\hat{F}_n(t_i) \leq x \quad \text{if and only if} \quad \underset{s}{\operatorname{argmin}} \{V_n(s) - xG_n(s)\} \geq t_i.$$

Thus, the limit distribution of $\hat{F}_n(t)$ can be derived by studying the locations of minimums of the sequence of processes $s \mapsto V_n(s) - xG_n(s)$. For each t , the value $\hat{F}_n(t)$ equals $\hat{F}_n(T_i)$ for some i , and hence, $\hat{F}_n(t) \leq x$ if and only if the argmin appearing in the display is larger than the observation time T_i that is just left of t . Define $\kappa_n(t) := \sup\{1 \leq k \leq n, T_k \leq t\}$. Then

$$\hat{F}_n(t) \leq x \quad \text{if and only if} \quad \underset{s}{\operatorname{argmin}} \{V_n(s) - xG_n(s)\} \geq T_{(\kappa_n(t))}.$$

Let $\{\ell_n, n \geq 1\}$ be a sequence of positive numbers satisfying that

$$1 \leq \ell_n \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\ell_n^{10} \log n}{n^{1/3}} = 0.$$

Let χ_n be nonnegative such that $\ell_n^{4.5} \chi_n \rightarrow 0$. Note that ℓ_n will be taken as b_n and 1 in the proofs of Theorems 3.1 and 3.2, respectively.

Then, for any $x \in \mathbb{R}$,

$$P\left(\frac{n^{1/3}}{\ell_n}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\right) = P\left(\operatorname{argmin}_{t \in [(t_0 + \chi_n)n^{1/3}, \infty)} Y_n^x(t_0 + \chi_n + n^{-1/3}t) < n^{1/3}(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n))\right),$$

where $\bar{\kappa}_n = \kappa_n(t_0 + \chi_n)$ and $Y_n^x(s) = V_n(s) - (F_0(t_0 + \chi_n) + n^{-1/3}\ell_n x)G_n(s)$.

Note that the decomposition of $Y_n^x(t_0 + \chi_n + n^{-1/3}t)$ on page 299 of [van der Vaart and Wellner \(1996\)](#) is not suitable for the application of strong approximation. In order to apply the strong approximation, we write $Y_n^x(t_0 + \chi_n + n^{-1/3}t)$ as

$$Z_n^x(t) = U_n(t) + r_n(t) + x\ell_n\gamma_n(t) + R_n(t) + x\ell_n\Gamma_n(t)$$

with

$$\left\{ \begin{aligned} U_n(t) &:= \frac{1}{n^{1/3}} \sum_{i=1}^n (I_{\{X_i \leq t_0 + \chi_n\}} - F_0(t_0 + \chi_n))(I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}), \\ r_n(t) &:= n^{2/3} E((I_{\{X_1 \leq T_1\}} - I_{\{X_1 \leq t_0 + \chi_n\}})(I_{\{T_1 \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_1 \leq t_0 + \chi_n\}})), \\ \gamma_n(t) &:= n^{1/3}(G(t_0 + \chi_n + n^{-1/3}t) - G(t_0 + \chi_n)), \\ R_n(t) &:= \frac{1}{n^{1/3}} \sum_{i=1}^n (I_{\{X_i \leq T_i\}} - I_{\{X_i \leq t_0 + \chi_n\}})(I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}) - r_n(t), \\ \Gamma_n(t) &:= \frac{1}{n^{2/3}} \sum_{i=1}^n (I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}) - \gamma_n(t). \end{aligned} \right.$$

This decomposition plays a major role in the proof of our main result. Therefore,

$$P\left(\frac{n^{1/3}}{\ell_n}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\right) = P\left(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) < n^{1/3}(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n))\right).$$

Similarly, for any $x \in \mathbb{R}$,

$$P\left(\frac{n^{1/3}}{\ell_n}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq -x\right) = P\left(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq n^{1/3}(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n))\right).$$

LEMMA 5.1. For any $\delta > 0$,

$$(5.1) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta) = -\infty.$$

PROOF. For any $\delta > 0$,

$$\begin{aligned} &P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta) \\ &= P(T_i \notin (t_0 + \chi_n - n^{-1/3}\ell_n^{-2}\delta, t_0 + \chi_n) \text{ for all } i = 1, \dots, n) \\ &= \left(1 - \int_{t_0 + \chi_n - n^{-1/3}\ell_n^{-2}\delta}^{t_0 + \chi_n} g(t) dt\right)^n. \end{aligned}$$

Thus,

$$\limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta) = -\infty. \quad \square$$

PROPOSITION 5.1. For each $\rho > 0$, we have

$$(5.2) \quad \begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{x \in [0, \rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ &\leq \limsup_{n \rightarrow \infty} \sup_{x \in [0, \rho]} \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}, \end{aligned}$$

$$(5.3) \quad \begin{aligned} &\liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ &\geq \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq -\ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}, \end{aligned}$$

$$(5.4) \quad \begin{aligned} &\liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ &\geq \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \end{aligned}$$

and

$$(5.5) \quad \begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{x \in [0, \rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ &\leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{x \in [0, \rho]} \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq -\ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}. \end{aligned}$$

PROOF. For each $\rho > 0$ and $0 \leq x \leq \rho$, we have

$$\begin{aligned} & \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq -\ell_n^{-2}\delta) - P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ & \leq \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ & \leq \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \end{aligned}$$

and

$$\begin{aligned} & \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ & \leq \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)} \\ & \leq \frac{P(\operatorname{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq -\ell_n^{-2}\delta) + P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}. \end{aligned}$$

Therefore, by Lemma 5.1, we obtain the conclusions of the proposition. \square

5.2. *A strong approximation theorem associated with the NPMLE.* In this subsection, we present an exponential approximation for a version of $U_n(t)$ by a two-sided Wiener process. A similar approximation is also derived for the process $Z_n^x(t)$. This estimate will play a pivotal role in our proof of the moderate deviation result.

LEMMA 5.2. *Let $Y_i = I_{\{X_i \leq t_0 + \chi_n\}} - F_0(t_0 + \chi_n)$. There exist a version $(\{Y_i^*, i \geq 1\}, \{T_i^*, i \geq 1\})$ of $(\{Y_i, i \geq 1\}, \{T_i, i \geq 1\})$, a two-sided Wiener process $W_n(t)$ with $W_n(0) = 0, n \geq 1$ and two positive absolute constants C, K such that for any $\delta > 0$,*

$$\begin{aligned} (5.6) \quad & P\left(\sup_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} |U_n^*(t) - n^{1/6}\sigma_U W_n(G_n^*(t))| \geq \delta\right) \\ & \leq \frac{n^{KC}}{K} \exp\{-K\delta n^{1/3}\}, \end{aligned}$$

where

$$U_n^*(t) := \frac{1}{n^{1/3}} \sum_{i=1}^n Y_i^* I_{\{t_0 + \chi_n < T_i^* \leq t_0 + \chi_n + n^{-1/3}t\}},$$

$$G_n^*(t) = G(t_0 + \chi_n + n^{-1/3}t) - G(t_0 + \chi_n)$$

and

$$\sigma_U^2 = E((I_{\{X_i \leq t_0 + \chi_n\}} - F_0(t_0 + \chi_n))^2) = F_0(t_0 + \chi_n)(1 - F_0(t_0 + \chi_n)).$$

PROOF. Since for any two-sided Wiener process W_n with $W_n(0) = 0$,

$$\{W_n(G(t_0 + \chi_n + n^{-1/3}t)) - W_n(G(t_0 + \chi_n)), t \in \mathbb{R}\} \stackrel{d}{=} \{W_n(G_n^*(t)), t \in \mathbb{R}\},$$

applying Theorem 5 in Maumy (2004) or Theorem B.3 in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)] to Y_i and $Z_i = G(T_i)$, there exist a version $(\{Y_i^*, i \geq 1\}, \{T_i^*, i \geq 1\})$ of $(\{Y_i, i \geq 1\}, \{T_i, i \geq 1\})$, a Wiener process $W_n(t), n \geq 1$ and two positive absolute constants C, K such that for all $\delta > 0$, (5.6) holds. \square

We can write

$$\begin{cases} R_n(t) = \sum_{i=1}^n \frac{1}{n} (g_n(X_i, T_i, t) - r_n(t)) =: \sum_{i=1}^n Z_{ni}(t), \\ \Gamma_n(t) = \sum_{i=1}^n \frac{1}{n} (h_n(T_i, t) - \gamma_n(t)) =: \sum_{i=1}^n \xi_{ni}(t), \end{cases}$$

where

$$\begin{cases} g_n(y, u, t) = n^{2/3} (I_{\{y \leq u\}} - I_{\{y \leq t_0 + \chi_n\}}) (I_{\{u \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{u \leq t_0 + \chi_n\}}), \\ h_n(u, t) = n^{1/3} (I_{\{u \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{u \leq t_0 + \chi_n\}}). \end{cases}$$

When $t \in [-\epsilon_0 n^{1/3}/2, \epsilon_0 n^{1/3}/2]$,

$$\begin{cases} r_n(t) = \int_0^t \int_0^u f_0(t_0 + \chi_n + n^{-1/3}s) ds g(t_0 + \chi_n + n^{-1/3}u) du, \\ \gamma_n(t) = \int_0^t g(t_0 + \chi_n + n^{-1/3}u) du. \end{cases}$$

The following lemma is straightforward.

LEMMA 5.3. (i) If $\ell_n = 1$ for all $n \geq 1$, then, under Assumption (A1),

$$(5.7) \quad \begin{cases} \lim_{n \rightarrow \infty} \sup_{|t| \leq N} \left| r_n(t) - \frac{f_0(t_0)g(t_0)t^2}{2} \right| = 0, \\ \lim_{n \rightarrow \infty} \sup_{|t| \leq N} |\gamma_n(t) - g(t_0)t| = 0. \end{cases}$$

(ii) If Assumptions (A1) and (A2) hold, then there exists a positive constant L such that, for any $N \geq 1$,

$$\begin{cases} \sup_{|t| \leq N \ell_n} \left| r_n(t) - \frac{f_0(t_0)g(t_0)t^2}{2} \right| \leq L(n^{-1/3}N^3 \ell_n^3 + n^{-2/3}N^4 \ell_n^4 + N^2 \ell_n^2 \chi_n), \\ \sup_{|t| \leq N \ell_n} |\gamma_n(t) - g(t_0)t| \leq L(N^2 \ell_n^2 n^{-1/3} + N \ell_n \chi_n). \end{cases}$$

In particular, (5.7) holds.

Set

$$\begin{cases} \sigma_{R,n,\lambda}^2 = E\left(\sup_{t \in [-1,1]} \sum_{i=1}^n |Z_{ni}(\lambda t)|^2\right), & \mu_{R,n,\lambda} = E\left(\sup_{t \in [-1,1]} |R_n(\lambda t)|\right), \\ \sigma_{\Gamma,n,\lambda}^2 = E\left(\sup_{t \in [-1,1]} \sum_{i=1}^n |\xi_{ni}(\lambda t)|^2\right), & \mu_{\Gamma,n,\lambda} = E\left(\sup_{t \in [-1,1]} |\Gamma_n(\lambda t)|\right). \end{cases}$$

Then by Proposition 2.1 in [Giné and Guillou \(2001\)](#) or Lemma B.1 in Appendix B of the Supplementary Material [[Gao, Xiong and Zhao \(2017\)](#)], there exists a positive constant L such that

$$(5.8) \quad \begin{cases} \sigma_{R,n,N\ell_n}^2 \leq LN^2 \ell_n^2 n^{-1/3}, & \mu_{R,n,N\ell_n} \leq LN \ell_n n^{-1/6} (\log n)^{1/2}, \\ \sigma_{\Gamma,n,N\ell_n}^2 \leq L \ell_n N n^{-2/3}, & \mu_{\Gamma,n,N\ell_n} \leq LN^{1/2} \ell_n^{1/2} n^{-1/3} (\log n)^{1/2}. \end{cases}$$

Now, define

$$\begin{aligned} \tilde{Z}_n^x(t) &= U_n(t) + \frac{f_0(t_0)g(t_0)t^2}{2} + x \ell_n g(t_0)t, \\ \tilde{Z}_n^{x,*}(t) &= U_n^*(t) + \frac{f_0(t_0)g(t_0)t^2}{2} + x \ell_n g(t_0)t \end{aligned}$$

and

$$\hat{Z}_n^x(t) = n^{1/6} \sigma_U W_n(G_n^*(t)) + \frac{f_0(t_0)g(t_0)t^2}{2} + x \ell_n g(t_0)t.$$

Then the processes \tilde{Z}_n^x and $\tilde{Z}_n^{x,*}$ have the same distribution.

PROPOSITION 5.2. *Suppose Assumption (A1) holds. If $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) holds, then for any $\delta > 0$ and $\rho > 0$,*

$$(5.9) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(\ell_n^{5/2} \sup_{|t| \leq N\ell_n} |Z_n^x(t) - \tilde{Z}_n^x(t)| \geq \delta\right) = -\infty$$

and

$$(5.10) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(\ell_n^{5/2} \sup_{|t| \leq N\ell_n} |\tilde{Z}_n^{x,*}(t) - \hat{Z}_n^x(t)| \geq \delta\right) = -\infty.$$

PROOF. First, by (5.6),

$$\limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log P\left(\ell_n^{5/2} \sup_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} |U_n^*(t) - n^{1/6} \sigma_U W_n(G_n^*(t))| \geq \delta\right) = -\infty.$$

Thus, (5.10) is valid.

Applying Talagrand’s inequality (see Theorem 1.4 in [Talagrand \(1996\)](#), or Theorem B.1 in Appendix B of the Supplementary Material [[Gao, Xiong and Zhao](#)]

(2017)] to $\{n^{1/3}R_n(N\ell_n t); t \in [-1, 1]\}$ and $\{n^{1/3}\ell_n\Gamma_n(N\ell_n t); t \in [-1, 1]\}$, there exists a constant $L \in (0, \infty)$ such that for all $n \geq 1$ and any $\delta > 0$,

$$P\left(\ell_n^{5/2} \left| \sup_{|t| \leq N\ell_n} |R_n(t)| - E\left(\sup_{|t| \leq N\ell_n} |R_n(t)|\right) \right| \geq \delta\right) \leq L \exp\left\{-\frac{\delta n^{1/3}/\ell_n^{5/2}}{L} \log\left(1 + \frac{L\delta/\ell_n^{5/2}}{n^{1/3}\sigma_{R,n,N\ell_n}^2}\right)\right\}$$

and

$$P\left(\ell_n^{7/2} \left| \sup_{|t| \leq N\ell_n} |\Gamma_n(t)| - E\left(\sup_{|t| \leq N\ell_n} |\Gamma_n(t)|\right) \right| \geq \delta\right) = P\left(\ell_n \left| \sup_{t \in [-1,1]} |n^{1/3}\Gamma_n(N\ell_n t)| - E\left(\sup_{t \in [-1,1]} |n^{1/3}\Gamma_n(N\ell_n t)|\right) \right| \geq \delta n^{1/3}/\ell_n^{5/2}\right) \leq L \exp\left\{-\frac{\delta n^{1/3}/\ell_n^{5/2}}{L} \log\left(1 + \frac{L\delta/\ell_n^{5/2}}{n^{1/3}\ell_n^2\sigma_{\Gamma,n,N\ell_n}^2}\right)\right\}.$$

Therefore, by (5.8), if $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) holds, then

$$\limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log P\left(\ell_n^{5/2} \sup_{|t| \leq N\ell_n} |R_n(t)| \geq \delta\right) = -\infty$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log P\left(\ell_n^{7/2} \sup_{|t| \leq N\ell_n} |\Gamma_n(t)| \geq \delta\right) = -\infty.$$

These yield (5.9). \square

5.3. Exponential equivalence. For each $x \in \mathbb{R}$, let

$$\left\{ \begin{array}{l} \tau_n^x = \operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} Z_n^x(t), \\ \hat{\tau}_n^x = \operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} \hat{Z}_n^x(t), \\ \tilde{\tau}_n^x = \operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} \tilde{Z}_n^x(t), \\ \tilde{\tau}_n^{x,*} = \operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} \tilde{Z}_n^{x,*}(t). \end{array} \right.$$

Then the processes $\tilde{\tau}_n^x$ and $\tilde{\tau}_n^{x,*}$ have the same distribution.

The main result in this subsection is the following exponential equivalence theorem.

PROPOSITION 5.3. *Suppose Assumption (A1) holds. If $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) holds with $\ell_n \rightarrow \infty$, then for any $\delta > 0$ and $\rho > 0$,*

$$(5.11) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(|\tilde{\tau}_n^{x,*} - \hat{\tau}_n^x| \geq \ell_n^{-2} \delta) = -\infty$$

and

$$(5.12) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(|\tau_n^x - \tilde{\tau}_n^x| \geq \ell_n^{-2} \delta) = -\infty.$$

Proposition 5.3 will be proved through the following two lemmas. Their proofs are based on the Talagrand deviation inequality and small ball estimates, and will be given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)].

LEMMA 5.4. *For each $\rho > 0$,*

$$(5.13) \quad \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(\max\{|\tilde{\tau}_n^x|, |\tau_n^x|\} \geq \ell_n N) = -\infty$$

and

$$(5.14) \quad \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(\max\{|\hat{\tau}_n^x|, |\tilde{\tau}_n^{x,*}|\} \geq \ell_n N) = -\infty.$$

LEMMA 5.5. *For each $\rho > 0$, $\delta > 0$ and $N \geq 1$,*

$$(5.15) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(|\hat{\tau}_n^x| \leq N \ell_n, \hat{Z}_n^x(\hat{\tau}_n^x) - \inf_{\substack{|s| \leq N \ell_n, \\ |s - \hat{\tau}_n^x| \geq \delta \ell_n^{-2}}} \hat{Z}_n^x(s) \geq \frac{-\epsilon}{\ell_n^{5/2}}\right) = -\infty$$

and

$$(5.16) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(|\tilde{\tau}_n^x| \leq N \ell_n, \tilde{Z}_n^x(\tilde{\tau}_n^x) - \inf_{\substack{|s| \leq N \ell_n, \\ |s - \tilde{\tau}_n^x| \geq \delta \ell_n^{-2}}} \tilde{Z}_n^x(s) \geq \frac{-\epsilon}{\ell_n^{5/2}}\right) = -\infty.$$

Finally, we complete the proof of Proposition 5.3.

PROOF OF PROPOSITION 5.3. We only show (5.11). For any $\delta > 0$, $N \geq 1$, and $\epsilon \in (0, 1/2)$, set

$$\Delta_n = \hat{Z}_n^x(\hat{\tau}_n^x) - \inf_{|s| \leq N \ell_n, |s - \hat{\tau}_n^x| \geq \delta \ell_n^{-2}} \hat{Z}_n^x(s).$$

Note that on $\{|\tilde{\tau}_n^{x,*}| \leq \ell_n N\}$, for any $|s| \leq \ell_n N$,

$$\begin{aligned} \hat{Z}_n^x(s) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) &= \hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) + \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) \\ &\geq \hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s) + \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) \\ &\geq -2 \sup_{|s| \leq \ell_n N} |\hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s)|. \end{aligned}$$

Then

$$\begin{aligned} P(|\tilde{\tau}_n^{x,*} - \hat{\tau}_n^x| \geq \ell_n^{-2} \delta) &\leq P(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \geq N \ell_n) + P(|\hat{\tau}_n^x| \leq N \ell_n, \Delta_n \geq -\epsilon/\ell_n^{5/2}) \\ &\quad + P(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \leq N \ell_n, \ell_n^{5/2} \Delta_n < -\epsilon, |\tilde{\tau}_n^{x,*} - \hat{\tau}_n^x| \geq \ell_n^{-2} \delta) \\ &\leq P(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \geq N \ell_n) + P(|\hat{\tau}_n^x| \leq N \ell_n, \Delta_n \geq -\epsilon/\ell_n^{5/2}) \\ &\quad + P(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \leq N \ell_n, \ell_n^{5/2}(\hat{Z}_n^x(\hat{\tau}_n^x) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*})) < -\epsilon) \\ &\leq P(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \geq N \ell_n) + P(|\hat{\tau}_n^x| \leq N \ell_n, \Delta_n \geq -\epsilon/\ell_n^{5/2}) \\ &\quad + P\left(2\ell_n^{5/2} \sup_{|s| \leq \ell_n N} |\hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s)| \geq \epsilon\right). \end{aligned}$$

Thus, by Lemmas 5.4 and 5.5 and Proposition 5.2, (5.11) holds. \square

5.4. *Proof of Theorem 3.1.* In this subsection we present the proof of Theorem 3.1. For $c > 0$, define $\tau(c) = \operatorname{argmin}_s \{W(s) + cs^2\}$ and $\tau := \tau(1)$. Noting that $c^{1/3}(W(s) + cs^2)$ and $W(c^{2/3}s) + (c^{2/3}s)^2$ have identical distributions, it then follows from $\tau(c) = \operatorname{argmin}_s \{c^{1/3}(W(s) + cs^2)\}$ that $\tau(c) \stackrel{d}{=} c^{-2/3}\tau$. Define

$$(5.17) \quad \begin{cases} \hat{Y}_n^x(s) = W_n(s) + \frac{f_0(t_0)g(t_0)}{2\sigma_U} p_n^2(n^{-1/3}s) + \frac{g(t_0)}{\sigma_U} x \ell_n p_n(n^{-1/3}s), \\ \quad n^{-1/3}s \in (-G(t_0 + \chi_n), 1 - G(t_0 + \chi_n)), \\ \hat{Y}_n^x(s) = W_n(s) + \frac{f_0(t_0)}{2\sigma_U g(t_0)} s^2 + \frac{1}{\sigma_U} x \ell_n s, \quad s \in \mathbb{R}, \end{cases}$$

where $p_n(s)$ is the inverse function of $q_n(u) := G(t_0 + \chi_n + n^{-1/3}u) - G(t_0 + \chi_n)$. Set $\hat{T}_n^x = \operatorname{argmin}_s \hat{Y}_n^x(s)$ and $T_n^x = \operatorname{argmin}_s Y_n^x(s)$. Then $\hat{T}_n^x = n^{1/3}q_n(\hat{\tau}_n^x)$, and

$$(5.18) \quad \lim_{n \rightarrow \infty} \ell_n^3 \sup_{|s| \leq N \ell_n} |n^{1/3}q_n(s) - g(t_0)s| = 0.$$

When $s \in [-\epsilon_0 n^{1/3}, \epsilon_0 n^{1/3}]$, we can write

$$p_n(n^{-1/3}s) = \int_0^{n^{-1/3}s} \frac{n^{1/3} du}{g(t_0 + \chi_n + n^{-1/3}u)} = \int_0^{n^{-2/3}s} \frac{n^{2/3} dv}{g(t_0 + \chi_n + v)}.$$

Thus,

$$\begin{aligned} & \sup_{|s| \leq N\ell_n} \left| p_n(n^{-1/3}s) - \frac{s}{g(t_0 + \chi_n)} \right| \\ & \leq \int_0^{n^{-2/3}N\ell_n} \frac{n^{2/3} |g(t_0 + \chi_n + v) - g(t_0 + \chi_n)| dv}{g(t_0 + \chi_n + v)g(t_0 + \chi_n)}, \end{aligned}$$

which implies that when $\ell_n = 1$, $\lim_{n \rightarrow \infty} \sup_{|s| \leq N\ell_n} |p_n(n^{-1/3}s) - \frac{s}{g(t_0)}| = 0$, and that when Assumptions (A1) and (A2) hold,

$$(5.19) \quad \lim_{n \rightarrow \infty} \ell_n^{3.5} \sup_{|s| \leq N\ell_n} \left| p_n(n^{-1/3}s) - \frac{s}{g(t_0)} \right| = 0.$$

Therefore, if Assumption (A1) holds and $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) also holds, then for each $\rho > 0$ and any $\delta > 0$,

$$(5.20) \quad \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P \left(\ell_n^{5/2} \sup_{|s| \leq \ell_n N} |\hat{Y}_n^x(s) - Y_n^x(s)| \geq \delta \right) = -\infty.$$

By the same arguments as used in the proofs of Lemma 5.4 and Proposition 5.3, the following result holds.

LEMMA 5.6. (i) For each $\rho > 0$,

$$(5.21) \quad \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(\max\{|\hat{T}_n^x|, |T_n^x|\} \geq \ell_n N) = -\infty.$$

(ii) If Assumption (A1) holds and $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) also holds, then for each $\rho > 0$ and any $\delta > 0$,

$$(5.22) \quad \limsup_{n \rightarrow \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P(|\hat{T}_n^x - T_n^x| \geq \delta \ell_n^{-2}) = -\infty.$$

Write

$$\begin{aligned} T_n^x = \operatorname{argmin}_s & \left\{ W_n \left(s - \frac{xg(t_0)\ell_n}{f_0(t_0)} \right) - W_n \left(-\frac{xg(t_0)\ell_n}{f_0(t_0)} \right) + \frac{f_0(t_0)}{2\sigma_U g(t_0)} s^2 \right\} \\ & - \frac{xg(t_0)\ell_n}{f_0(t_0)}. \end{aligned}$$

Thus, T_n^x and $-\frac{xg(t_0)\ell_n}{f_0(t_0)} + \left(\frac{f_0^2(t_0)}{4F_0(t_0)(1-F_0(t_0))g^2(t_0)} \right)^{-1/3} \tau$ are identical in distribution.

Now we are ready to complete the proof of the theorem.

PROOF OF THEOREM 3.1. We only show (3.1). Take $\ell_n = b_n$. Then for each $\rho > 0$, by (5.18), Lemma 5.6 and (2.1), we have

$$\begin{aligned} & \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(\hat{\tau}_n^{-x} < -b_n^{-2}\delta) \\ &= \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} \\ & \quad \times P(\hat{T}_n^{-x} < n^{1/3}q_n(-b_n^{-2}\delta)) \\ &= \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(T_n^{-x} < -b_n^{-2}\delta) \\ &= \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} \\ & \quad \times P(\tau < -b_n^{-2}\delta - b_n(\sigma_F^2(t_0))^{-1/3}x) \\ &= 1 \end{aligned}$$

and

$$\limsup_{n \rightarrow \infty} \sup_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(\hat{\tau}_n^{-x} \leq 0) = 1.$$

Thus, by Propositions 5.3 and 5.1, this yields

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \rho b_n]} \left| \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \geq x\right) - 1 \right| = 0.$$

In particular, we obtain that for any $x > 0$,

$$(5.23) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n^3} \log P\left(\pm \frac{1}{b_n} \frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} > x\right) = -\frac{2}{3}x^3.$$

Equation (5.23) is equivalent to the statement that the sequence $\{\frac{n^{1/3}}{b_n}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)), n \geq 1\}$ satisfies the MDP in \mathbb{R} with speed b_n^3 and rate function $I(x)$ [cf. Dembo and Zeitouni (1998), page 7]. \square

5.5. Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to that of Theorem 3.1. Take $\ell_n \equiv 1$. Then for $x \in \mathbb{R}$, by Propositions 5.1 and 5.3, Lemma 5.6

and (5.18), we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq x) \\ & \leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P(\tau_n^{-x} > -\delta) \\ & = \lim_{\delta \downarrow 0} P\left(\frac{xg(t_0)}{f_0(t_0)} + \left(\frac{f_0^2(t_0)}{4F_0(t_0)(1-F_0(t_0))g^2(t_0)}\right)^{-1/3} \tau > -\delta\right) \\ & = P\left(\left(\frac{4f_0(t_0)F_0(t_0)(1-F_0(t_0))}{g(t_0)}\right)^{1/3} \tau \leq x\right) \end{aligned}$$

and

$$\begin{aligned} & \liminf_{n \rightarrow \infty} P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq x) \\ & \geq \liminf_{n \rightarrow \infty} P(\tau_n^{-x} \geq 0) \\ & = P\left(\left(\frac{4f_0(t_0)F_0(t_0)(1-F_0(t_0))}{g(t_0)}\right)^{1/3} \tau \leq x\right). \end{aligned}$$

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SUPPLEMENTARY MATERIAL

Supplement to “Moderate deviations and nonparametric inference for monotone functions” (DOI: [10.1214/17-AOS1583SUPP](https://doi.org/10.1214/17-AOS1583SUPP); .pdf). The supplement [Gao, Xiong and Zhao (2017)] contains all remaining technical proofs omitted from the main text due to space constraints, in which we prove Lemmas 5.4 and 5.5, Proposition 2.2, Theorem 2.3 and its Corollaries 2.1–2.3.

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