

TESTING WHETHER JUMPS HAVE FINITE OR INFINITE ACTIVITY

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We propose statistical tests to discriminate between the finite and infinite activity of jumps in a semimartingale discretely observed at high frequency. The two statistics allow for a symmetric treatment of the problem: we can either take the null hypothesis to be finite activity, or infinite activity. When implemented on high-frequency stock returns, both tests point toward the presence of infinite-activity jumps in the data.

1. Introduction. Traditionally, models with jumps in finance have relied on Poisson processes, as in Merton (1976), Ball and Torous (1983) and Bates (1991). These jump-diffusion models allow for a finite number of jumps in a finite time interval, with the idea that the Brownian-driven diffusive part of the model captures normal asset price variations while the Poisson-driven jump part of the model captures large market moves in response to unexpected information. More recently, financial models have been proposed that allow for infinitely many jumps in finite time intervals, using a variety of specifications, such as the variance gamma model of Madan and Seneta (1990) and Madan, Carr and Chang (1998), the hyperbolic model of Eberlein and Keller (1995), the CGMY model of Carr et al. (2002) and the finite moment log stable process of Carr and Wu (2003a). These models can capture both small and frequent jumps, as well as large and infrequent ones.

In this paper, we develop statistical procedures to discriminate empirically between the two situations of finite and infinite number of jumps, while allowing in both cases for the presence of a continuous component in the model. While many theoretical models make use of one or the other type of jumps, no statistical test has been proposed so far that can identify the most likely type in a given data set, as existing tests have focused on the issue of testing for the presence of jumps but not distinguishing between different types of jumps; see Aït-Sahalia (2002), Carr and Wu (2003b), Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), Andersen, Bollerslev and Diebold (2007), Jiang and Oomen (2008), Lee and Mykland (2008), Aït-Sahalia and Jacod (2009b) and Lee and Hannig (2010), among others.

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The setup we consider is one where a univariate process X is observed on a fixed time interval $[0, T]$, at discretely and regularly spaced times $i \Delta_n$. In a typical high-frequency financial application, X will be the log of an asset price, the length of observation T ranges from, say, one day to one year, while the sampling interval Δ_n is small, typically measured in seconds. Assuming that the observed path has jumps, we want to test whether there are a finite number of jumps or not on that path, two properties commonly referred to as “finite activity” or “infinite activity” for the jump component of X .

Our aim is to provide asymptotic testing procedures, as the time lag Δ_n between successive observations goes to 0, allowing to decide which of these two hypotheses is more likely; that is, we want to decide in which of the two complementary subsets

$$(1) \quad \begin{aligned} \Omega_T^f &= \{\omega : t \mapsto X_t(\omega) \text{ has finitely many jumps in } [0, T]\}, \\ \Omega_T^i &= \{\omega : t \mapsto X_t(\omega) \text{ has infinitely many jumps in } [0, T]\} \end{aligned}$$

of the sample space Ω we are in.

More specifically, we want to find tests with a prescribed asymptotic significance level, and with asymptotic power going to 1, to test the null hypothesis that ω is in Ω_T^f , and also the symmetric null hypothesis that the observed ω is in Ω_T^i . We will need some assumptions on the process X , basically that it is an Itô semimartingale. However, we wish to keep the solution as nonparametric as possible, and in particular we do not want to specify the structure of the volatility or of the jumps.

The simple idea behind the two test statistics we propose is common with our earlier work on testing whether jumps are present or not, or whether a continuous component is present. We compute certain power variations of the increments, suitably truncated and/or sampled at different frequencies. Related methodologies are being utilized by other authors. For example, [Todorov and Tauchen \(2010\)](#) use the test statistics of [Aït-Sahalia and Jacod \(2009b\)](#), study its logarithm for different values of the power argument and contrast the behavior of the plot above two and below two in order to identify the presence of a Brownian component. [Cont and Mancini \(2009\)](#) use threshold or truncation-based estimators of the continuous component of the quadratic variation, originally proposed in [Mancini \(2001\)](#), in order to test for the presence of a continuous component in the price process. The resulting test is applicable when the jump component of the process has finite variation, and a test for whether the jump component indeed has finite variation is also proposed. In terms of the Blumenthal–Gettoor index β , this corresponds to testing whether $\beta < 1$.

We aim here to construct test statistics which are simple to compute and have the desirable property of being model-free. In particular, no feature of the dynamics of the underlying asset price, which can be quite complex with potentially jumps of various activity levels, stochastic volatility, jumps in volatility, etc., need to be

estimated in order to compute either the statistic or its distribution under the null hypothesis. In fact, implementing the two tests we propose requires nothing more than the computation of various truncated power truncations.

We consider two testing problems, one where the null hypothesis is finite jump activity and its “dual” where the null hypothesis is infinite jump activity. Under the null hypothesis of finite-activity jumps, the test statistic we propose is similar to the simpler statistic S_n of Aït-Sahalia and Jacod (2009b) which was employed to test for the presence of jumps, with an additional truncation step. An appropriately selected truncation mechanism eliminates finite-activity jumps, so that the probability limit of the statistic S_n post-truncation is the same in this paper as that of the simpler statistic in the earlier work, under a purely continuous model. While the result is indeed in that case that “the answer is the same,” this is not completely obvious a priori and still needs to be established mathematically. And the commonality is limited to probability limits: the two statistics have different asymptotic distributions.

Under the reverse scenario, where the null hypothesis is that jumps are infinitely active, then the statistic S'_n we propose for this purpose is radically new and so is its asymptotic behavior. That second statistic has no relationship to previous work.

As we will see below, when implemented on high-frequency stock returns, both tests point toward the presence of infinitely active jumps in the data. That is, in the test where Ω_T^f is the null hypothesis, we reject the null; in the test where Ω_T^i is the null hypothesis, we fail to reject the null. This is in line with the empirical results of a companion paper, Aït-Sahalia and Jacod (2009a), which contains an extension to Itô semimartingales of the classical Blumenthal–Gettoor index β for Lévy processes and estimators for β ; see also Belomestny (2010) for different estimators. This parameter β takes values between 0 and 2 and plays the role of a “degree of jump activity” for infinitely active jump processes. Then if the estimator of β is found to be “high” in its range $[0, 2]$, with a confidence interval excluding 0, as it is the case in the empirical findings of Aït-Sahalia and Jacod (2009a), it is a strong evidence against finite activity. However, finite activity implies $\beta = 0$, but the converse fails, so using estimators of β can at the best allow for tests when the null is “infinite activity,” and even for this it does not allow for determining the asymptotic level of the test. Thus in fact the present paper and the other one are complementary, both aiming to have a picture as complete as possible of a continuous-time process which is discretely observed at increasing frequencies. Finally we can also mention that here the assumptions are significantly weaker than in Aït-Sahalia and Jacod (2009a), in the sense that the test proposed here is nonparametric, where the estimator of β proposed there is parametric.

The paper is organized as follows. Section 2 describes our model and the statistical problem. Our testing procedure is described in Section 3, and Sections 4 and 5 are devoted to a simulation study of the tests and an empirical implementation on high-frequency stock returns. Technical results are gathered in the supplemental article [Aït-Sahalia and Jacod (2011)].

2. The model. The underlying process X which we observe at discrete times is a one-dimensional process which we specify below. Observe that taking a one-dimensional process is not a restriction in our context since, if it were multidimensional, infinitely many jumps on $[0, T]$ means that at least one of its components has infinitely many jumps, so the tests below can be performed separately on each of the components. In all the paper the terminal time T is fixed. However, it is convenient, and not a restriction, to assume that the process X is defined over the whole half-line.

Our structural assumption is that X is an Itô semimartingale on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which means that its characteristics (B, C, ν) are absolutely continuous with respect to Lebesgue measure. B is the drift, C is the quadratic variation of the continuous martingale part, and ν is the compensator of the jump measure μ of X . In other words, we have

$$(2) \quad \begin{aligned} B_t(\omega) &= \int_0^t b_s(\omega) ds, \\ C_t(\omega) &= \int_0^t \sigma_s(\omega)^2 ds, \quad \nu(\omega, dt, dx) = dt F_t(\omega, dx). \end{aligned}$$

Here b and σ are optional process, and $F = F_t(\omega, dx)$ is a transition measure from $\Omega \times \mathbb{R}_+$ endowed with the predictable σ -field into $\mathbb{R} \setminus \{0\}$. One may then write X as

$$(3) \quad \begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \\ &\quad + \underbrace{\int_0^t \int x 1_{\{|x| \leq 1\}} (\mu - \nu)(ds, dx)}_{\text{small jumps}} + \underbrace{\int_0^t \int x 1_{\{|x| > 1\}} \mu(ds, dx)}_{\text{large jumps}}, \end{aligned}$$

where W is a standard Wiener process. It is also possible to write the last two terms above as integrals with respect to a Poisson measure and its compensator, but we will not need this here.

The cutoff level 1 used to distinguish small and large jumps is arbitrary; any fixed jump size $\varepsilon > 0$ will do. In terms of the definition (3), changing the cutoff level amounts to an adjustment to the drift of the process. Ultimately, the question we are asking about the finite or infinite degree of activity of jumps is a question about the behavior of the compensator ν near 0. There are always a finite number of big jumps. The question is whether there are a finite or infinite number of small jumps. This is controlled by the behavior of ν near 0.

2.1. The basic assumptions. The assumptions we make depend on the null hypothesis we want to test. We start with a very mild (local) boundedness assumption. Recall that a process a_t is pre-locally bounded if $|a_t| \leq n$ for $t < T_n$, for a sequence (T_n) of stopping times increasing to $+\infty$.

ASSUMPTION 1. *The processes b_t , σ_t and $F_t(\mathbb{R})1_{\{F_t(\mathbb{R}) < \infty\}}$ and $\int(x^2 \wedge 1)F_t(dx)$ are pre-locally bounded.*

In some cases we will need something more about the drift b and the volatility σ .

ASSUMPTION 2. *The drift process b_t is càdlàg, and the volatility process σ_t is an Itô semimartingale satisfying Assumption 1.*

Under this assumption we can write σ_t as (3), with a Wiener process W' which may be correlated with W . Another (equivalent) way of writing this is

$$(4) \quad \sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + N_t + \sum_{s \leq t} \Delta \sigma_s 1_{\{|\Delta \sigma_s| > 1\}},$$

where N is a local martingale which is orthogonal to the Brownian motion W and has jumps bounded by 1. Saying that σ satisfies Assumption 1 implies that the compensator of the process $[N, N]_t + \sum_{s \leq t} 1_{\{|\Delta \sigma_s| > 1\}}$ has the form $\int_0^t n_s ds$, and the processes \tilde{b}_t and n_t are pre-locally bounded, and the process $\tilde{\sigma}_t$ is càdlàg.

Next, we need conditions on the Lévy measures F_t , which are quite stronger than what is in Assumption 1. We state here a relatively restrictive assumption.

ASSUMPTION 3. *The Lévy measure $F_t = F_t(\omega, dx)$ is of the form*

$$(5) \quad F_t(dx) = \frac{\gamma'_t (\log(1/|x|))^{\delta_t}}{|x|^{1+\gamma_t}} (a_t^{(+)} 1_{\{0 < x \leq z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)} \leq x < 0\}}) dx + F'_t(dx),$$

where, for some pre-locally bounded process $L_t \geq 1$:

(i) $a_t^{(+)}$, $a_t^{(-)}$, $z_t^{(+)}$ and $z_t^{(-)}$ are nonnegative predictable processes satisfying

$$(6) \quad \frac{1}{L_t} \leq z_t^{(+)} \leq 1, \quad \frac{1}{L_t} \leq z_t^{(-)} \leq 1, \quad A_t := a_t^{(+)} + a_t^{(-)} \leq L_t,$$

(ii) γ_t , γ'_t and δ_t are predictable processes, satisfying for some constant $\delta \geq 0$

$$(7) \quad 0 \leq \gamma_t < 2 - 1/L_t, \quad |\delta_t| \leq L_t, \quad \gamma_t = 0 \Rightarrow \delta_t = \delta, \\ \gamma'_t = \begin{cases} \gamma_t, & \text{if } \gamma_t > 0, \\ 1, & \text{if } \gamma_t = 0, \end{cases}$$

(iii) $F'_t = F'_t(\omega, dx)$ is a signed measure, whose absolute value $|F'_t|$ satisfies, for some increasing continuous function $\phi : [0, 1] \mapsto [0, 1]$ with $\phi(0) = 0$ and some constant $c \in [0, 1)$:

$$(8) \quad \gamma_t = 0 \quad \text{or} \quad A_t = 0, \quad x > 0 \Rightarrow |F'_t|([-x, x]) \leq L_t \phi(x \wedge 1), \\ \gamma_t > 0 \quad \text{and} \quad A_t > 0 \Rightarrow \int (|x|^{c\gamma_t} \wedge 1) |F'_t|(dx) \leq L_t.$$

Equivalently, one could take $\gamma'_t = 1$ identically, provided in (6) $A_t \leq L_t$ is substituted with $A_t(1_{\{\gamma_t=0\}} + \frac{1}{\gamma_t} 1_{\{\gamma_t>0\}}) \leq L_t$.

Since F'_t is allowed to be a signed measure, (5) does *not* mean that $F_t(dx)$ restricted to $(0, z_t^{(+)})$, say, has the density $f_t^{(+)}(x) = a_t^{(+)} \gamma'_t (\log(1/|x|))^{\delta_t} / |x|^{1+\gamma_t}$; it simply means that the “leading part” of F_t on a small interval $(0, \varepsilon]$ has a density behaving as $f_t^{(+)}(x)$ as $x \downarrow 0$, and likewise on the negative side.

In all models with jumps of which we are aware in financial economics, such as those cited in the first paragraph of the **Introduction**, the Lévy measure has a density around 0, which behaves like $a_t^{(\pm)} (\log(1/|x|))^{\delta_t} / |x|^{1+\gamma_t}$ as $x \downarrow 0$ or $x \uparrow 0$ (in most cases with γ_t and δ_t constant). Thus all these models satisfy Assumption 3. For instance, it is satisfied if the discontinuous part of X is a stable process of index $\beta \in (0, 2)$, with $\gamma_t = \beta$ and $\delta_t = 0$ and $z_t^{(\pm)} = 1$, and $a_t^{(\pm)}$ being constants; in this case the residual measure F'_t is the restriction of the Lévy measure to the complement of $[-1, 1]$, and (8) holds for any $c \in (0, 1)$. When the discontinuous part of X is a tempered stable process the assumption is also satisfied with the same processes as above, but now the residual measure F'_t is not positive in general, although it again satisfies (8) with any $c \in (0, 1)$. Gamma and two-sided Gamma processes also satisfy this assumption, take $\gamma_t = 0$ and $\delta_t = 0$ and $z_t^{(\pm)} = 1$, and $a_t^{(\pm)}$ being constant.

This assumption also accounts for a stable or tempered stable or Gamma process with time-varying intensity, when γ_t, δ_t and $z_t^{(\pm)}$ are as above, but $a_t^{(\pm)}$ are genuine processes. It also accounts for a stable with time-varying index process, as well as for X being the sum of a stable or tempered stable process with jump activity index β plus another process whose jumps have activity strictly less than β . Furthermore, any process of the form $Y_t = Y_0 + \int_0^t w_s dX_s$ satisfies Assumption 3 as soon as X does and w_t is locally bounded and predictable. As is easily checked (see Section 1 of the supplemental article [Aït-Sahalia and Jacod (2011)] for a formal proof), under Assumption 3 the set Ω_T^i of (1) is (almost surely)

$$(9) \quad \Omega_T^i = \{\bar{A}_T > 0\} \quad \text{where } \bar{A}_t = \int_0^t A_s ds.$$

The previous assumption is designed for the test for which the null is “finite activity.” For the symmetric test, the assumption we need is stronger:

ASSUMPTION 4. *We have Assumption 3 with $\gamma_t = \beta$ [a constant in $(0, 2)$], and $\delta_t = 0$.*

The reason we need a stronger assumption under the null of infinite jump activity is that the asymptotic distribution of the test statistic under the null is now driven by the behavior of F_t near 0, whereas in the previous situation where the null has finite jump activity it is the Brownian motion that becomes the driving process for the behavior of the statistic under the null.

Assumptions 3 and 4 have the advantage of being easily interpretable and also easy to check for any concrete model. But as a matter of fact it is possible to substantially weaken them, and we do this in the next subsection. The reader who is satisfied with the degree of generality of Assumption 3 can skip the next subsection and go directly to the description of the tests in Section 3.

2.2. *Some weaker assumptions.* For a better understanding of what follows, let us first recall the notion of Blumenthal–Gettoor index (in short, BG index). There are two distinct notions. First, we have a (random) global BG index over the interval $[0, t]$ defined by

$$(10) \quad \Gamma_t = \inf \left(p > 0 : \int_0^t ds \int (|x|^p \wedge 1) F_s(dx) < \infty \right).$$

This is a nondecreasing $[0, 2]$ -valued process. It is not necessarily right-continuous, nor left-continuous, but it is always optional. Second, we have an instantaneous BG index γ_t , which is the BG index of the Lévy measure F_t , defined as the following (random) number:

$$(11) \quad \gamma_t = \inf \left(p > 0 : \int (|x|^p \wedge 1) F_t(dx) < \infty \right),$$

which necessarily belongs to $[0, 2]$. As a process, γ_t is predictable. The symmetrical tail function $\overline{F}_t(x) = F_t((-x, x)^c)$ of F_t (defined for $x > 0$) satisfies for all ω and t :

$$(12) \quad \begin{aligned} p > \gamma_t &\Rightarrow \lim_{x \rightarrow 0} x^p \overline{F}_t(x) = 0, \\ p < \gamma_t &\Rightarrow \limsup_{x \rightarrow 0} x^p \overline{F}_t(x) = \infty. \end{aligned}$$

In the latter case, $x^p \overline{F}_t(x)$ does not necessarily converge to ∞ when $x \rightarrow 0$. When $x^p \overline{F}_t(x) \rightarrow \infty$ as $x \rightarrow 0$ for all $p < \gamma_t$, or if $F_t = 0$, we say that the measure F_t is *regular*: this is the case when, for example, the function \overline{F}_t is regularly varying at 0.

The connections between these two indices are not completely straightforward; they are expressed in the next lemma, where ΔX_s denotes the jump of X at time s :

LEMMA 1. *Outside a \mathbb{P} -null set, we have for all t :*

$$(13) \quad \Gamma_t = \inf \left(p > 0 : \sum_{s \leq t} |\Delta X_s|^p < \infty \right).$$

Moreover if λ denotes the Lebesgue measure we have, outside a \mathbb{P} -null set again,

$$(14) \quad \Gamma_t(\omega) \geq \lambda - \text{ess sup}(\gamma_s(\omega) : s \in [0, t]),$$

and this inequality is an equality as soon as $\sup_{s \in [0, t], x \in (0, 1]} x^{\gamma_s + \varepsilon} \overline{F}_s(x) < \infty$ for all $\varepsilon > 0$.

Our general assumption involves two functions with the following properties [ϕ is indeed like in (8)]:

$$(15) \quad \left\{ \begin{array}{l} \phi : [0, 1] \rightarrow [0, \infty) \text{ is continuous increasing with } \phi(0) = 0, \\ \psi : (0, 1] \rightarrow [1, \infty) \text{ is continuous decreasing with } \psi(0) := \lim_{x \rightarrow 0} \psi(x) \\ \text{being either } 1 \text{ [then } \psi(x) = 1 \text{ for all } x], \text{ or } +\infty, \text{ in which case} \\ x^\varepsilon \psi(x) = 0 \text{ as } x \rightarrow 0 \text{ for all } \varepsilon > 0. \end{array} \right.$$

ASSUMPTION 5. *The global BG index Γ_t takes its values in $[0, 2)$, and there are a constant $a \in (0, 1]$, two functions ϕ and ψ as in (15) and pre-locally bounded processes $L(\varepsilon)$ and $L'(p)$, such that for all $\varepsilon > 0$ and $p \geq 2$ and $\mathbb{P} \otimes \lambda$ -almost all (ω, t) we have*

$$(16) \quad \begin{aligned} x \in (0, 1] &\Rightarrow x^{\Gamma_t + \varepsilon} \overline{F}_t(x) \leq L(\varepsilon)_t, \\ x, y \in (0, 1] &\Rightarrow \overline{F}_t(x) - \overline{F}_t(x(1+y)) \leq \begin{cases} \frac{L(\varepsilon)_t}{x^{\Gamma_t + \varepsilon}} (y + x^{a(\Gamma_t + \varepsilon)}), \\ \text{if } \Gamma_t > 0, \\ \frac{L(\varepsilon)_t}{x^{\Gamma_t + \varepsilon}} (y + x^{\Gamma_t + \varepsilon} \phi(x)), \\ \text{if } \Gamma_t = 0, \end{cases} \\ 0 < u \leq 1, \quad \Gamma_t = 0 &\Rightarrow \int_{\{|x| \leq u\}} |x|^p F_t(dx) \leq L'(p)_t u^p \psi(u). \end{aligned}$$

This assumption looks complicated, but it is just a mild local boundedness assumption, which is made even weaker by the fact that we use the global BG index Γ_t instead of the (perhaps more natural) instantaneous index γ_t .

Now we introduce the set Ω_T^{ii} which will be the alternative for our first test when the null is “finitely many jumps.” This set has a somewhat complicated description, which goes as follows, and for which we introduce the notation (for $p \geq 2$):

$$(17) \quad G(p, u)_t = \frac{1}{u^p \psi(u)} \int_{\{|x| \leq u\}} |x|^p F_t(dx).$$

Then we set

$$(18) \quad \Omega_T^{ii} = \Omega_T^{i, \Gamma > 0} \cup \Omega_T^{i, \Gamma = 0} \quad \text{where} \\ \Omega_T^{i, \Gamma > 0} = \text{the set on which } \Gamma_T > 0 \text{ and, for all } a' \in (0, 1),$$

$$(19) \quad \lambda\left(\left\{s \in [0, T] : \lim_{x \rightarrow 0} x^{a' \Gamma_T} \overline{F}_s(x) = \infty\right\}\right) > 0,$$

$$\Omega_T^{i, \Gamma = 0} = \text{the set on which } \Gamma_T = 0 \text{ and, for all } p \geq 2,$$

$$(20) \quad \lambda\left(\left\{s \in [0, T] : \liminf_{u \rightarrow 0} G(p, u)_s > 0\right\}\right) > 0.$$

We will see in Section 1 of the supplemental article [Aït-Sahalia and Jacod (2011)] that $\Omega_T^{ii} \subset \Omega_T^i$, but equality may fail. In view of Lemma 1, and if the measures F_t are regular [see after (13)], the set $\Omega_T^{i,\Gamma>0}$ is equal to $\{\Gamma_T > 0\}$, which is also the set where there are infinitely many jumps due to a positive BG index. The interpretation of the set $\Omega_T^{i,\Gamma=0}$ is more delicate: observe that $G(p, u)_t \leq L'(p)_t$ by (14); then saying that $\omega \in \Omega_T^{i,\Gamma=0}$ amounts to saying that for “enough” values of $t \leq T$ the variables $G(p, u)_t$ are not small.

The following is proved there:

LEMMA 2. *Assumption 3 implies Assumption 5, and we have $\Omega_T^{ii} = \Omega_T^i$ a.s.*

As for Assumption 4, it can be weakened as follows:

ASSUMPTION 6. *There are two constants $\beta \in (0, 2)$ and $\beta' \in [0, \beta)$ and a pre-locally bounded process L_t , such that the Lévy measure F_t is of the form*

$$(21) \quad F_t(dx) = \frac{\beta}{|x|^{1+\beta}} (a_t^{(+)} 1_{\{0 < x \leq z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)} \leq x < 0\}}) dx + F_t'(dx)$$

[the same as (5) with $\delta_t = 0$ and $\gamma_t = \beta$] with (6) and the (signed) measure F_t' satisfying

$$(22) \quad \int (|x|^{\beta'} \wedge 1) |F_t'(dx)| \leq L_t.$$

We associate with this assumption the following increasing process and set:

$$(23) \quad \bar{A}_t = \int_0^t A_s ds, \quad \Omega_T^{i\beta} = \{\bar{A}_T > 0\},$$

where the exponent in $\Omega_T^{i\beta}$ stands for “infinite activity for the jumps associated with the part of the Lévy measure having index of jump activity β ,” and we have $\Omega_T^{i\beta} \subset \Omega_T^i$, up to a null set. Again, the set $\Omega_T^i \setminus \Omega_T^{i\beta}$ is not necessarily empty.

Assumption 4 obviously implies Assumption 6, with the same β and with $\beta' = c\beta$, and in this case $\Omega_T^{i\beta} = \Omega_T^i$. However, Assumption 6, which is exactly the assumption under which the estimation of the BG index is performed in Aït-Sahalia and Jacod (2009a), does not require the measure F_t' to be finite when $A_t = 0$, so it is weaker than Assumption 4.

3. The two tests.

3.1. *Defining the hypotheses to be tested.* Ideally, we would like to construct tests in the following two situations:

$$(24) \quad \begin{aligned} H_0 : \Omega_T^f & \text{ vs. } H_1 : \Omega_T^i, \\ H_0 : \Omega_T^i & \text{ vs. } H_1 : \Omega_T^f. \end{aligned}$$

In order to derive the asymptotic distributions of the test statistics which we construct below, we need to slightly restrict these testing hypotheses. Besides the sets defined in (18) and (23), we also define two other complementary sets:

$$(25) \quad \Omega_T^W = \left\{ \int_0^T \sigma_s^2 ds > 0 \right\}, \quad \Omega_T^{\text{no}W} = \left\{ \int_0^T \sigma_s^2 ds = 0 \right\}.$$

That is, Ω_T^W is the set on which the continuous martingale part of X is not degenerate over $[0, T]$, and the exponents in Ω_T^W and $\Omega_T^{\text{no}W}$ stand for “the Wiener process is present,” and “no Wiener process is present,” respectively.

We will provide tests for the following assumptions (below we state the assumptions needed for the null to have a test with a given asymptotic level, and those needed for the alternative if we want the test to have asymptotic power equal to 1):

$$(26) \quad \begin{aligned} H_0 : \Omega_T^f \cap \Omega_T^W \text{ (Assumptions 1 and 2)} & \quad \text{vs.} \quad H_1 : \begin{cases} \Omega_T^i \text{ (Assumptions 1 and 3),} \\ \Omega_T^{ii} \text{ (Assumptions 1 and 5),} \end{cases} \\ H_0 : \begin{cases} \Omega_T^{i\beta} \text{ (Assumptions 1 and 6),} \\ \Omega_T^i \text{ (Assumptions 1 and 4)} \end{cases} & \quad \text{vs.} \quad H_1 : \Omega_T^f \cap \Omega_T^W \text{ (Assumptions 1).} \end{aligned}$$

Note that this lets aside the two sets $\Omega_T^f \cap \Omega_T^{\text{no}W}$ and $\Omega_T^i \setminus \Omega_T^{i\beta}$ in the null hypothesis: in the context of high-frequency data, no semimartingale model where $\Omega_T^f \cap \Omega_T^{\text{no}W}$ is not empty has been used, to our knowledge. Indeed, on this set the path of X over the interval $[0, T]$ is a pure drift plus finitely many jumps. It also lets aside the set $\Omega_T^i \setminus \Omega_T^{i\beta}$, under Assumptions 1 and 6, which is more annoying. Note that the set Ω_T^f contains those ω for which $X(\omega)$ is continuous on $[0, T]$, although we would not test against infinite activity if we did not know beforehand that there were some jumps.

Next, we specify the notion of testing when the null and alternative hypotheses are families of possible outcomes. Suppose now that we want to test the null hypothesis “we are in a subset Ω_0 ” of Ω , against the alternative “we are in a subset Ω_1 ,” with of course $\Omega_0 \cap \Omega_1 = \emptyset$. We then construct a critical (rejection) region C_n at stage n , that is, when the time lag between observations is Δ_n . This critical region is itself a subset of Ω , which should depend only on the observed values of the process X at stage n . We are not really within the framework of standard statistics, since the two hypotheses are themselves random.

We then take the following as our definition of the asymptotic size:

$$(27) \quad a = \sup \left(\limsup_n \mathbb{P}(C_n \mid A) : A \in \mathcal{F}, A \subset \Omega_0 \right).$$

Here $\mathbb{P}(C_n \mid A)$ is the usual conditional probability knowing A , with the convention that it vanishes if $\mathbb{P}(A) = 0$. If $\mathbb{P}(\Omega_0) = 0$, then $a = 0$, which is a natural

convention since in this case we want to reject the assumption whatever the outcome ω is. Note that a features a form of uniformity over all subsets $A \subset \Omega_0$. As for the asymptotic power, we define it as

$$(28) \quad P = \inf \left(\liminf_n \mathbb{P}(C_n | A) : A \in \mathcal{F}, A \subset \Omega_1, \mathbb{P}(A) > 0 \right).$$

Again, this is a number.

Clearly, and as in all tests in high-frequency statistics, at any given stage n it is impossible to distinguish between finitely many or infinitely many jumps (or, for that matter, between a continuous and a discontinuous path). So testing such hypotheses can only have an asymptotic meaning, as the mesh Δ_n goes to 0. Now, our definition of the asymptotic level is the usual one, apart from the fact that we test a given family of outcomes rather than a given family of laws. For the asymptotic power, it is far from the typical usual statistical understanding. Namely saying that $P = 1$, as will often be the case below, does not mean anything like the infimum of the power over all possible alternatives is 1; it is rather a form of consistency on the set of alternatives.

3.2. Truncated power variations. Before stating the results, we introduce some notation, to be used throughout. We introduce the observed increments of X as

$$(29) \quad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n},$$

to be distinguished from the (unobservable) jumps of the process, $\Delta X_s = X_s - X_{s-}$. In a typical application, X is a log-asset price, so $\Delta_i^n X$ is the recorded log-return over Δ_n units of time.

We take a sequence u_n of positive numbers, which will serve as our thresholds or cutoffs for truncating the increments when necessary, and will go to 0 as the sampling frequency increase. There will be restrictions on the rate of convergence of this sequence, expressed in the form

$$(30) \quad \sup_n \Delta_n^\rho / u_n < \infty$$

for some $\rho > 0$: this condition becomes weaker when ρ increases. Two specific values for ρ , in connection with the power $p > 2$ which is used below, are of interest to us:

$$(31) \quad \rho_1(p) = \frac{p-2}{2p}, \quad \rho_2(p) = \frac{p-2}{4p-4} \wedge \frac{2p-4}{11p-10}.$$

These quantities increase when p increases [on $(2, \infty)$], and $\rho_1(p) > \rho_2(p) > 0$.

Finally, with any $p > 0$ we associate the increasing processes

$$(32) \quad B(p, u_n, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \leq u_n\}}$$

consisting of the sum of the p th absolute power of the increments of X , truncated at level u_n , and sampled at time intervals Δ_n . These truncated power variations, used in various combinations, will be the key ingredients in the test statistics we construct below.

3.3. *The finite-activity null hypothesis.* We first set the null hypothesis to be finite activity, that is, $\Omega_0 = \Omega_T^f \cap \Omega_T^W$, whereas the alternative is $\Omega_1 = \Omega_T^i$.

We choose an integer $k \geq 2$ and a real $p > 2$. We then propose the test statistic, which depends on k , p , and on the truncation level u_n , and on the time interval $[0, T]$, defined as follows:

$$(33) \quad S_n = \frac{B(p, u_n, k\Delta_n)_T}{B(p, u_n, \Delta_n)_T}.$$

That is, we compute the truncated power variations at two different frequencies in the numerator and denominator, but otherwise use the same power p and truncation level u_n . Since k is an integer, both truncated power variations can be computed from the same data sample. If the original data consist of log-returns sampled every Δ_n units of time, then sampling every $k\Delta_n$ units of time involves simply retaining every k th observation in that same sample.

The first result gives the limiting behavior of the statistic S_n , in terms of convergence in probability:

THEOREM 1. (a) *Under Assumption 1 and if the sequence u_n satisfies (30) with some $\rho < 1/2$, we have*

$$(34) \quad S_n \xrightarrow{\mathbb{P}} k^{p/2-1} \quad \text{on the set } \Omega_T^f \cap \Omega_T^W.$$

(b) *Under Assumptions 1 and 3 (resp., Assumption 5) and if the sequence u_n satisfies (30) with some $\rho < \rho_1(p)$, we have*

$$(35) \quad S_n \xrightarrow{\mathbb{P}} 1 \quad \text{on the set } \Omega_T^i \text{ (resp., } \Omega_T^{ii}).$$

As the result shows, the statistic S_n behaves differently depending upon whether the number of jumps is finite or not. Intuitively, if the number of jumps is finite, then at some point along the asymptotics the truncation eliminates them and the residual behavior of the truncated power variation is driven by the continuous part of the semimartingale. More specifically, $B(p, u_n, \Delta_n)_T$ is of order $O_p(\Delta_n^{p/2-1} A(p)_T)$ where $A(p)_T = m_p \int_0^T |\sigma_s|^p ds$ is the continuous variation of order p and m_p is a constant defined below. It follows from this that the ratio in S_n has the limit given in (34) since the numerator and denominator tend to zero but at different rates.

By setting $\rho < 1/2$ in the truncation rate, we are effectively retaining all the increments of the continuous part of the semimartingale, and so we indeed obtain

the “full” continuous variation $A(p)_T$ after truncation. Note that by contrast, the untruncated power variation converges when $p > 2$ to the discontinuous variation of order p , namely $B(p)_T = \sum_{s < T} |\Delta X_s|^p$, and so we would not have been able to distinguish finite or infinite jump activity without truncation, as long as jumps (of any activity) are present.

If the jumps have infinite activity, on the other hand, that is, under the alternative hypothesis, the asymptotic behavior of the truncated power variation is driven by the small jumps, whether the Brownian motion is present or not, and the truncation rate matters. We will see, for example, that, under Assumption 4 or 6, the truncated power variation is of order $O_p(u_n^{p-\beta})$. Since here we are truncating at the same level u_n for the two sampling frequencies Δ_n and $k\Delta_n$, this fact has no consequence on the behavior of the ratio S_n , which tends to 1 as stated by (35) even when Assumption 6 fails.

Theorem 1 implies that for the test at hand an a priori reasonable critical region is $C_n = \{S_n < c_n\}$, for c_n between 1 and $k^{p/2-1}$, and in particular if we choose $c_n = c$ in the interval $(1, k^{p/2-1})$ the asymptotic level and power are, respectively, 0 and 1 if the model satisfies Assumption 5.

For a more refined version of this test, with a prescribed level $a \in (0, 1)$, we need a central limit theorem associated with the convergence in (34), and a standardized version goes as follows [we use $\xrightarrow{\mathcal{L}^{-(s)}}$ to denote the stable convergence in law; see, e.g., Jacod and Shiryaev (2003) for this notion]:

THEOREM 2. *Assume Assumptions 1 and 2, and that the sequence u_n satisfies (30) with some $\rho < 1/2$. Then*

$$(36) \quad (S_n - k^{p/2-1})/\sqrt{V_n} \xrightarrow{\mathcal{L}^{-(s)}} \mathcal{N}(0, 1) \quad \text{in restriction to } \Omega_T^f \cap \Omega_T^W,$$

where

$$(37) \quad V_n = N(p, k) \frac{B(2p, u_n, \Delta_n)_T}{(B(p, u_n, \Delta_n)_T)^2},$$

$$(38) \quad N(p, k) = \frac{1}{m_{2p}} (k^{p-2}(1+k)m_{2p} + k^{p-2}(k-1)m_p^2 - 2k^{p/2-1}m_{k,p}),$$

and $m_{p,k} = \mathbb{E}(|U|^p | U + \sqrt{k-1}V|^p)$ and $m_p = \mathbb{E}(|U|^p)$ for U and V two independent $\mathcal{N}(0, 1)$ variables.

Note that to implement this asymptotic distribution in practice, and hence the test, we simply need to compute truncated power variations for various powers (specifically p and $2p$, in this case) and at two different sampling frequencies (Δ_n and $k\Delta_n$). No other estimation is required. In particular, we do not need to estimate any aspect of the dynamics of the X , such as its drift or volatility processes or its jump measure. In that sense, the test statistic is nonparametric, or model-free.

We are now ready to exhibit a critical region for testing $H_0 : \Omega_T^f \cap \Omega_T^W$ vs. $H_1 : \Omega_T^i$, or Ω_T^{ii} , or $\Omega_T^{i,\Gamma>0}$ (depending on the assumptions), using S_n with a prescribed asymptotic level $a \in (0, 1)$. Denoting by z_a the a -quantile of $\mathcal{N}(0, 1)$, that is, $\mathbb{P}(U > z_a) = a$ where U is $\mathcal{N}(0, 1)$, we set

$$(39) \quad C_n = \{S_n < k^{p/2-1} - z_a \sqrt{V_n}\}.$$

More precisely, we can state the level and power of the test under selected alternatives as follows:

THEOREM 3. *Under Assumptions 1 and 2, and if the sequence u_n satisfies (30) with some $\rho < 1/2$, the asymptotic level of the critical region defined by (39) for testing the null hypothesis “finite jump activity” (i.e., $\Omega_T^f \cap \Omega_T^W$) equals a . Moreover the asymptotic power of this test is 1 in the following cases:*

- under Assumption 3 with $\delta > 0$, for the alternative Ω_T^i ,
- under Assumption 5 with $\psi(0) = \infty$, for the alternative Ω_T^{ii} ,
- under Assumption 5, for the alternative $\Omega_T^{i,\Gamma>0}$.

Of course this test has the same asymptotic level a for testing $\Omega_T^f \cap \Omega_T^W$ against its complement $(\Omega_T^f \cap \Omega_T^W)^c$, under Assumptions 1 and 2; but in this case the asymptotic power is probably not 1, and may well be 0.

3.4. The infinite-activity null hypothesis. In the second case, we assume Assumption 6 (or Assumption 4) and we set the null hypothesis to be infinite activity, that is, $\Omega_0 = \Omega_T^{i\beta}$ (or Ω_T^i), whereas the alternative is $\Omega_1 = \Omega_T^f \cap \Omega_T^W$. Unfortunately, we cannot simply use the same test statistic S_n we proposed in (33) for this second testing problem. The reason is that, while the distribution of S_n is model-free under the null hypothesis of the first testing problem, it is no longer model-free under the null hypothesis of the second testing problem: its distribution when jumps have infinite activity depends upon the degree of jump activity, β . While it is possible to estimate β consistently [see Aït-Sahalia and Jacod (2009a)], it would be preferable to construct a statistic whose implementation does not require a preliminary estimate of the degree of jump activity.

In order to design a test statistic which is model-free under the null of infinite activity, we choose three reals $\gamma > 1$ and $p' > p > 2$ and then define a family of test statistics as follows:

$$(40) \quad S'_n = \frac{B(p', \gamma u_n, \Delta_n)_T B(p, u_n, \Delta_n)_T}{B(p', u_n, \Delta_n)_T B(p, \gamma u_n, \Delta_n)_T}.$$

In other words, unlike the previous statistic S_n , we now play with different powers p and p' , and different levels of truncation u_n and γu_n , but otherwise sample at the same frequency Δ_n . Once more, the first result states the limiting behavior of S'_n in terms of convergence in probability:

THEOREM 4. Assume Assumptions 1 and 4 (resp., Assumptions 1 and 6).

(a) If the sequence u_n satisfies (30) with some $\rho < \rho_1(p)$, we have

$$(41) \quad S'_n \xrightarrow{\mathbb{P}} \gamma^{p'-p} \quad \text{on the set } \Omega_T^i \text{ (resp., } \Omega_T^{i\beta}).$$

(b) If the sequence u_n satisfies (30) with some $\rho < 1/2$, we have

$$(42) \quad S'_n \xrightarrow{\mathbb{P}} 1 \quad \text{on the set } \Omega_T^f \cap \Omega_T^W.$$

That is, as was the case in Theorem 1, the test statistic S'_n tends to 1 under the alternative and to a value different from 1 under the null. Intuitively, under the alternative of finite jump activity, the behavior of each one of the four truncated power variations in (40) is driven by the continuous part of the semimartingale. The truncation level is such that essentially all the Brownian increments are kept. Then the truncated power variations all tend to zero at rate $O_p(\Delta_n^{p/2-1})$ and $O_p(\Delta_n^{p'/2-1})$, respectively, and by construction the (random) constants of proportionality $A(p)_T$ and $A(p')_T$ cancel out in the ratios, producing the limit 1 given in (42).

If, on the other hand, jumps have infinite activity, then the small jumps are the ones that matter and the truncation level becomes material, producing four terms that all tend to zero but at the different orders $O_p(u_n^{p-\beta})$, $O_p(u_n^{p'-\beta})$, $O_p((\gamma u_n)^{p-\beta})$ and $O_p((\gamma u_n)^{p'-\beta})$, respectively, resulting in the limit (41). By design, the limit is independent of β .

A reasonable critical region is $C_n = \{S'_n < c_n\}$ with c_n between 1 and $\gamma^{p'-p}$, and if $c_n = c$ is in the interval $(1, \gamma^{p'-p})$ the asymptotic level and power are, respectively, 0 and 1. For a test with a prescribed level we again need a standardized central limit theorem associated with the convergence in (41):

THEOREM 5. Assume Assumptions 1 and 4 with $c < 1/2$ (resp., Assumption 6 with $\beta' < \beta/2$), and if the sequence u_n satisfies (30) with some $\rho < \rho_2(p)$, we have

$$(43) \quad (S'_n - \gamma^{p'-p}) / \sqrt{V'_n} \xrightarrow{\mathcal{L}^{-(s)}} \mathcal{N}(0, 1) \quad \text{in restriction to } \Omega_T^i \text{ (resp., } \Omega_T^{i\beta}),$$

where

$$(44) \quad V'_n = \gamma^{2p'-2p} \left(\frac{B(2p, u_n, \Delta_n)_T}{(B(p, u_n, \Delta_n)_T)^2} + (1 - 2\gamma^{-p}) \frac{B(2p, \gamma u_n, \Delta_n)_T}{(B(p, \gamma u_n, \Delta_n)_T)^2} \right. \\ \left. + \frac{B(2p', u_n, \Delta_n)_T}{(B(p', u_n, \Delta_n)_T)^2} + (1 - 2\gamma^{-p'}) \frac{B(2p', \gamma u_n, \Delta_n)_T}{(B(p', \gamma u_n, \Delta_n)_T)^2} \right. \\ \left. - 2 \frac{B(p + p', u_n, \Delta_n)_T}{B(p, u_n, \Delta_n)_T B(p', u_n, \Delta_n)_T} \right. \\ \left. - 2(1 - \gamma^{-p} - \gamma^{-p'}) \frac{B(p + p', \gamma u_n, \Delta_n)_T}{B(p, \gamma u_n, \Delta_n)_T B(p', \gamma u_n, \Delta_n)_T} \right).$$

As was the case for S_n in Theorem 2, the asymptotic distribution of S'_n under the null is again model-free. While the expression (44) looks complicated, implementing it simply requires the computation of truncated power variations B of order p , p' , $p + p'$, $2p$ and $2p'$, and at truncation levels u_n and γu_n . No other aspects of the dynamics of the X process, such as its degree of jump activity β , for instance, need to be estimated.

The critical region for testing $H_0 : \Omega_T^i$ or $\Omega_T^{i\beta}$ vs. $H_1 : \Omega_T^f$ using S'_n with a prescribed asymptotic level $a \in (0, 1)$ will be

$$(45) \quad C'_n = \{S'_n < \gamma^{p'-p} - z_a \sqrt{V'_n}\}.$$

We can state more precisely the level and power of the test as follows:

THEOREM 6. *Under Assumptions 1 and 4 with $c < 1/2$ (resp., Assumption 6 with $\beta' < \beta/2$), and if the sequence u_n satisfies (30) with some $\rho < \rho_2(p)$, the asymptotic level of the critical region defined by (45) for testing the null hypothesis “infinite activity for jumps” (i.e., Ω_T^i , resp., $\Omega_T^{i\beta}$, against $\Omega_T^f \cap \Omega_T^W$) equals a . Moreover the asymptotic power of this test is 1.*

Under the null hypothesis the rate of convergence is $1/u_n^{\beta/2}$ (contrary to the situation of Theorem 3, where the rate was $1/\sqrt{\Delta_n}$ whatever β and u_n were). So, although asymptotically u_n does not explicitly show in the test itself, one should probably take u_n as small as possible (we have no choice as to β , of course). That is, one should take ρ as large as possible, which in turn results in choosing p as large as possible [recall (31)]. We discuss actual choices in practice in Sections 4 and 5 below.

3.5. Microstructure noise. In practice, the tests presented here need a lot of data to be effective, that is, we need a high sampling frequency. However, in this case, it is well established that the so-called *microstructure noise* may be a relevant consideration, and in some cases may even dominate at ultra high frequencies. It is outside the scope of this paper to provide a complete analysis of what happens when noise is present, or to contemplate constructing effective testing procedures in the presence of noise. However, as a first step, it may be enlightening to determine at least the limiting behavior (in probability) of our test statistics in the presence of noise since this may help guide the interpretation of the empirical results when the test is implemented in practice.

We start first with pure additive noise, which is the type of noise considered by much of the literature, primarily for reasons of tractability, although it may not account very well for the microstructure noise encountered in practice. It gives, however, an insight on what can happen in the presence of noise. In this situation, at any given observation time t we actually observe log-returns perturbed by a noise term ε_t in

$$(46) \quad Z_t = X_t + \varepsilon_t.$$

To avoid intricate statements, we make a few basic, and mild, assumptions on the noise process ε_t : those variables are i.i.d., centered, independent of the underlying process, with the following property:

$$(47) \quad \text{if } s \neq t \text{ the variable } \varepsilon_t - \varepsilon_s \text{ has a density } f, \text{ continuous and positive at } 0.$$

Note that no moment condition is required, except $\mathbb{E}(\varepsilon_t) = 0$, because we consider only truncated increments.

We write $\overline{B}(p, u_n, \Delta_n)$ for the variables introduced in (32) if we replace $X_{i\Delta_n}$ by $Z_{i\Delta_n}$, and likewise for \overline{S}_n and \overline{S}'_n if we do the same substitution in (33) and (40) (those are associated with some integer k and some numbers $p, p' > 2$ and $\gamma > 1$). Under the above assumption on the noise, we have the following:

THEOREM 7. *Let the sequence u_n satisfy (30) with some $\rho < \frac{1}{3}$. Then, under Assumption 1 we have*

$$(48) \quad \overline{S}_n \xrightarrow{\mathbb{P}} \frac{1}{k}, \quad \overline{S}'_n \xrightarrow{\mathbb{P}} \gamma^{p'-p}.$$

It is remarkable that the assumptions in this theorem are much weaker than in the previous results, as far as X is concerned. This is of course because the noise, when present, becomes the prevalent factor. This has the important consequence that the statistics \overline{S}_n and \overline{S}'_n are no longer able to distinguish between the two hypotheses of finite or infinite activity when noise dominates. When we use \overline{S}_n we get a limit which differs from both limits in Theorem 1, so when we test for the null hypothesis of finite activity and the empirical value of \overline{S}_n is close to $1/k$ we can in principle tell that this is due to the noise, whereas if it is close to $k^{p/2-1}$ or to 1 this is probably due to finite activity, or infinite activity. When we use \overline{S}'_n the situation is worse, because noise plus finite activity leads to accept the hypothesis that we have infinite activity whether the jumps have finite or infinite activity.

Alternatively, it might be closer to the reality to model the microstructure noise as a pure rounding noise. There, instead of observing $X_{i\Delta_n}$, we observe $[X_{i\Delta_n}]_\alpha$, that is, $X_{i\Delta_n}$ rounded to the nearest multiple of α , where α is the tick size: typically $\alpha = 1$ cent for a decimalized stock, or $\alpha = 1/32$ nd of a dollar, for bond prices. Let us denote by $\widehat{B}(p, u_n, \Delta_n)_t$ the variables of (32) when we replace $X_{i\Delta_n}$ by $[X_{i\Delta_n}]_\alpha$. Theorems 1 and 4 were based on limit theorems for the truncated power variations $B(p, u_n, \Delta_n)_t$. In the rounded case we indeed have something radically different: as soon as $u_n < \alpha$, we eliminate *all* increments, because increments are multiples of α . Therefore $\widehat{B}(p, u_n, \Delta_n)_t = 0$ for all n large enough, and of course the ratios (33) and (40) make no sense. It follows that in the case of a rounding noise, the statistics proposed in this paper are totally meaningless. And as a matter of fact, even when the whole (rounded) path of X is observed, we cannot decide whether we have finitely many or infinitely many jumps. This means that not only

our statistics do not work for this problem, but there cannot exist asymptotically consistent tests for this problem.

Naturally, these two idealized descriptions of microstructure noise do not exhaust the possibilities for modeling the noise. One can, for example, use a mixed model which mixes additive noise and rounding, or more general forms, as in Jacod et al. (2009), for example. At present, however, it is not clear how our statistics theoretically behave in these more general cases, nor how to construct asymptotically consistent tests, nor even if such tests exist at all.

4. Simulation results. We now report simulation results documenting the finite sample performance of the test statistics S_n and S'_n in finite samples under their respective null and alternative hypotheses. We calibrate the values to be realistic for a liquid stock trading on the NYSE. We use observation lengths ranging from $T = 1$ day, consisting of 6.5 hours of trading, that is, $n = 23,400$ seconds, to $T = 1$ year. The sampling frequencies we consider range from $\Delta_n = 1$ second to 5 minutes. The highest sampling frequencies serve to validate the asymptotic theory contained above, while the lower frequencies serve as proxies for situations where sparse sampling is employed as a means to reduce the adverse impact of market microstructure noise.

The tables and graphs that follow report the results of 5,000 simulations. In order to validate the asymptotic theory developed above, we start with the highest sampling frequency of $\Delta_n = 1$ second. Below, we will examine the accuracy of the tests as a function of the sampling frequencies ranging from 5 seconds to 5 minutes. The data generating process is the stochastic volatility model $dX_t = \sigma_t dW_t + \theta dY_t$, with $\sigma_t = v_t^{1/2}$, $dv_t = \chi(\eta - v_t) dt + \xi v_t^{1/2} dB_t + dJ_t$, $\mathbb{E}(W_t B_t) = \bar{\rho} t$, $\eta^{1/2} = 0.25$, $\xi = 0.5$, $\chi = 5$, $\bar{\rho} = -0.5$, J is a compound Poisson jump process with jumps that are uniformly distributed on $[-30\%, 30\%]$ and $X_0 = 1$. The jump process Y is either a β -stable process with $\beta = 1$, that is, a Cauchy process (which has infinite activity, and will be our model under $\Omega_T^{i\beta}$) or a compound Poisson process (which has finite activity, and will be our model under Ω_T^f). In the infinite-activity case, empirical estimates of β reported in Aït-Sahalia and Jacod (2009a) are higher than 1; simulation results using β -stable processes with such values are qualitatively similar. In the finite-activity case, the jump size of the compound Poisson process is drawn from a truncated Normal distribution with mean 0 and standard deviation 0.10, designed to produce jumps greater in magnitude than 0.05. The estimator is implemented with a truncation rate $u_n = \alpha \Delta_n^{\varpi}$ where $\varpi = 0.20$. In the results, we report the level of truncation u_n indexed by the number α of standard deviations of the continuous martingale part of the process, defined in multiples of the long-term volatility parameter $\eta^{1/2}$. That is, α is such that $u_n = \alpha \eta^{1/2} \Delta_n^{1/2}$.

The statistic S_n is implemented with $k = 2$ and $p = 4$. When $p = 4$, the constants appearing in (38) are

$$(49) \quad N(4, k) = \frac{16}{3}k(2k^2 - k - 1).$$

For the second test, S'_n is implemented with $p' = 4$, $p = 3$ and the second truncation level γu_n related to the first according to $\alpha' = 2\alpha$, that is, $\gamma = 2$.

Given η , the scale parameter θ [or equivalently A_t in (6), which here is a constant] of the stable process in simulations is calibrated to deliver different various values of the tail probability $TP = \mathbb{P}(|\Delta Y_t| \geq 4\eta^{1/2}\Delta_n^{1/2})$ reported in the tables as low ($TP = 0.01$), medium ($TP = 0.05$) and high ($TP = 0.10$). For the Poisson process, it is the value of the arrival rate parameter λ that is set to generate the desired level of jump tail probability as low ($\lambda = 2/23,400$), medium ($\lambda = 10/23,400$) or high ($\lambda = 50/23,400$). In the various simulations' design, we hold η fixed. Therefore the tail probability parameter controls the relative scale of the jump component of the semimartingale relative to its continuous counterpart.

Table 1 reports the Monte Carlo rejection rates of the test of $H_0 : \Omega_T^f \cap \Omega_T^W$ vs. $H_1 : \Omega_T^{i\beta}$ at the 10% and 5% level, using the test statistic S_n , for various levels of truncation α . Recall that for concreteness α is expressed as a number of standard deviations of the Brownian part of X . We find that the test behaves well, with empirical test levels close to their theoretical counterparts, and so for a wide range of values of α .

The limit in probability of S_n is $k^{p/2-1} = 2$ with our choice of k and p , provided $u_n/\sqrt{\Delta_n} \rightarrow \infty$. With our notation α denoting the threshold expressed as the number of (normalized) standard deviations of the Brownian part, this means that this limit holds when $\alpha = \alpha_n$ goes to infinity. In practice we choose α between 6 and 15, and this introduces a bias. To evaluate this bias, in Figure 1, we plot as a function of α the limiting value of S_n under H_0 , when α stays fixed instead

TABLE 1
Monte Carlo rejection rate for the test of finite activity using the statistic S_n (testing $H_0 : \Omega_T^f \cap \Omega_T^W$ vs. $H_1 : \Omega_T^{i\beta}$ using S_n)

Finite jump intensity	Test theoretical level	Test empirical level for a degree of truncation α						
		6	7	8	9	10	12	15
Low	10%	9.6%	9.6%	9.5%	9.5%	9.5%	9.5%	9.5%
	5%	5.6%	4.5%	4.4%	4.4%	4.4%	4.4%	4.4%
Medium	10%	11.5%	10.7%	10.5%	10.5%	10.5%	10.4%	10.4%
	5%	6.1%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%
High	10%	11.8%	10.4%	10.3%	10.3%	10.3%	10.2%	10.2%
	5%	5.9%	5.1%	5.0%	5.0%	5.0%	5.0%	5.0%

Behavior of S Under the Null Hypothesis
 H_0 : Finite Jump Activity

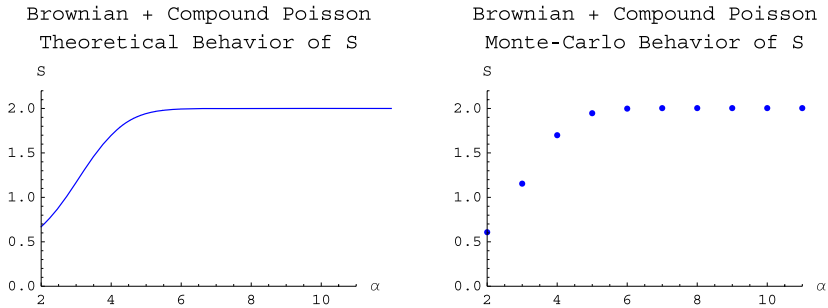


FIG. 1. Behavior of the test statistic S_n under $H_0: \Omega_T^f \cap \Omega_T^W$ for varying degrees of truncation α .

of increasing to infinity, and compare it with the theoretical limit 2 (in the simple model used for the simulations it is possible to numerically compute it with any desired accuracy); this is the left graph, and on the right graph we draw the corresponding average value of S_n from the Monte Carlo simulations, both as functions of α . It is worth noting that the behavior of S_n is driven by that of the Brownian motion component, since the truncation effectively eliminates the finite-activity jumps. For very small values of α , $\alpha < 5$, S_n starts below 2 as predicted by the theoretical behavior of the continuous martingale part of X . Once α gets above 5, the theoretical limit of S_n will then remain at 2 as long as α is not so large as to start including some jumps.

Figure 2 plots the limiting value of S_n (as a function of α again) under H_1 , where the model consists of a Brownian and a Cauchy components as described above. The upper left graph is the theoretical behavior of S_n for the Brownian component taken in isolation, the upper right graph the theoretical behavior for the Cauchy component taken in isolation, the lower left graph the theoretical behavior for the sum of the two components, that is, the model we simulate from, and the lower right graph the corresponding results from the Monte Carlo simulations. When the Cauchy process is present, the three curves in the figure correspond to the low (solid curve), medium (long dashes) and high (short dashes) scale of the jump component relative to the continuous component. For very small values of α , such as $\alpha < 5$, the behavior of S_n tracks that of the Brownian component, which is increasing in α toward 2. As α increases, the curves then start reversing course and tend to 1 instead, the limit driven by the infinite-activity jump process as shown in (35). The higher the scale of the jump component relative to the continuous component, the more the curve in the lower left graph approximates the corresponding one in the upper right graph, that is, the more it resembles that of a pure Cauchy process. For the Monte Carlo results, the low, medium and high jump scales are represented by circles, squares and diamonds, respectively. In all

Behavior of S Under the Alternative Hypothesis
 H_1 : Infinite Jump Activity

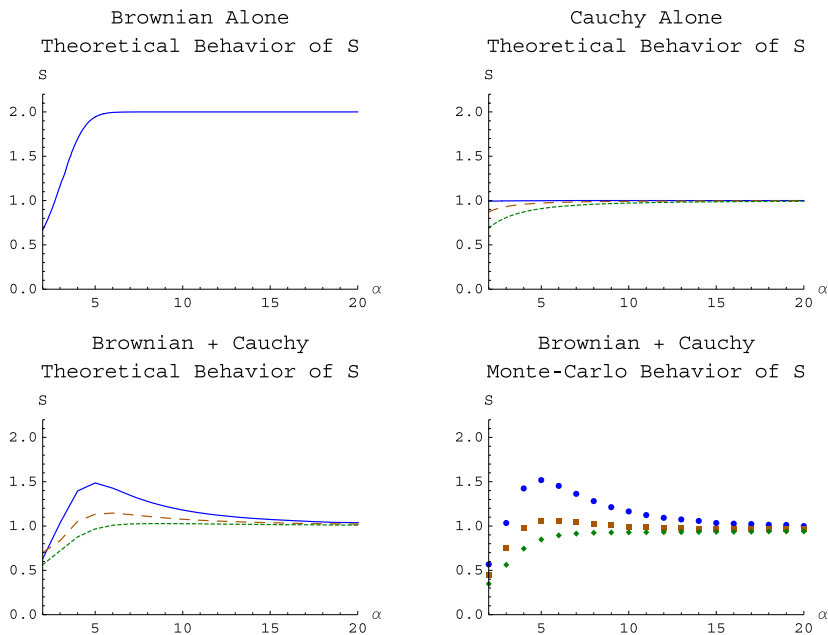


FIG. 2. Behavior of the test statistic S_n under $H_1 : \Omega_T^{i\beta}$ for varying degrees of truncation α .

three cases, they track the predicted theoretical limits closely for each value of the truncation level α .

Finally, we report in Figure 3 histograms of the values of the unstandardized S_n computed under H_0 , centered in 2 as expected from (34), and H_1 , centered in 1 as expected from (35), respectively. Figure 4 reports the Monte Carlo distribution of the statistic S_n , standardized according to (36), compared to the limiting $\mathcal{N}(0, 1)$ distribution.

Next, we turn to the symmetric problem, that of testing of $H_0 : \Omega_T^{i\beta}$ vs. $H_1 : \Omega_T^f \cap \Omega_T^W$, which we do using the statistic S'_n . The results are reported in Table 2 for the test rejection rate under the null hypothesis. Figure 5 shows the limiting value of S'_n under H_0 , as expected from the theoretical limit of $\gamma^{p'-p} = 2$ given in (41) (lower left graph for the Brownian plus Cauchy model) and the corresponding average value of S'_n from the Monte Carlo simulations (lower right graph). The upper graphs in the figure correspond to the Brownian alone (left, with a limit of 1) and Cauchy alone (right, with a limit of 2) situations. As was the case for the previous test, higher values of the jump scale parameter make the resulting semimartingale model approximate the behavior of its infinite-activity jump component more closely, while for lower scale parameters and low values of α , the behavior is determined by the continuous component. In all cases, we find that the Monte Carlo

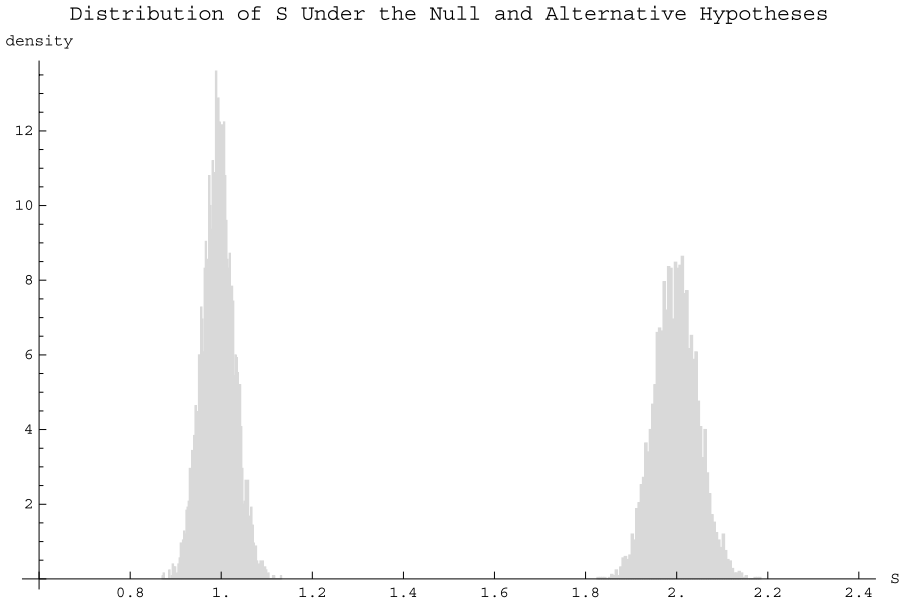


FIG. 3. Monte Carlo distributions of the unstandardized test statistic S_n under $H_0: \Omega_T^f \cap \Omega_T^W$ (right histogram) and $H_1: \Omega_T^{i\beta}$ (left histogram).

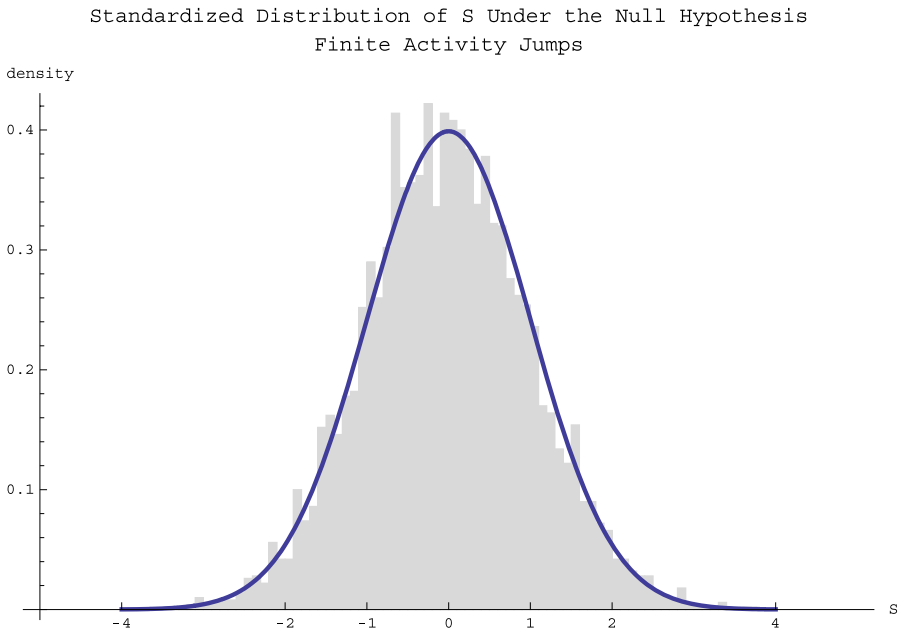


FIG. 4. Monte Carlo and asymptotic distributions of the standardized test statistic S_n under $H_0: \Omega_T^f \cap \Omega_T^W$.

TABLE 2
Monte Carlo rejection rate for the test of infinite jump activity using the statistic S'_n (testing $H_0 : \Omega_T^{i\beta}$ vs. $H_1 : \Omega_T^f$ using S'_n)

Infinite jump activity	Test theoretical level	Test empirical level
Low	10%	8.2%
	5%	3.6%
Medium	10%	8.9%
	5%	3.8%
High	10%	10.0%
	5%	4.9%

results track the predicted theoretical limits for each value of the truncation level α . Compared to Figure 2, we find that the limit is approached with less precision than was the case of S_n , and requires larger values of α than was the case for the other test when the infinite-activity jump process is mixed with the continuous component.

Behavior of S' Under the Null Hypothesis
 $H_0 : \text{Infinite Jump Activity}$

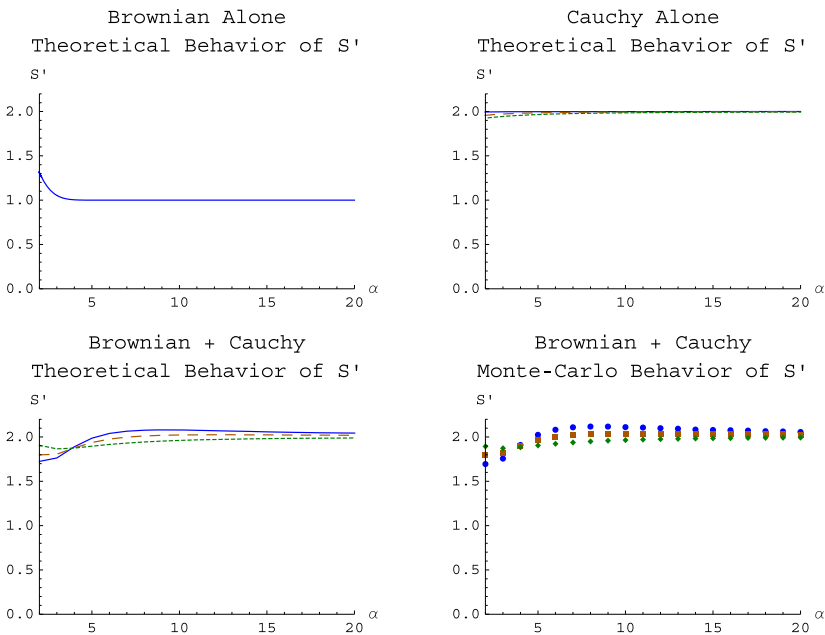


FIG. 5. Behavior of the test statistic S'_n under $H_0 : \Omega_T^{i\beta}$ for varying degrees of truncation α .

Behavior of S' Under the Alternative Hypothesis
 H_1 : Finite Jump Activity

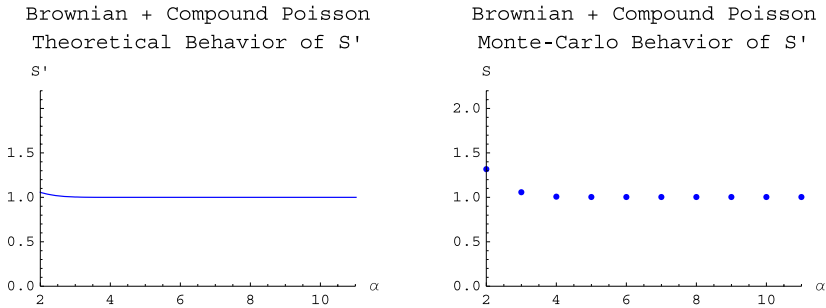


FIG. 6. Behavior of the test statistic S'_n under $H_1 : \Omega_T^f \cap \Omega_T^W$ for varying degrees of truncation α .

Figure 6 shows the point limit results for S'_n under H_1 . There, the theoretical limit of 1 from (42) is reached quickly, as was the case in Figure 1. This is the case provided that the values of α are not so large that jumps start being retained in the calculations. In particular, the value of S'_n can become large if α is such that a jump is just included at the u_n cutoff but not yet at the γu_n cutoff. But in most cases the statistic is very close to 1 when the number of jumps is finite, simply because once n is large enough for the truncation level to have eliminated all the jumps at the higher level of truncation γu_n , then there are very few Brownian increments between truncation levels u_n and γu_n ; therefore $B(p, \gamma u_n, \Delta_n)_T$ is only marginally larger than $B(p, u_n, \Delta_n)_T$, and similarly for power p' .

Figure 7 reports the Monte Carlo distributions of S'_n under H_0 and H_1 ; they are centered at 2 and 1 as expected. Under H_1 , we note that S'_n displays very little variability with the provision above regarding the value of α . As a general rule, the test based on S'_n appears more sensitive to values of α than the test based on S_n . Figure 8 reports the Monte Carlo and asymptotic distribution of the standardized S'_n under H_0 , according to (43).

Finally, we examine the accuracy of the tests as a function of the sampling frequencies. We consider for this purpose two experiments. In the first one, we fix the observation length to $T = 5$ days (one week) and consider sampling frequencies from 5 seconds to 5 minutes. The sample size n therefore decreases by a factor 60 over the range of values of Δ_n considered. In the second experiment, we consider the same sampling frequencies but increase the length of the observation window T from 5 days to 1 year, in order to keep the sample size approximately constant over the range of values of Δ_n . The results of the first experiment are reported in Figures 9 and 10, respectively. We find little size distortion or power loss. The results of the second experiment are, not surprisingly, better and are not shown here in order to save space.

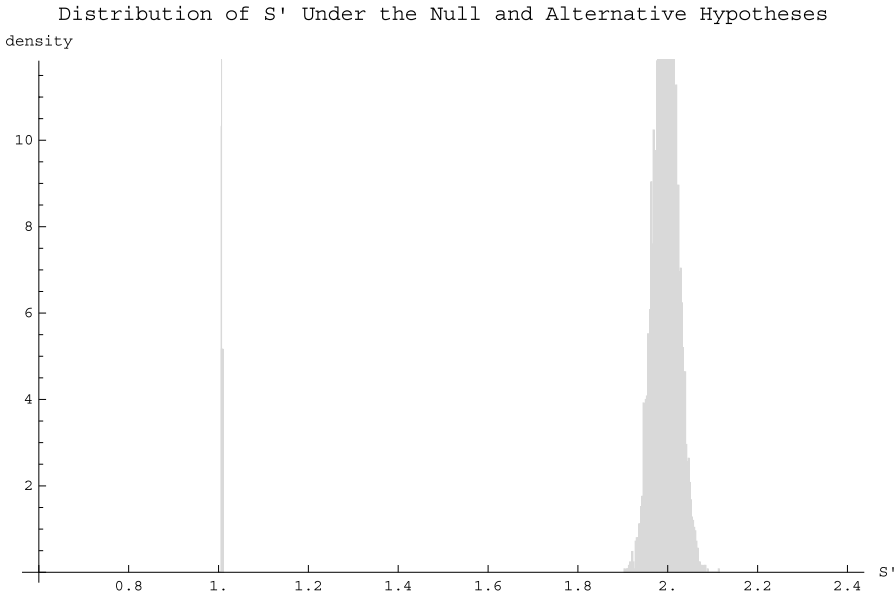


FIG. 7. Monte Carlo distributions of the unstandardized test statistic S'_n under $H_0: \Omega_T^{i\beta}$ (right histogram) and $H_1: \Omega_T^f$ (left histogram).

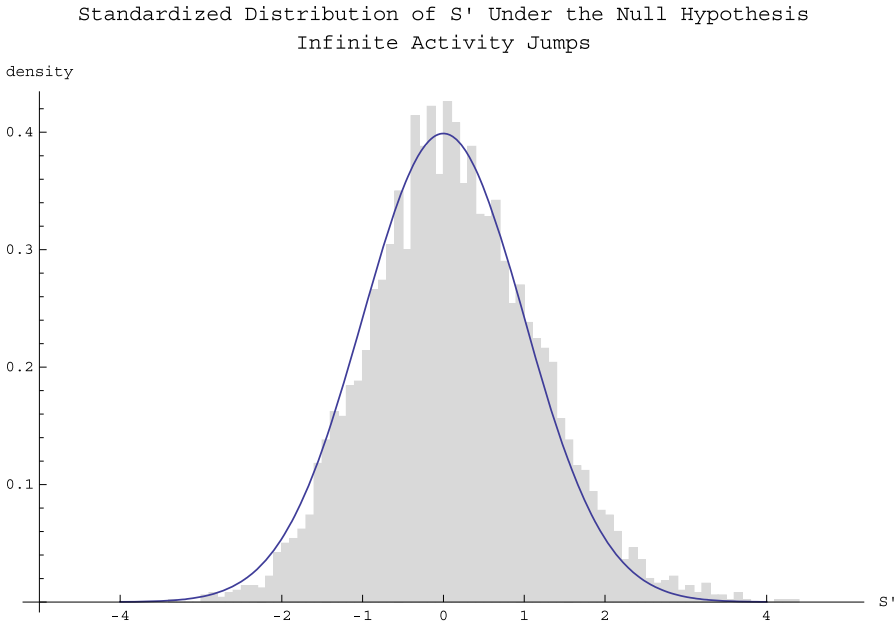


FIG. 8. Monte Carlo and asymptotic distributions of the standardized test statistic S'_n under $H_0: \Omega_T^{i\beta}$.

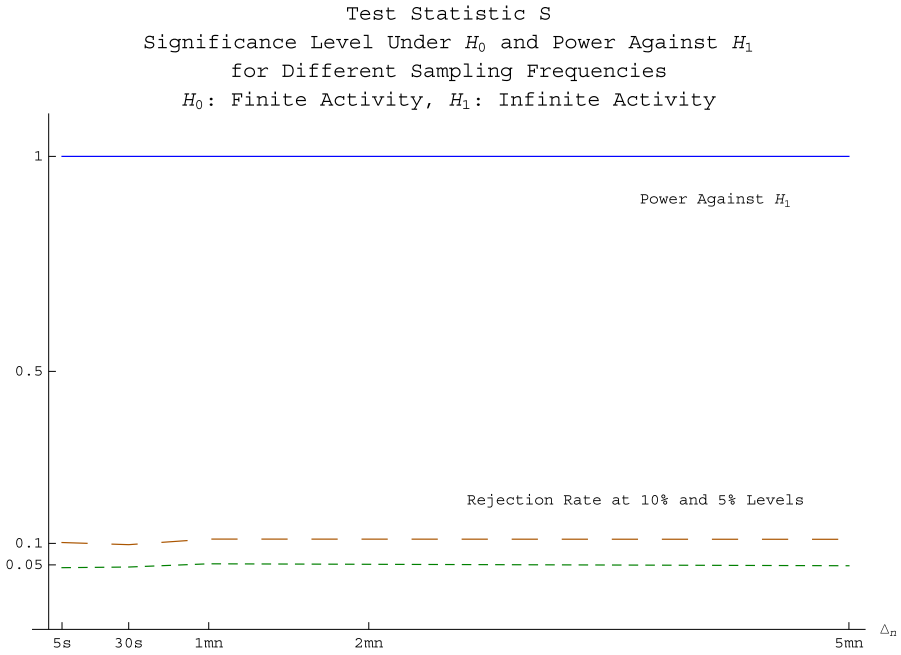


FIG. 9. Level and power of the test based on the statistic S as a function of the sampling frequency.

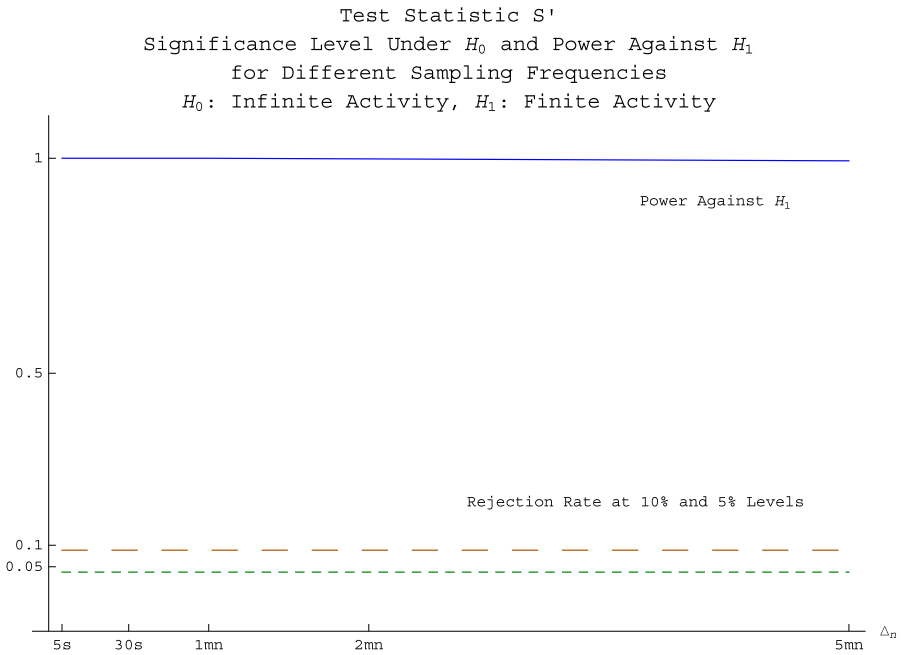


FIG. 10. Level and power of the test based on the statistic S' as a function of the sampling frequency.

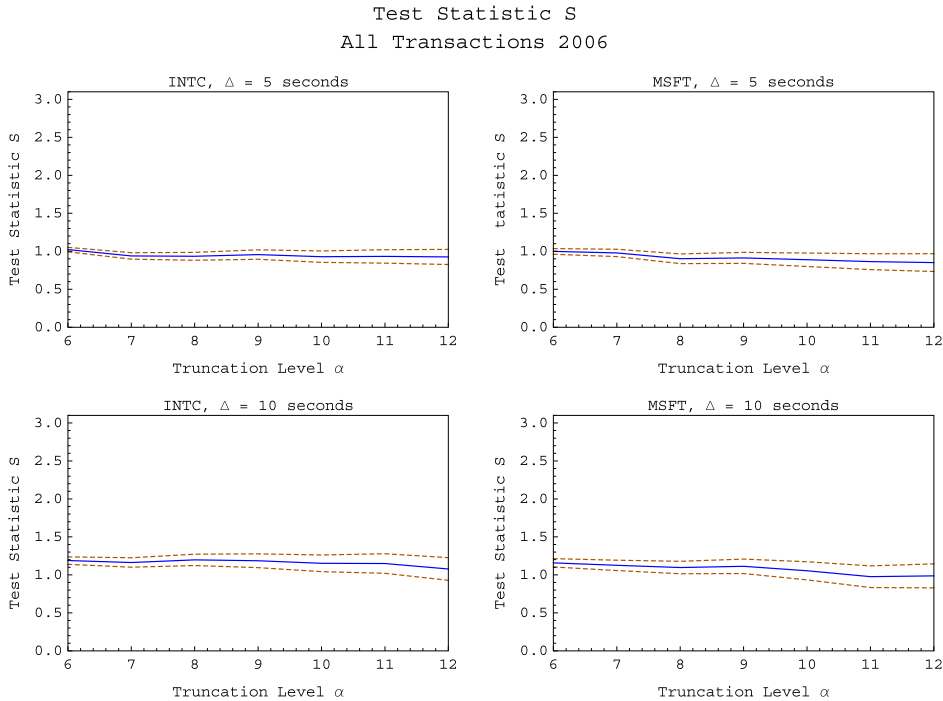


FIG. 11. Values of the test statistic S_n computed for all 2006 transactions of INTC and MSFT, sampled at 5 and 10 second intervals.

5. Empirical results. In this section, we apply our two test statistics to real data, consisting of all transactions recorded during the year 2006 on two of the most actively traded stocks, Intel (INTC) and Microsoft (MSFT). The data source is the TAQ database. Using the correction variables in the data set, we retain only transactions that are labeled “good trades” by the exchanges: regular trades that were not corrected, changed, or signified as canceled or in error; and original trades which were later corrected, in which case the trade record contains the corrected data for the trade. Beyond that, no further adjustment to the data is made.

We first consider the test where the null hypothesis consists of finite jump activity. Figure 11 shows the values of the test statistic S_n , along with a 95% confidence interval computed from the asymptotic distribution (36) under the finite-activity null hypothesis, plotted for a range of values of the truncation index α . The truncation level plays the same role as that of a bandwidth parameter in classical nonparametric estimator, and it is therefore important that it be properly scaled initially. For this purpose, the values of α are indexed in terms of standard deviations of the continuous martingale part of the log-price: we first estimate the volatility of the continuous part of X using the sum of squared increments that are smaller than $\Delta_n^{1/2}$ (meaning that we would retain the increments of up to four standard deviations if the annualized volatility of the stock were 25% per year) and then

use that estimate to form the initial cutoff level used in the construction of the test statistic. To account for potential time series variation in the volatility process σ_t , that procedure is implemented separately for each day and we compute the sum, for that day, of the increments that are smaller than the cutoff, to the appropriate power p required by the test statistic. For the full year, we then add the power variations computed for each day. We then compute the results corresponding to a range of values of u_n indexed by α .

In order to account for the presence of market microstructure noise in the data, we compute the limiting values of our test statistics in the case where the noise is a pure additive noise; the limit is then $1/k$, independent of p . We also consider the dependence of the test statistics as the sampling interval increases, and the signal-to-noise ratio presumably improves.

The test statistic is implemented with $p = 4$ and $k = 2$, using $\Delta_n = 5$ seconds in the upper panels, and $\Delta_n = 10$ seconds in the lower panels. As a result, from Theorem 1, S_n should go to $k^{p/2-1} = 2$ under the null of finite activity, and to 1 under the alternative of infinite activity. As the plots show, we find that S_n is close to 1, which leads us to reject the null hypothesis of finite activity.

Next, we turn to the test where the null hypothesis consists of infinite jump activity. Figure 12 shows the values of the test statistic S'_n , along with a 95% con-

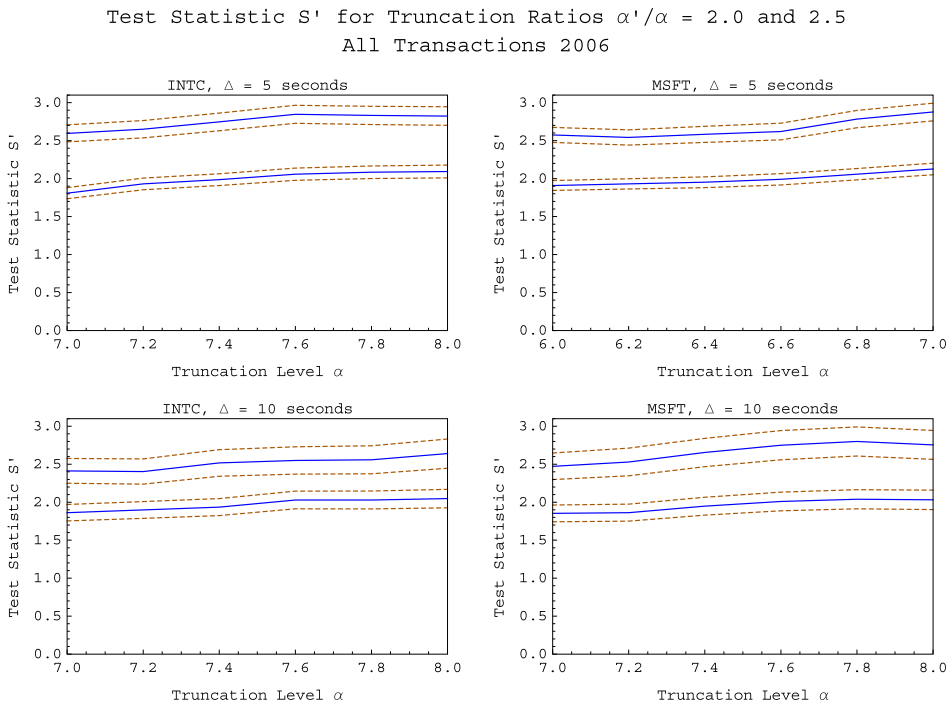


FIG. 12. Values of the test statistic S'_n computed for all 2006 transactions of INTC and MSFT, sampled at 5 and 10 second intervals.

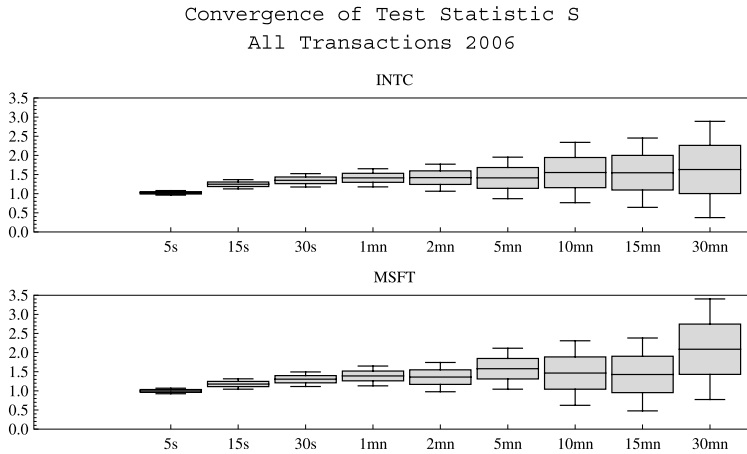


FIG. 13. Box plot for the test statistic S_n computed for all 2006 transactions of INTC and MSFT, sampled at time intervals ranging from 5 seconds to 30 minutes.

fidence interval computed from the asymptotic distribution (43) under the infinite-activity null hypothesis, plotted for a range of values of the truncation index α . The two curves correspond to values of $\gamma = \alpha'/\alpha$ equal to 2.0 and 2.5, respectively. The test statistic is implemented with $p' = 4$ and $p = 3$, using $\Delta_n = 5$ seconds in the upper panels, and $\Delta_n = 10$ seconds in the lower panels. Recall from Theorem 4 that under the null of infinite activity, S'_n should go to $\gamma^{p'-p} = 2.0$ and 2.5, respectively, for the two curves, and to 1 under the alternative of finite activity. We find that S'_n is close to the predicted value $\gamma^{p'-p}$, which leads us to not reject the null hypothesis of infinite activity.

To summarize, the answer from both tests appears indicative of infinite jump activity in those data: using S_n , we reject the null of finite activity, while using S'_n we do not reject the null of infinite activity. Finally, we illustrate in Figure 13 the convergence of S_n to 1 as the sampling interval decreases, indicating that the null hypothesis of finite activity is rejected when high-frequency data (of the order of seconds) are used. On the other hand, we see that using longer sampling intervals (of the order of minutes) makes it impossible to reject the null hypothesis of finite activity using S_n . This is compatible with the fact that small jumps occurring over short time intervals can be aggregated or smoothed out over longer time intervals. For the second testing situation, Figure 14 shows the convergence of S'_n to 2 as the sampling interval decreases from 30 minutes to 5 seconds, indicating that the null hypothesis of infinite activity is not rejected at high frequency. As in the first test, lower frequency data tend to be more compatible with finite jump activity. In both cases, longer sampling intervals Δ_n over the same sampling length T lead to a reduction in the sample size $n = T/\Delta_n$, which generally leads to an increase in the variance of the test statistic, making it more difficult *ceteris paribus* to reject the null hypothesis.

Convergence of Test Statistic S' for Truncation Ratio $\alpha'/\alpha = 2.0$
All Transactions 2006

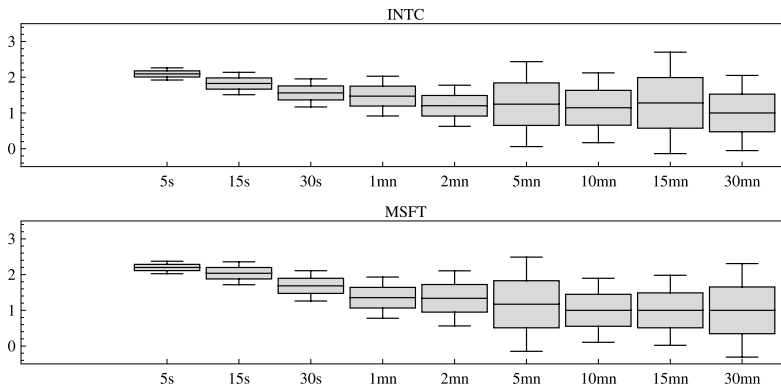


FIG. 14. Box plot for the test statistic S'_n computed for all 2006 transactions of INTC and MSFT, sampled at time intervals ranging from 5 seconds to 30 minutes.

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SUPPLEMENTARY MATERIAL

Supplement to “Testing whether jumps have finite or infinite activity” (DOI: [10.1214/11-AOS873SUPP](https://doi.org/10.1214/11-AOS873SUPP); .pdf). This supplementary article contains a few additional technical details about the assumptions made in this paper, and the proofs of all results.

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