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ESTIMATION FOR LÉVY PROCESSES FROM HIGH FREQUENCY DATA WITHIN A LONG TIME INTERVAL

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In this paper, we study nonparametric estimation of the Lévy density for Lévy processes, with and without Brownian component. For this, we consider n discrete time observations with step Δ. The asymptotic framework is: n tends to infinity, $\Delta = \Delta_n$ tends to zero while $n\Delta_n$ tends to infinity. We use a Fourier approach to construct an adaptive nonparametric estimator of the Lévy density and to provide a bound for the global $\mathbb{L}^2$-risk. Estimators of the drift and of the variance of the Gaussian component are also studied. We discuss rates of convergence and give examples and simulation results for processes fitting in our framework.

1. Introduction. Let $(L_t, t \geq 0)$ be a real-valued Lévy process, that is, a process with stationary independent increments and càdlàg sample paths. The distribution of $(L_t, t \geq 0)$ is completely specified by the characteristic function $\psi_t(u) = \mathbb{E}(\exp iuL_t)$ of the random variable L_t which has the form

$$\psi_t(u) = \exp t\left(iu\tilde{b} - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}/\{0\}} (e^{iux} - 1 - iux1_{|x|\leq 1})N(dx)\right), \tag{1.1}$$

where $\tilde{b} \in \mathbb{R}$, $\sigma^2 \geq 0$ and $N(dx)$ is a positive measure on $\mathbb{R}/\{0\}$ satisfying $\int_{\mathbb{R}/\{0\}} x^2 \wedge 1 N(dx) < \infty$ [see, e.g., Bertoin (1996) or Sato (1999)]. Thus, the statistical problem for Lévy processes is the estimation of its characteristic triple $(\tilde{b}, \sigma^2, N)$ where appears a finite-dimensional parameter $(\tilde{b}, \sigma^2)$ and an infinite-dimensional parameter N, the Lévy measure. In most recent contributions, authors consider a discrete time observation of the sample path, with regular sampling interval Δ. Therefore, statistical procedures are based on the i.i.d. sample composed of the increments $(Z_k = Z_k^\Delta = L_{k\Delta} - L_{(k-1)\Delta}, k = 1, \dots, n)$. In the general case, the distribution of the r.v. Z_k is not explicitly given as a function of $(\tilde{b}, \sigma^2, N)$. This is why authors rather use the relationship between the characteristic function ψ_Δ of Z_k and the characteristic triple. Assuming that $N(dx) = n(x)\,dx$ admits a density, several papers concentrate on the estimation of the Lévy density under various assumptions on the characteristic triple, including the case of $\tilde{b} = \sigma^2 = 0$ or assuming stronger integrability conditions on the Lévy density [see, e.g., Watteel

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and Kulperger (2003), Jongbloed and van der Meulen (2006), van Es, Gugushvili and Spreij (2007), Figueroa-López (2009) and the references therein, Comte and Genon-Catalot (2009, 2010a, 2010b)]. The joint estimation of $(\tilde{b}, \sigma^2, N)$ is investigated in Neumann and Reiss (2009) or Gugushvili (2009). The methods and results differ according to the asymptotic point of view. One may consider that the sampling interval Δ is fixed and that n tends to infinity (low frequency data). This approach, which is quite natural, raises mathematical difficulties and does not take into account the underlying continuous time model properties. One may consider that $\Delta = \Delta_n$ tends to 0 as n tends to infinity (high frequency data). Under the assumption that Δ_n tends to 0 within a fixed length time interval ($n\Delta_n = t$ fixed), the estimation of σ has been widely investigated for Lévy processes [see, e.g., Woerner (2006), Barndorff-Nielsen, Shephard and Winkel (2006), Jacod (2007)]. However, the Lévy density cannot be identified from observations within a finite-length time interval. To identify all parameters in the high-frequency context, one has to assume both that Δ_n tends to 0 and $n\Delta_n$ tends to infinity. This is the point of view adopted in this paper. Our main focus is the nonparametric estimation of the Lévy density $n(\cdot)$ by an adaptive deconvolution method which generalizes the study of Comte and Genon-Catalot (2009). We also study estimators of the other parameters. More precisely, we assume that the Lévy density satisfies

$$\text{(H1)} \qquad \int_{\mathbb{R}} x^2 n(x)\,dx < \infty.$$

For statistical purposes, this assumption, which was proposed in Neumann and Reiss (2009), has several useful consequences. First, for all t, $\mathbb{E}L_t^2 < +\infty$ and as $\int_{\mathbb{R}}(e^{iux} - 1 - iux)n(x)\,dx$ is well defined, we get the following expression for (1.1):

$$\psi_t(u) = \exp t\left(iub - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}}(e^{iux} - 1 - iux)n(x)\,dx\right), \tag{1.2}$$

where $b = \mathbb{E}L_1$ has a statistical meaning (contrary to $\tilde{b}$).

In Section 2, we present our main assumptions and some preliminary properties. In Section 3, we assume that $\sigma = 0$ and study the estimation of the function $h(x) = x^2 n(x)$. Using a sample of size $2n$, we build two collections of estimators $(\hat{h}_m, \bar{h}_m)_{m>0}$ indexed by a cut-off parameter m. The collections are obtained by Fourier inversion of two different estimators of the Fourier transform h^* of the function h. The estimators of h^* are built using empirical estimators of the characteristic function ψ_Δ and its first two derivatives. First, we give a bound for the $\mathbb{L}^2$-risk of $(\hat{h}_m, \bar{h}_m)$ for fixed m. Then, introducing an adequate penalty, we propose a data-driven choice of the cut-off parameter which yields an estimator $(\hat{h}_{\hat{m}}, \bar{h}_{\bar{m}})$ for each collection. The $\mathbb{L}^2$-risk of these estimators is studied. We discuss the rates of convergence reached on Sobolev classes of regularity for the function h. In Section 4, we consider the general case. To reach the Lévy density and get rid of the unknown σ^2, we must now use derivatives of ψ_Δ up to the order 3 and we estimate the function $p(x) = x^3 n(x)$ developing the Fourier inversion approach and adaptive choice of the cut-off parameter as for h. It is worth stressing that the point of view of small sampling interval is crucial to our study. Indeed, it helps obtaining simple estimators of ψ_Δ and its successive derivatives which are used to estimate the Fourier transform p^* of p. Section 5 is devoted to the estimation of (b, σ). We study classical empirical means of the observations. This gives an estimator of b but cannot give estimators of σ. To estimate σ, we consider power variation estimators, introduced in Woerner (2006), Barndorff-Nielsen, Shephard and Winkel (2006), Jacod (2007), Aït-Sahalia and Jacod (2007), under the asymptotic framework of high frequency data within a long time interval. In Section 6, we give examples of Lévy models satisfying our set of assumptions. We provide numerical simulation results in Section 7. Section 8 contains the main proofs. In the Appendix, two classical results, used in proofs, are recalled.

2. Assumptions and preliminary properties. Let us consider the two functions

$$h(x) = x^2 n(x), \qquad p(x) = x^3 n(x),$$

and the assumptions

$$\text{(H2)}(k) \qquad \int_{\mathbb{R}} |x|^k n(x)\, dx < \infty,$$

$$\begin{cases} \text{(H3)} & h \text{ belongs to } \mathbb{L}^2(\mathbb{R}) \\ \text{(H4)} & \int x^8 n^2(x)\, dx = \int x^4 h^2(x)\, dx < \infty \end{cases}$$

or

$$\begin{cases} \text{(H5)} & p \text{ belongs to } \mathbb{L}^2(\mathbb{R}) \\ \text{(H6)} & \int x^{12} n^2(x)\, dx = \int x^6 p^2(x)\, dx < \infty. \end{cases}$$

Assumption (H2)(k) is a moment assumption. Indeed, according to Sato [(1999), Section 5.25, Theorem 5.23], $\mathbb{E}|L_t|^k < \infty$ is equivalent to $\int_{|x|>1} |x|^k n(x)\, dx < \infty$. Below, for each stated result, the required value of k is given. Under (H1), the function h is integrable and Section 3 is devoted to the nonparametric estimation of h under the additional assumptions (H3)–(H4) when $\sigma^2 = 0$. Assumption (H4) is only required for the adaptive result. Under (H1)–(H2)(3), the function p is integrable and Section 4 concerns the estimation of p under (H5)–(H6) when $\sigma^2 \neq 0$.

Properties of the moments of $L_\Delta = Z_1^\Delta = Z_1$ for small Δ are used in the proofs below.

LEMMA 2.1. *Let $k \geq 1$ be an integer and assume* (H1)–(H2)(k) *with* $k = 3$ (*or* $k \geq 3$). *Then*, $\mathbb{E}(|Z_1|^k) < +\infty$ *and* $\mathbb{E}(Z_1) = b\Delta$, $\mathrm{Var}(Z_1) = \Delta(\sigma^2 + \int x^2 n(x)\, dx)$ *and for* $3 \leq \ell \leq k$, $\mathbb{E}(Z_1^\ell) = \Delta c_\ell + o(\Delta)$ *where* $c_\ell = \int x^\ell n(x)\, dx$.

Thus, under (H1), (H2)(k), $\mathbb{E}(Z_1^\ell/\Delta)$ is bounded for all $\ell \le k$, for all Δ.

In the sequel, results on the behavior of the characteristic function ψ_Δ [see (1.2)] for small Δ are needed.

LEMMA 2.2. *Under* (H1), $|\psi_\Delta(u) - 1| \le \Delta|u|(c(u) + \sigma^2|u|)$ *where* $c(u) = |b| + |\int_0^u |h^*(v)|\,dv|$, $h^*(v) = \int e^{ivx}h(x)\,dx$ *denotes the Fourier transform of* h. *If* h^* *is integrable on* $\mathbb{R}$, *then*

$$|\psi_\Delta(u) - 1| \le \Delta|u|(|b| + |h^*|_1 + |u|\sigma^2).$$

PROOF. By formula (1.2), under (H1), ψ_Δ is C^1 with $\psi'_\Delta(u) = \Delta\psi_\Delta(u) \times (\phi(u) - \sigma^2 u)$, where we have set, using that $e^{iux} - 1 = ix\int_0^u e^{ivx}\,dv$,

$$\phi(u) = ib - \int_0^u h^*(v)\,dv. \tag{2.1}$$

We have $|\phi(u)| \le |b| + |\int_0^u |h^*(v)|\,dv|$ and by the Taylor formula, $\psi_\Delta(u) - 1 = u\psi'_\Delta(c_u u)$ for some $c_u \in (0, 1)$. The result follows. $\square$

3. Case of no Gaussian component. In this section, we consider the case $\sigma^2 = 0$ and focus on the nonparametric estimation of h. For reasons that will appear below, we suppose that we have at our disposal a $2n$-sample, $(Z_k)_{1\le k\le 2n}$, with $Z_k = Z_k^\Delta = L_{k\Delta} - L_{(k-1)\Delta}$. We assume that $\Delta = \Delta_n$ tends to 0 and $n\Delta_n$ tends to infinity. Hence, Δ and Z_k depend on n. However, to simplify notation, we omit the dependence on n and simply write Δ, Z_k.

3.1. *Definition of estimators depending on a cut-off parameter.* For a complex valued function f belonging to $\mathbb{L}^1(\mathbb{R})$, we denote its Fourier transform by $f^*(u) = \int e^{iux} f(x)\,dx$. For integrable and square integrable functions f, f_1, f_2, we use the following notation:

$$\|f\| = \int |f(x)|^2\,dx, \qquad \langle f_1, f_2\rangle = \int f_1(x)\bar{f}_2(x)\,dx$$

($\bar{z}$ denotes the conjugate of the complex number z). We have: $(f^*)^*(x) = 2\pi f(-x)$ and $\langle f_1, f_2\rangle = 1/(2\pi)\langle f_1^*, f_2^*\rangle$.

By formula (1.2), under (H1), ψ_Δ is C^2 and we have, as $\sigma^2 = 0$ [see (2.1)];

$$\frac{\psi'_\Delta(u)}{\psi_\Delta(u)} = i\Delta\left(b + \int \frac{e^{iux} - 1}{x} h(x)\,dx\right) = \Delta\phi(u).$$

Derivating again gives

$$h^*(u) = -\frac{1}{\Delta}\left(\frac{\psi''_\Delta(u)\psi_\Delta(u) - (\psi'_\Delta(u))^2}{\psi_\Delta^2(u)}\right), \tag{3.1}$$

where, for all u, $\lim_{\Delta\to 0}\psi_\Delta(u)=1$. By splitting the $2n$-sample into two independent subsamples of n observations, we introduce the following empirical unbiased estimators of $\psi_\Delta, \psi'_\Delta, \psi''_\Delta$:

$$\hat{\psi}^{(j)}_{\Delta,q}(u)=\frac{1}{n}\sum_{k=1+(q-1)n}^{qn}(iZ_k)^j e^{iuZ_k}, \qquad j=0,1,2, q=1,2.$$

We also define, based on the full sample, the estimator of ψ''_Δ

$$\hat{\psi}^{(2)}_{\Delta}(u)=\frac{1}{2n}\sum_{k=1}^{2n}(iZ_k)^2 e^{iuZ_k}.$$

We now build estimators of the Fourier transform h^* of h. Considering the expression of h^* in (3.1), we replace $\psi_\Delta, \psi'_\Delta, \psi''_\Delta$ in the numerator by the empirical estimators built on the two independent subsamples of size n. In the denominator, ψ^2_Δ is simply replaced by 1. This yields

$$\hat{h}^*(u)=\frac{1}{\Delta}\big(\hat{\psi}^{(1)}_{\Delta,1}(u)\hat{\psi}^{(1)}_{\Delta,2}(u)-\hat{\psi}^{(2)}_{\Delta,1}(u)\hat{\psi}^{(0)}_{\Delta,2}(u)\big). \tag{3.2}$$

Hence, using independence of the two subsamples,

$$\mathbb{E}\hat{h}^*(u)=\frac{1}{\Delta}\big((\psi'_\Delta(u))^2-\psi''_\Delta(u)\psi_\Delta(u)\big)=h^*(u)+h^*(u)\big(\psi^2_\Delta(u)-1\big).$$

Introducing a cut-off parameter m, we define an associated estimator of h

$$\hat{h}_m(x)=\frac{1}{2\pi}\int_{-\pi m}^{\pi m}e^{-iux}\hat{h}^*(u)\,du.$$

This means that $\hat{h}^*_m(u)=\hat{h}^*(u)1_{[-\pi m,\pi m]}(u)$. By integration, the following expression is available:

$$\hat{h}_m(x)=\frac{1}{n^2\Delta}\sum_{1\le j,k\le n}(Z_k^2-Z_kZ_{n+j})\frac{\sin(\pi m(Z_k+Z_{j+n}-x))}{\pi(Z_k+Z_{j+n}-x)}.$$

We also define another estimator of h^* of h by setting

$$\bar{h}^*(u)=-\frac{1}{\Delta}\hat{\psi}^{(2)}_\Delta(u). \tag{3.3}$$

Here, using (3.1), we get

$$\mathbb{E}\bar{h}^*(u)=-\frac{1}{\Delta}\psi''_\Delta(u)=h^*(u)+h^*(u)\big(\psi_\Delta(u)-1\big)-\Delta\psi_\Delta(u)\phi^2(u). \tag{3.4}$$

Thus, $\bar{h}^*$ is simpler but has an additional bias term. We set

$$\bar{h}_m(x)=\frac{1}{2\pi}\int_{-\pi m}^{\pi m}e^{-iux}\bar{h}^*(u)\,du=\frac{1}{2n\Delta}\sum_{k=1}^{2n}Z_k^2\frac{\sin(\pi m(Z_k-x))}{\pi(Z_k-x)}. \tag{3.5}$$

3.2. *Risk for a fixed cut-off parameter.* Next, let us define

$$h_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux} h^*(u)\,du.$$

Then we can prove the following result.

PROPOSITION 3.1. *Assume that* (H1)–(H2)(4) *and* (H3) *hold. Then*

$$\mathbb{E}(\|\hat{h}_m - h\|^2) \le \|h_m - h\|^2 + 72\mathbb{E}(Z_1^4/\Delta)\frac{m}{n\Delta} + \frac{4\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2 c^2(u)|h^*(u)|^2\,du, \tag{3.6}$$

$$\mathbb{E}(\|\bar{h}_m - h\|^2) \le \|h_m - h\|^2 + \mathbb{E}(Z_1^4/\Delta)\frac{m}{n\Delta} + \frac{2\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2 c^2(u)|h^*(u)|^2\,du + C\Delta^2 B_m, \tag{3.7}$$

with C a constant, $c(u)$ is defined in Lemma 2.2, $B_m = (2/\pi)\int_{-\pi m}^{\pi m} |\phi(u)|^4\,du$ [see (2.1)] satisfies $B_m = O(m)$ if $h^* \in \mathbb{L}_1(\mathbb{R})$ and $B_m = O(m^5)$ otherwise.

REMARK 3.1. We stress that the estimator $\hat{h}_m$ is more complicated to study, but $\bar{h}_m$ has an additional bias term.

3.3. *Rates of convergence in Sobolev classes.* The following result concerns classes of functions h belonging to

$$\mathcal{C}(a, L) = \left\{ f \in (\mathbb{L}^1 \cap \mathbb{L}^2)(\mathbb{R}), \int (1+u^2)^a |f^*(u)|^2\,du \le L \right\}. \tag{3.8}$$

PROPOSITION 3.2. *Assume that* (H1)–(H2)(4) *and* (H3) *hold and that h belongs to $\mathcal{C}(a, L)$ with $a > 1/2$. Consider the asymptotic setting where $n \to +\infty$, $\Delta \to 0$, $n\Delta \to +\infty$ and assume that $m \le n\Delta$. If $n\Delta^2 \le 1$, then, for the choice $m = O((n\Delta)^{1/(2a+1)})$, we have*

$$\mathbb{E}(\|\hat{h}_m - h\|^2) \le O\big((n\Delta)^{-2a/(2a+1)}\big).$$

If $a \ge 1$, the condition $n\Delta^2 \le 1$ can be replaced by $n\Delta^3 \le 1$. The same result holds for $\bar{h}_m$.

REMARK 3.2. We can also discuss the case where $a \in (0, 1/2]$. If $a \le 1/2$, $|\int_0^u |h^*(v)|\,dv| = O(|u|^{1/2-a})$. Hence, the last term in (3.6) is of order $\Delta^2 m^{3-4a}$ which is less than m^{-2a} if $\Delta^2 m^{3-2a} \le 1$ and thus $\Delta^2 m^3 \le 1$. This requires $n\Delta^{5/3} \le 1$. The same holds for $\bar{h}_m$.

Note that no lower bound result is available for this problem. A benchmark for comparison could be the problem of density estimation for i.i.d. observations without noise: if the density f belongs to $\mathcal{C}(a, L)$, the optimal minimax rate is of order $O(n^{-2a/(2a+1)})$ [see Ibragimov and Khas'minskij (1980)].

3.4. *Model selection.* The estimators $\hat{h}_m, \bar{h}_m$ are deconvolution estimators that can also be described as minimum contrast estimators and projection estimators. For details, the reader is referred to Comte and Genon-Catalot (2009, 2010b). For $m > 0$, let

$$S_m = \{f \in \mathbb{L}^2(\mathbb{R}), \operatorname{support}(f^*) \subset [-\pi m, \pi m]\}.$$

The space S_m is generated by an orthonormal basis, the sinus cardinal basis, defined by

$$\varphi_{m,j}(x) = \sqrt{m}\varphi(mx - j), \ j \in \mathbb{Z}, \qquad \varphi(x) = \frac{\sin \pi x}{\pi x} \ \big(\varphi(0) = 1\big).$$

This is due to the fact that $\varphi^*_{m,j}(u) = (e^{iuj/m}/\sqrt{m})1_{[-\pi m, \pi m]}(u)$, $j \in \mathbb{Z}$. For a function $f \in \mathbb{L}^2(\mathbb{R})$, $f_m(x) = (2\pi)^{-1} \int_{-\pi m}^{\pi m} e^{-iux} f^*(u)\,du$ is the orthogonal projection of f on S_m. Introducing, for a function $t \in S_m$,

$$\gamma_n(t) = \|t\|^2 - \frac{1}{\pi}\langle \hat{h}^*, t^*\rangle = \|t\|^2 - 2\langle \hat{h}_m, t\rangle,$$

we get

$$\hat{h}_m = \arg\min_{t \in S_m} \gamma_n(t),$$

and $\gamma_n(\hat{h}_m) = -\|\hat{h}_m\|^2$. We have

$$\hat{h}_m = \sum_{j \in \mathbb{Z}} \hat{a}_{m,j}\varphi_{m,j} \qquad \text{with } \hat{a}_{m,j} = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} \hat{h}^*(u)\varphi^*_{m,j}(-u)\,du$$

and $\|\hat{h}_m\|^2 = 1/(2\pi)\int_{-\pi m}^{\pi m} |\hat{h}^*(u)|^2\,du$. The coefficients $\hat{a}_{m,j}$ of the series as well as $\|\hat{h}_m\|^2$ can be explicitly computed by integration. In the same way, we set

$$\Gamma_n(t) = \|t\|^2 - \frac{1}{\pi}\langle \bar{h}^*, t^*\rangle = \|t\|^2 - 2\langle \bar{h}_m, t\rangle,$$

and obtain

$$\bar{h}_m = \arg\min_{t \in S_m} \Gamma_n(t).$$

Analogously, $\bar{h}_m$ has a series expansion on the sinus cardinal basis with explicit coefficients and $\|\bar{h}_m\|^2$ has a closed-form formula. We give the explicit expression of $\|\bar{h}_m\|^2$ which is less cumbersome than $\|\hat{h}_m\|^2$:

$$\|\bar{h}_m\|^2 = \frac{m}{4n^2\Delta^2} \sum_{1 \le k,l \le 2n} Z_k^2 Z_l^2 \varphi\big(m(Z_k - Z_l)\big). \tag{3.9}$$

Now, we need to select the best m as possible, in a set $\mathcal{M}_n = \{m \in \mathbb{N}, 1 \le m \le n\Delta\} = \{1, \dots, m_n\}$. For the estimators $\hat{h}_m$, we propose to take

$$\hat{m} = \arg\min_{m\in\mathcal{M}_n} \left(-\|\hat{h}_m\|^2 + \mathrm{pen}(m)\right) \tag{3.10}$$

with

$$\mathrm{pen}(m) = \kappa \frac{m}{n\Delta^2}\left(\left(\frac{1}{n}\sum_{k=1}^{n} Z_k^2\right)\left(\frac{1}{n}\sum_{k=n+1}^{2n} Z_k^2\right) + \frac{1}{n}\sum_{k=1}^{n} Z_k^4\right).$$

The intuition for this choice is the following. The expression of $\mathrm{pen}(m)$ is an estimator of the variance term of the risk bound (3.6) as close as possible of the variance [see (8.2)]. The term $-\|\hat{h}_m\|^2$ is an estimator of $-\|h_m\|^2 = \|h-h_m\|^2 - \|h\|^2$, which is up to a constant, the bias term of the bound (3.6). This is why $\hat{m}$ mimics the optimal bias-variance compromise.

For the estimators $\bar{h}_m$, we define

$$\bar{m} = \arg\min_{m\in\mathcal{M}_n}\left(-\|\bar{h}_m\|^2 + \kappa' \frac{m}{n\Delta^2}\left(\frac{1}{2n}\sum_{k=1}^{2n} Z_k^4\right)\right). \tag{3.11}$$

The following result shows that the above data-driven choices of the cut-off parameter lead to an automatic optimization of the risk.

THEOREM 3.1. *Assume* (H1)–(H2)(16)–(H3)–(H4). *If, moreover,* $h^* \in \mathbb{L}^1(\mathbb{R})$ *and* $n\Delta^3 \le 1$, *there exist numerical constants* κ, κ' *such that*

$$\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2) \le C \inf_{m\in\mathcal{M}_n}\left(\|h-h_m\|^2 + \kappa\left(\Delta\mathbb{E}^2\left(\frac{Z_1^2}{\Delta}\right) + \mathbb{E}\left(\frac{Z_1^4}{\Delta}\right)\right)\frac{m}{n\Delta}\right)$$
$$+ \frac{\Delta^2}{\pi}\int_{-\pi m_n}^{\pi m_n} u^2|h^*(u)|^2\,du + C\frac{\ln^2(n\Delta)}{n\Delta},$$

$$\mathbb{E}(\|\bar{h}_{\bar{m}} - h\|^2) \le C \inf_{m\in\mathcal{M}_n}\left(\|h-h_m\|^2 + \kappa'\mathbb{E}\left(\frac{Z_1^4}{\Delta}\right)\frac{m}{n\Delta}\right)$$
$$+ \frac{\Delta^2}{\pi}\int_{-\pi m_n}^{\pi m_n} u^2|h^*(u)|^2\,du + \Delta^2 B_{m_n} + C\frac{\ln^2(n\Delta)}{n\Delta},$$

where $B_{m_n} = O(m_n)$ (B_{m_n} *is defined in Proposition* 3.1).

The numerical constants κ, κ' have to be calibrated via simulations [see discussion in Comte and Genon-Catalot (2009)].

By computations analogous to those in the proof of Proposition 3.2, we obtain the following corollary.

COROLLARY 3.1. *Assume that the assumptions of Theorem* 3.1 *are fulfilled. If, for some positive* L, $h \in \mathcal{C}(a, L)$ *with* $a > 1/2$, *then* $\mathbb{E}(\|\hat{h}_{\hat{m}} - h\|^2) = O((n\Delta)^{-2a/(2a+1)})$ *provided that* $n\Delta^2 \le 1$. *The same holds for* $\mathbb{E}(\|\bar{h}_{\bar{m}} - h\|^2)$. *If* $a \ge 1$, *the constraint* $n\Delta^3 \le 1$ *is enough.*

4. Study of the general case ($\sigma^2 \neq 0$). In this section, we assume (H1)–(H2)(3) and study the estimation of the function

$$p(x) = x^3 n(x).$$

We suppose that we have a sample of size n, $(Z_k)_{1\le k\le n}$, $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$.

4.1. *Definition of the estimators.* We compute the three first derivatives of ψ_Δ [see (2.1)]:

$$\frac{\psi'_\Delta(u)}{\psi_\Delta(u)} = \Delta\left(ib - u\sigma^2 + i\int \frac{e^{iux}-1}{x} h(x)\,dx\right) = \Delta\big(\phi(u) - u\sigma^2\big).$$

Derivating again gives

$$\frac{\psi''_\Delta(u)\psi_\Delta(u) - (\psi'_\Delta(u))^2}{(\psi_\Delta(u))^2} = \Delta\big(\phi'(u) - \sigma^2\big) = -\Delta\left(\sigma^2 + \int e^{iux} x^2 n(x)\,dx\right),$$

and last

$$p^*(u) = \frac{i}{\Delta}\left(\frac{\psi^{(3)}_\Delta(u)}{\psi_\Delta(u)} - 3\frac{\psi''_\Delta(u)\psi'_\Delta(u)}{\psi^2_\Delta(u)} + 2\frac{[\psi'_\Delta(u)]^3}{\psi^3_\Delta(u)}\right).$$

Let

$$\bar{p}^*(u) = \frac{i}{\Delta}\hat{\psi}^{(3)}_\Delta(u) \qquad \text{with } \hat{\psi}^{(3)}_\Delta(u) = \frac{1}{n}\sum_{k=1}^n (iZ_k)^3 e^{iuZ_k}.$$

Then

$$\bar{p}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-iux}\bar{p}^*(u)\,du = \frac{1}{n\Delta}\sum_{k=1}^n Z_k^3 \frac{\sin(\pi m(Z_k - x))}{\pi(Z_k - x)}. \tag{4.1}$$

Let us set

$$\tilde{\phi}(u) = \phi(u) - u\sigma^2 = ib - \int_0^u h^*(v)\,dv - u\sigma^2. \tag{4.2}$$

Using $\psi'_\Delta(u) = \Delta\psi_\Delta(u)\tilde{\phi}(u)$ and some computations, we get

$$\begin{aligned}\mathbb{E}\bar{p}^*(u) - p^*(u) = {} & \big(\psi_\Delta(u) - 1\big)p^*(u) - 3i\Delta\psi_\Delta(u)\tilde{\phi}(u)\big(\sigma^2 + h^*(u)\big) \\ & + i\Delta^2\psi_\Delta(u)(\tilde{\phi}(u))^3.\end{aligned} \tag{4.3}$$

REMARK 4.1. By a method analogous to the one used for h, considering a sample of size $3n$, we can build another estimator of p^* which is less biased but more complicated to study.

4.2. *Risk of the estimators*. The risk of the estimator with fixed cut-off parameter is bounded as follows.

PROPOSITION 4.1. *Under* (H1)–(H2)(6) *and* (H5),

$$\begin{aligned}\mathbb{E}(\|\bar{p}_m - p\|^2) &\le \|p - p_m\|^2 + \mathbb{E}(Z_1^6/\Delta)\frac{m}{n\Delta} \\ &\quad + C\left(\Delta^2 \int_{-\pi m}^{\pi m} u^2(1+u^2)|p^*(u)|^2\,du + \Delta^2 m^3 + \Delta^4 m^7\right),\end{aligned} \tag{4.4}$$

where $p_m(x) = (2\pi)^{-1}\int_{-\pi m}^{\pi m} e^{-iux} p^*(u)\,du$ *denotes the orthogonal projection of* p *on* S_m.

We can state the result analogous to the one of Proposition 3.2.

PROPOSITION 4.2. *Assume that* (H1), (H2)(6), (H5) *hold and that* p *belongs to* $\mathcal{C}(a, L)$. *Consider the asymptotic setting where* $n \to +\infty$, $\Delta \to 0$ *and* $n\Delta \to +\infty$. *If* $n\Delta^{11/7} \le 1$, *then*

$$\mathbb{E}(\|\bar{p}_m - p\|^2) \le O\big((n\Delta)^{-2a/(2a+1)}\big).$$

If $a \ge 1/2$, *the condition* $n\Delta^{7/5} \le 1$ *can be replaced by* $n\Delta^2 \le 1$.

4.3. *Model selection strategy.* The data driven selection of the best possible m imposes here a restricted collection of models. We choose $M_n = \{m \in \mathbb{N}/\{0\}, m \le \sqrt{n\Delta} := \mu_n\}$.

We can consider the estimator $\bar{p}_{\bar{m}}$ where

$$\begin{aligned}\bar{m} &= \arg\min_{m\in M_n}\big(-\|\bar{p}_m\|^2 + \overline{\mathrm{pen}}(m)\big) \\ &\quad \text{with } \overline{\mathrm{pen}}(m) = \kappa'\frac{m}{n\Delta^2}\left(\frac{1}{n}\sum_{k=1}^{n} Z_k^6\right).\end{aligned} \tag{4.5}$$

We can prove the following result.

THEOREM 4.1. *Under assumptions* (H1), (H2)(24), (H5), (H6) *and with* $n\Delta^2 \le 1$, *there exists a numerical constant* κ *such that* (*with* $\mu_n = \sqrt{n\Delta}$)

$$\begin{aligned}&\mathbb{E}(\|\bar{p}_{\bar{m}} - p\|^2) \\ &\quad \le C\inf_{m\in M_n}\left(\|p - p_m\|^2 + \kappa'\mathbb{E}\left(\frac{Z_1^6}{\Delta}\right)\frac{m}{n\Delta}\right) \\ &\qquad + C\left(\frac{\Delta^2}{\pi}\int_{-\pi\mu_n}^{\pi\mu_n} u^2(1+u^2)|p^*(u)|^2\,du + \Delta^2\mu_n^3 + \Delta^4\mu_n^7 + \frac{\ln^2(n\Delta)}{n\Delta}\right).\end{aligned}$$

The consequence of Theorem 4.1 is that the adaptive estimators reach automatically the expected rate of convergence when p belongs to a Sobolev class. This can be seen by computations analogous to those of Proposition 4.2.

5. Parameter estimation. Under (H1), the observed process may be written as $L_t = bt + \sigma W_t + X_t$ where (W_t) is a standard Brownian motion, (X_t) is a Lévy process, independent of (W_t), of the form

$$X_t = \int_{]0,t]} \int_{\mathbb{R}/\{0\}} x\big(\hat{p}(ds, dx) - ds\, n(x)\, dx\big),$$

where $\hat{p}(ds, dx)$ is the random jump measure of (L_t) [and (X_t)].

If moreover $\int |x| n(x)\, dx < \infty$, then $L_t = b_0 t + \sigma W_t + \Gamma_t$ where $b_0 = b - \int x n(x)\, dx$ and

$$\Gamma_t = \int_{]0,t]} \int_{\mathbb{R}} x \hat{p}(ds, dx) = X_t + t \int x n(x)\, dx = \sum_{s \le t} \Gamma_s - \Gamma_{s_-}$$

is of bounded variation on compact sets. We consider here a sample of size n. By using empirical means of the data Z_k^ℓ, it is possible to obtain consistent and asymptotically Gaussian estimators of b ($\ell = 1$) and, under suitable integrability assumptions on the Lévy density, of $\int x^\ell n(x)\, dx$ for $\ell \ge 3$. But this method fails to estimate σ for $\ell = 2$ (see below). For this, one has to use another approach based on power variations.

5.1. *Some small time properties.* To study estimators of b and σ, small time properties of moments of L_Δ are needed. For simple moments, the result is stated in Lemma 2.1. For absolute moments, we refer, for example, to Figueroa-López (2008): if $\int_{\{|x|>1\}} |x|^r n(x)\, dx < +\infty$, and $r > 2$, $\Delta^{-1}\mathbb{E}(|L_\Delta|^r) \to \int |x|^r n(x)\, dx$ as $\Delta \to 0$. For the case of $|x|^r$ with $r < 2$, we state the following proposition.

PROPOSITION 5.1. (i) *Let (Γ_t) be a Lévy process with no continuous component and Lévy measure $n(\gamma)\, d\gamma$. If $\int |\gamma| n(\gamma)\, d\gamma < \infty$, $b = \int \gamma n(\gamma)\, d\gamma$ and for $r \le 1$, $\int |\gamma|^r n(\gamma)\, d\gamma < \infty$. There exists a constant C such that, for all Δ, $\mathbb{E}|\Gamma_\Delta|^r \le C\Delta$. [Under the assumption, (Γ_t) has finite mean and bounded variation on compact sets.]*

(ii) *Let $X_t = B_{\Gamma_t}$ where (Γ_t) is a subordinator with Lévy density n_Γ satisfying $b = \int_0^{+\infty} \gamma n_\Gamma(\gamma)\, d\gamma < \infty$ and (B_t) is a Brownian motion independent of (Γ_t). The Lévy measure of (X_t) has a density given by*

$$n_X(x) = \int_0^{+\infty} e^{-x^2/2\gamma} \frac{1}{\sqrt{2\pi\gamma}} n_\Gamma(\gamma)\, d\gamma. \tag{5.1}$$

Consequently, if $C = \int_0^{+\infty} \gamma^{r/2} n_\Gamma(\gamma)\, d\gamma < \infty$ with $r \le 2$, then $\mathbb{E}|X_\Delta|^r \le C\Delta$.

(iii) *Let (X_t) be a Lévy process with no Gaussian component. Then $X_\Delta/\sqrt{\Delta}$ converges to 0 as Δ tends to 0 in probability and in $\mathbb{L}^r$ for all $r < 2$.*

5.2. *Estimator of b*. Consider a Lévy process (L_t) satisfying (H1) and set $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$ as above. Let us define the empirical means

$$\hat{b} = \frac{1}{n\Delta}\sum_{k=1}^{n} Z_k, \qquad \hat{c}_\ell = \frac{1}{n\Delta}\sum_{k=1}^{n} Z_k^\ell \qquad \text{for } \ell \geq 2. \tag{5.2}$$

We prove now that $\hat{b}$, $\hat{c}_\ell$, $\ell \geq 2$ are consistent and asymptotically Gaussian estimators of the quantities b, c_ℓ, $\ell \geq 2$ where

$$c_2 = \sigma^2 + \int x^2 n(x)\,dx, \qquad c_\ell = \int x^\ell n(x)\,dx \qquad \text{for } \ell \geq 3.$$

PROPOSITION 5.2. *Assume* (H1) *and n tends to infinity, Δ tends to* 0, *$n\Delta$ tends to infinity.*

(i) *Under* (H2)$(2+\varepsilon)$ *for some positive ε,*

$$\sqrt{n\Delta}(\hat{b} - b) \text{ converges in distribution to } \mathcal{N}(0, c_2).$$

(ii) *Under* (H2)$(2(\ell+\varepsilon))$ *for some positive ε, and if $n\Delta^3$ tends to* 0, *$\sqrt{n\Delta}(\hat{c}_\ell - c_\ell)$ converges in distribution to $\mathcal{N}(0, c_{2\ell})$.*

We stress that this method provides an estimator of b which is easy to compute and very good in practice (see Section 7), but cannot provide an estimator of σ^2.

5.3. *Estimation of σ with power variations*. Estimators of σ based on power variations of (L_t) have been proposed and mostly studied in the case where $n\Delta = 1$. They are studied for high frequency data within a long time interval in Aït-Sahalia and Jacod (2007). In the latter paper, the context is more general than ours, which implies that proofs are of high complexity. For Lévy processes fitting in our set of assumptions, we can derive the asymptotic properties of power variations estimators with a specific proof given in Section 8. Consider the family of estimators of σ given by

$$\hat{\sigma}(r) = \big[\hat{\sigma}_n^{(r)}\big]^{1/r} \qquad \text{with } \hat{\sigma}_n^{(r)} = \frac{1}{m_r n\Delta^{r/2}}\sum_{k=1}^{n} |Z_k|^r, \tag{5.3}$$

where $m_r = \mathbb{E}|X|^r$ for X a standard Gaussian variable (recall that $Z_k = L_{k\Delta} - L_{(k-1)\Delta}$).

PROPOSITION 5.3. *As n tends to infinity, Δ tends to* 0 *and $n\Delta$ tends to infinity, if $n\Delta^{2-r} = o(1)$, $\sqrt{n}(\hat{\sigma}_n^{(r)} - \sigma^r)$ converges in distribution to a $\mathcal{N}(0, \sigma^{2r}(m_{2r}/m_r^2 - 1))$ for*:

(i) *(L_t) a Lévy process satisfying* (H1) *and such that $\int |x| n(x)\,dx < \infty$ and $\int |x|^r n(x)\,dx < \infty$ for $r < 1$.*

(ii) ($L_t = bt + \sigma W_t + X_t$), *with* $X_t = B_{\Gamma_t}$, *where* W, B, Γ *are independent processes*, W, B *are Brownian motions*, Γ *is a subordinator with Lévy measure* n_Γ *satisfying* $b = \int_0^{+\infty} \gamma n_\Gamma(\gamma)\, d\gamma < \infty$ *and* $\int_0^{+\infty} \gamma^{r/2} n_\Gamma(\gamma)\, d\gamma < \infty$ *for* $r < 1$.

Consequently, $\sqrt{n}(\hat{\sigma}(r) - \sigma)$ *converges in distribution to a* $\mathcal{N}(0, (\sigma^2/r^2)(m_{2r}/m_r^2 - 1))$.

For other cases of Lévy processes, the result depends on the rate of convergence to 0 of $\mathbb{E}|X_\Delta|^r/\Delta^{r/2}$ [see Proposition 5.1(iii)] and will still hold if $\sqrt{n\Delta}\mathbb{E}|X_\Delta|^r/\Delta^{r/2}$ tends to 0.

REMARK 5.1. It is worth noting that the rate of convergence is $\sqrt{n}$. For $r = 1$, the estimator $\hat{\sigma}_n^{(1)}$ is consistent but not asymptotically Gaussian (because of its asymptotic bias). We have implemented these estimators for $r = 1/2$, $r = 1/4$ (see Section 7) for processes satisfying $\int |x|^r n(x)\, dx < +\infty$ for all positive r. Note that we always give integrability conditions on $\mathbb{R}$ for the Lévy density. This simplifies the presentation but induces some redundancies. One should distinguish integrability conditions near 0 and near infinity to avoid them.

6. Examples. In this section, we give examples of models fitting in our framework.

EXAMPLE 1. Drift + Brownian motion + Compound Poisson process.

Let

$$L_t = b_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{6.1}$$

where N_t is a Poisson process with constant intensity c and Y_i is a sequence of i.i.d. random variables with density f, independent of the process (N_t). Then, $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process and (L_t) is a Lévy process with Lévy density $n(x) = cf(x)$. Note that $\mathbb{E}L_1 = b = b_0 + \int xn(x)\, dx$. For the estimation of p, the rates that can be obtained depend on the density f provided that f satisfies the assumptions of Theorem 4.1, which are essentially here moment assumptions for the r.v.'s Y_i. Any order can be obtained as shown in Table 1 where rates are computed for f a standard Gaussian, an exponential with parameter 1 and a Beta distribution with parameters $(1, 3)$ (for p to be regular enough).

As $\int |x|^r n(x)\, dx < \infty$ for all $r < 1$ (actually, for all $r \leq 2$), estimation of σ is possible using $\hat{\sigma}(r)$ for any value of $0 < r < 1$ [provided that $n\Delta^{2-r} = o(1)$].

EXAMPLE 2. Drift + Brownian motion + Lévy–Gamma process.

Consider $L_t = b_0 t + \sigma W_t + \Gamma_t$ where (Γ_t) is a Lévy gamma process with parameters (β, α), that is, is a subordinator such that, for all $t > 0$, Γ_t has distribution Gamma with parameters $(\beta t, \alpha)$ and density: $\alpha^{\beta t} x^{\beta t - 1} e^{-\alpha x}/\Gamma(\beta t)\mathbb{1}_{x \geq 0}$.

TABLE 1
Rates for different "Drift + Brownian motion + Compound Poisson processes"

$f(x)$	$\mathcal{N}(0,1)$	$\mathcal{E}(1)$	$\beta(1,3)$
$p(x) = cx^3 f(x)$	$\propto x^3 e^{-x^2}$	$\propto x^3 e^{-x} \mathbb{1}_{x>0}$	$\propto x^3 (1-x)^2 \mathbb{1}_{[0,1]}(x)$
$p^*(u)$	$\propto (u^3 - 3u) e^{-u^2/2}$	$\propto 1/(1-iu)^4$	$O(1/\vert u\vert^3)$ for large $\vert u\vert$
$\int_{\vert u\vert \geq \pi m} \vert p^*(u)\vert^2 \, du$	$O((\pi m)^5 e^{-(\pi m)^2})$	$O((\pi m)^{-7})$	$O((\pi m)^{-5})$
$\int_{\vert u\vert \leq \pi \mu_n} u^4 \vert p^*(u)\vert^2 \, du$	$O(1)$	$O(1)$	$O(1)$
$\breve{m}$ (best choice of m)	$\sqrt{\log(n\Delta) - \frac{5}{2} \log\log(n\Delta)}/\pi$	$O((n\Delta)^{1/8})$	$O((n\Delta)^{1/6})$
Rate $\propto$	$\frac{\sqrt{\log(n\Delta)}}{n\Delta}$	$(n\Delta)^{-7/8}$	$(n\Delta)^{-5/6}$

The Lévy density of (L_t) is $n(x) = \beta x^{-1} e^{-\alpha x} \mathbb{1}_{x>0}$. We have $\mathbb{E}L_1 = b = b_0 + \int x n(x)\, dx$ and $p(x) = \beta x^2 e^{-\alpha x} \mathbb{1}_{x>0}$.

We find $p^*(u) = 2\beta/(\alpha - iu)^3$, $\int_{|u|\geq \pi m} |p^*(u)|^2 \, du = O(m^{-5})$ and $\int_{-\pi\mu_n}^{\pi\mu_n} u^4 |p^*(u)|^2 \, du = O(1)$. Therefore, the rate for estimating p is $O((n\Delta)^{-5/6})$ for a choice $\breve{m} = O((n\Delta)^{1/6})$.

As for all $r > 0$, $\int x^r n(x)\, dx < \infty$, $\hat{\sigma}(r)$ is authorized, for any value of $0 < r < 1$, to estimate σ.

EXAMPLE 2 (Continued). Drift + Brownian motion + A specific class of subordinators.

Let $L_t = b_0 t + \sigma W_t + \Gamma_t$ where (Γ_t) is a subordinator of pure jump type with Lévy density of the form $n(x) = \beta x^{\delta - 1/2} x^{-1} e^{-\alpha x} \mathbb{1}_{x>0}$ with $\delta > -1/2$ (thus, $\int x n(x)\, dx < \infty$). This class of subordinators includes compound Poisson processes ($\delta > 1/2$) and Lévy Gamma processes ($\delta = 1/2$). When $\delta > 0$, the function $xn(x)$ is both integrable and square integrable. This case was discussed in Comte and Genon-Catalot (2009) where the estimation of $xn(x)$, when $b_0 = 0$, $\sigma = 0$, is studied. Here, we consider the case $-1/2 < \delta \leq 0$ which includes the Lévy Inverse Gaussian process ($\delta = 0$). Assumptions (H1)–(H6) are satisfied. The function $p(x) = x^3 n(x)$ can be estimated in presence (or not) of additional drift and Brownian component. We can compute

$$p^*(u) = \beta \frac{\Gamma(\delta + 5/2)}{(\alpha - iu)^{\delta + 5/2}}.$$

Thus, $\int_{|u|\geq \pi m} |p^*(u)|^2 \, du = O(m^{-(2\delta+4)})$. As $2\delta + 1 \leq 1$, $u^4 |p^*(u)|^2$ is not integrable and we have $\Delta^2 \int_{|u| \leq \pi\mu_n} u^4 |p^*(u)|^2 \, du = \Delta^2 o(\mu_n) = o(\Delta^{3/2})$. The best rate for estimating p is $O((n\Delta)^{-(2\delta+4)/(2\delta+5)})$ for a choice $\breve{m} = O((n\Delta)^{1/(2\delta+5)})$. Note that $\Delta^{3/2} \leq (n\Delta)^{-(2\delta+4)/(2\delta+5)}$ for $n\Delta^2 \leq 1$ and $-1/2 < \delta \leq 0$.

We have $\int x^r n(x)\, dx < \infty$ for $r > 1/2 - \delta$. Hence, to estimate σ using $\hat{\sigma}(r)$, we must choose $1/2 - \delta < r < 1$.

EXAMPLE 3. Drift + Brownian motion + Pure jump martingale.

Consider $L_t = bt + \sigma W_t + B_{\Gamma_t}$ where W, B, Γ are independent processes, W, B are standard Brownian motion, and Γ is a pure-jump subordinator with Lévy density $n_\Gamma(\gamma) = \beta\gamma^{\delta-1/2}\gamma^{-1}e^{-\alpha\gamma}\mathbb{1}_{\gamma>0}$ as above (assuming $\delta > -1$). The Lévy density $n(\cdot)$ of (L_t) [and of $(X_t = B_{\Gamma_t})$] is linked with n_Γ [see (5.1)] and can be computed as the norming constant of a Generalized Inverse Gaussian distribution

$$n(x) = \frac{2\beta}{\sqrt{2\pi}} K_{\delta-1}(\sqrt{2\alpha}|x|)\left(\frac{|x|}{\sqrt{2\alpha}}\right)^{\delta-1},$$

where K_ν is a Bessel function of third kind (MacDonald function) [see, e.g., Barndorff-Nielsen and Shephard (2001)]. For $\delta = 1/2$, B_{Γ_t} is a symmetric bilateral Lévy Gamma process [see Madan and Seneta (1990), Küchler and Tappe (2008)]. For $\delta = 0$, B_{Γ_t} is a normal inverse Gaussian Lévy process [see Barndorff-Nielsen and Shephard (2001)]. The relation (5.1) allows to check that the function $p(x) = x^3 n(x)$ belongs to $\mathbb{L}^1 \cap \mathbb{L}^2$ and satisfies (H6) for $\delta > -3/4$. Moreover, we can obtain

$$p^*(u) = -i\beta\left(\frac{u^3\Gamma(\delta+5/2)}{(\alpha+u^2/2)^{5/2}} - 3\frac{u\Gamma(\delta+3/2)}{(\alpha+u^2/2)^{3/2}}\right).$$

Thus, $\int_{|u|\geq\pi m}|p^*(u)|^2\,du = O(m^{-3})$ and $\Delta^2\int_{|u|\leq\pi\mu_n}u^4|p^*(u)|^2\,du = \Delta^2 O(\mu_n) = O(\Delta^{3/2})$. The best rate for estimating p is $O((n\Delta)^{-3/4})$ obtained for $\breve{m} = O((n\Delta)^{1/4)})$. We have $\Delta^{3/2} \leq (n\Delta)^{-3/4}$ as $n\Delta^2 \leq 1$. As $\int \gamma^{r/2}n_\Gamma(\gamma)\,d\gamma < \infty$ for $r > 1-\delta/2$, the estimation of σ by $\hat{\sigma}(r)$ requires $1-\delta/2 < r < 1$. Therefore, we must have $\delta > 0$.

7. Simulations. In this section, we present numerical results for simulated Lévy processes corresponding to Examples 1 and 2 (see Section 6). For these models, the functions $g(x) = xn(x)$, h and p belong to $\mathbb{L}^1 \cap \mathbb{L}^2(\mathbb{R})$. Thus, we can apply the method of Comte and Genon-Catalot (2009), to estimate g when $b_0 = 0$, $\sigma = 0$, and the method developed here to estimate h when $\sigma = 0$ and p when $\sigma \neq 0$. We have implemented the estimators $\bar{h}_{\bar{m}}$, $\bar{p}_{\bar{m}}$ defined by (3.5)–(3.11) and (4.1)–(4.5). The numerical constant κ' appearing in the penalties has been set to 7.5 for g, 4 for h and 3 for p; its calibration is done by preliminary experiments. The cutoff $\bar{m}$ is chosen among 100 equispaced values between 0 and 10.

Figure 1 shows estimated curves for models with jump part coming from compound Poisson processes [see (6.1)] where the Y_i's are standard Gaussian, Exponential $\mathcal{E}(1)$, and $\beta(3,3)$ rescaled on $[-4,4]$. The intensity c is equal to 0.5.

Figure 2 shows estimated curves for jump part of Lévy Gamma and bilateral Lévy Gamma type. The bilateral Lévy Gamma process is the difference $\Gamma_t - \Gamma'_t$ of two independent Lévy Gamma processes.

On top of each graph, we give the mean value of the selected cutoff with its standard deviation in parentheses. This value is surprisingly small. As expected,

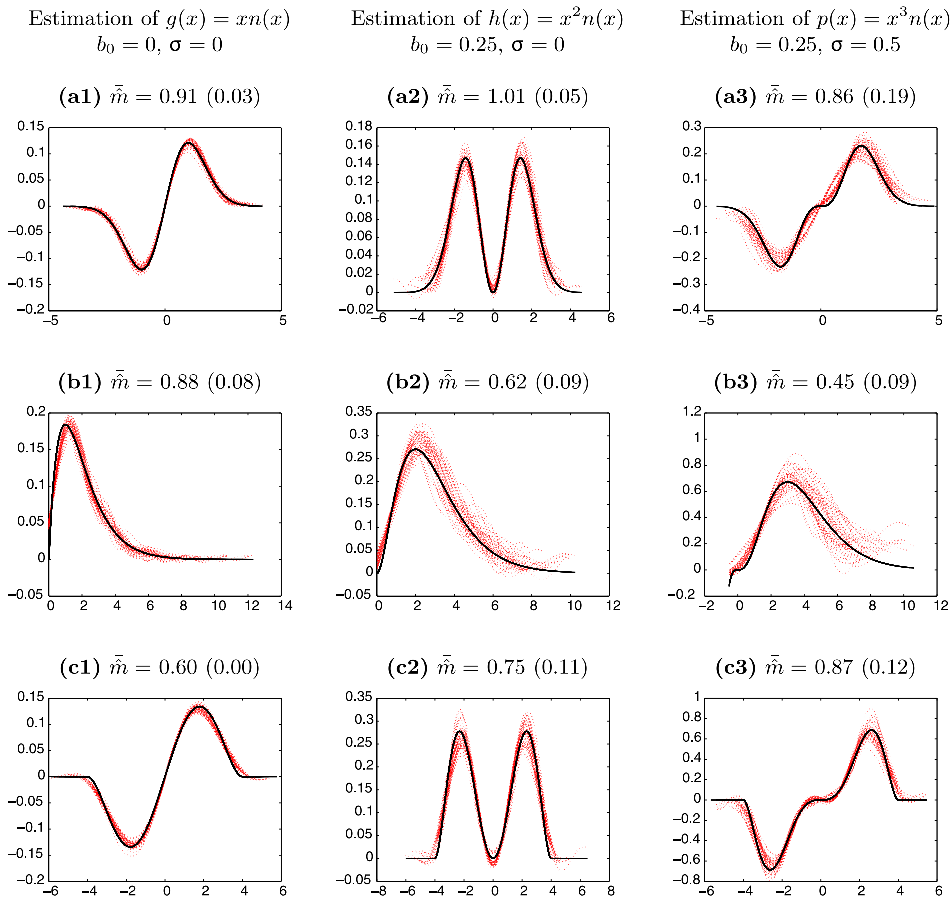

FIG. 1. *Variability bands for the estimation of* g, h, p *for a compound Poisson process with Gaussian* (first line), *Exponential* $\mathcal{E}(1)$ (second line) *and* $\beta(3,3)$ *rescaled on* $[-4,4]$ (third line) Y_i*'s, with* $c = 0.5$. *True* (*bold black line*) *and* 50 *estimated curves* (*dotted red*), $\Delta = 0.05$, $n = 5.10^4$.

the presence of a Gaussian component deteriorates the estimation, which remains satisfactory on the whole.

We estimate the product of a power of x and the Lévy density whereas other authors estimate $n(\cdot)$ on a compact set separated from the origin, see [12], Figueroa-Lopez (2009). Therefore, our point of view coincides with the usual one. Moreover we have, an obvious inequality; setting $\hat{n}(x) = \bar{h}(x)/x^2$ as $n(x) = h(x)/x^2$, we get

$$\mathbb{E}(\|(\hat{n} - n)1_{\mathbb{R}/[-a,a]}\|^2) \leq \frac{1}{a^2}\mathbb{E}(\|\bar{h} - h\|^2).$$

Analogous inequalities hold for $\hat{n}(x) = \hat{g}(x)/x$ or $\hat{n}(x) = \bar{p}(x)/x^3$. In Figure 3, we plot the estimator of $n(\cdot)$ deduced by dividing by the correct power of x and by excluding an interval $[-a, a]$ around zero. To obtain correct representations,

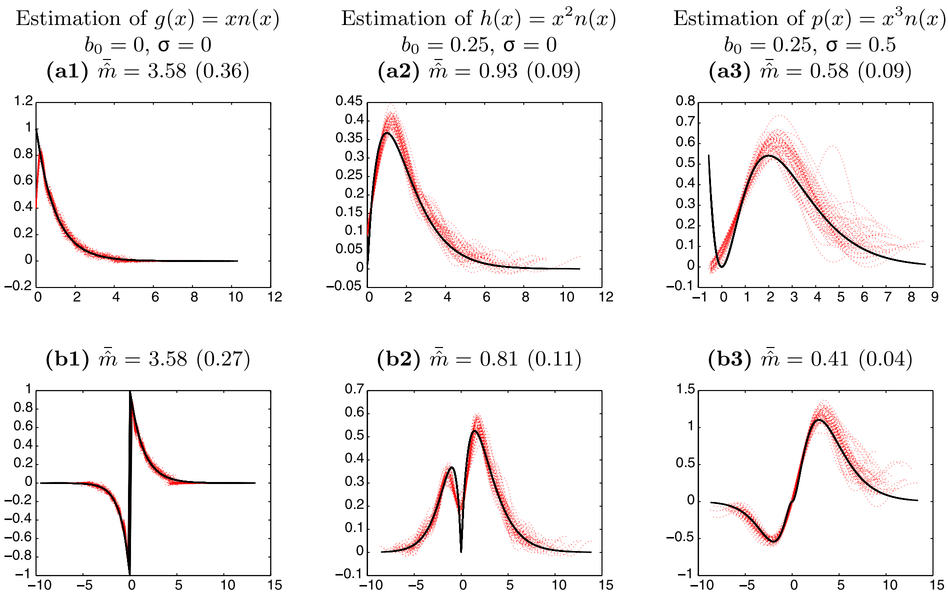

FIG. 2. *Variability bands for the estimation of* g, h, p *for jumps from a Lévy–Gamma process with* $\beta = 1, \alpha = 1$ (first line), *a bilateral Lévy–Gamma process with* $(\beta, \alpha) = (0.7, 1), (\beta', \alpha') = (1, 1)$ (second line). *True* (*bold black line*) *and* 50 *estimated curves* (*dotted red*), $\Delta = 0.05$, $n = 5.10^4$.

$a = 0.1$ suits for $\hat{g}(x)/x$, $a = 0.5$ for $\bar{h}(x)/x^2$ and $a = 1$ for $\bar{p}(x)/x^3$. The results are satisfactory and in accordance with the difficulty of estimating $n(\cdot)$ without or with Gaussian component.

Tables 2 and 3 show the means of the estimation results for $b = \mathbb{E}(L_1) = b_0 + \int xn(x)\,dx$ [see (5.2)] and σ, with standard deviations in parentheses.

The estimation of b is good in all cases, and especially when $n\Delta$ is large. The estimation of σ is clearly more difficult, with noticeable differences according to the values of n and Δ. When Δ is not small enough, the estimation can be heavily biased. In accordance with the theory, when r is smaller, the estimator of σ is slightly better (smaller bias). Table 4 shows the values of $n\Delta^2$ and $n\Delta^{2-r}$, which should be small for the performance of the estimator to be satisfactory. It is worth noting that σ is constantly over estimated.

8. Proofs.

8.1. *Proof of Proposition* 3.1. First, the Parseval formula gives $\|\hat{h}_m - h\|^2 = (1/(2\pi))\|\hat{h}_m^* - h^*\|^2$ and we can note that $h^*(u) - h_m^*(u) = h^*(u)\mathbb{1}_{|u| \geq \pi m}$ is orthogonal to $\hat{h}_m^* - h_m^*$ which has its support in $[-\pi m, \pi m]$. Thus,

$$\|\hat{h}_m - h\|^2 = \frac{1}{2\pi}(\|h^* - h_m^*\|^2 + \|h_m^* - \hat{h}_m^*\|^2).$$

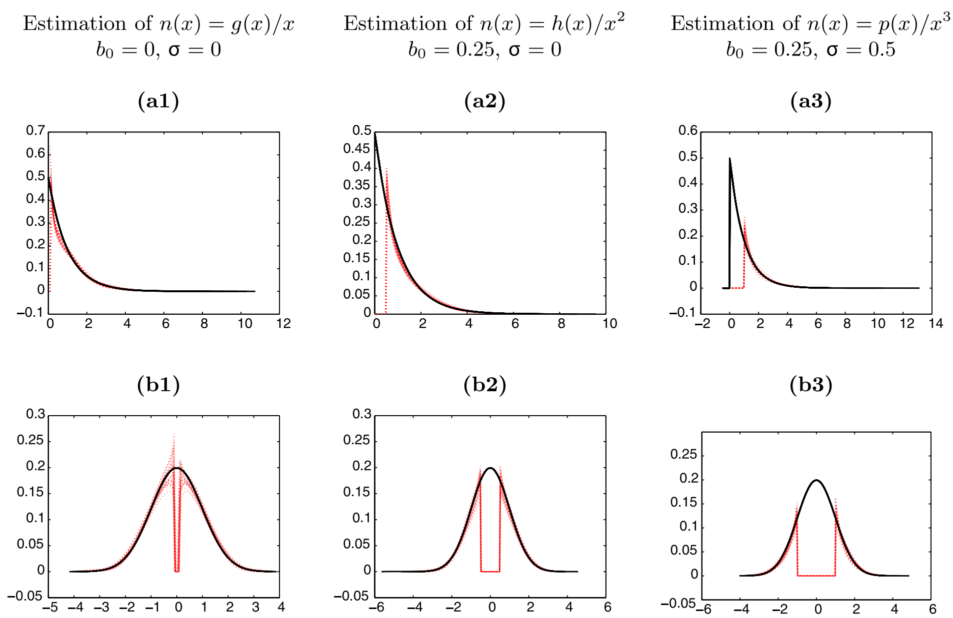

FIG. 3. *Estimation of* $n(\cdot)\mathbb{1}_{[-a,a]^c}$ *with* $a = 0.1$ (first column), $a = 0.5$ (second column), $a = 1$ (third column). *In all cases*, $\lambda = 0.5$, $n = 50{,}000$, $\Delta = 0.05$; 25 *estimated curves* (*thin dotted*) + *the true* (*bold line*).

The first term $(1/(2\pi))\|h^* - h_m^*\|^2 = \|h - h_m\|^2$ is a classical squared bias term. Next,

$$\begin{aligned}
\hat{h}_m^*(u) - h_m^*(u) &= [\hat{h}_m^*(u) - \mathbb{E}(\hat{h}_m^*(u))] + [\mathbb{E}(\hat{h}_m^*(u)) - h_m^*(u)] \\
&= [\hat{h}_m^*(u) - \mathbb{E}(\hat{h}_m^*(u))] + [\psi_\Delta^2(u) - 1]h^*(u)\mathbb{1}_{|u|\le\pi m}.
\end{aligned}$$

Bounding the norm of $\|\hat{h}_m^* - h_m^*\|^2$ by twice the sum of the norms of the two elements of the decomposition, we get

$$\begin{aligned}
\mathbb{E}(\|\hat{h}_m - h_m\|^2) &\le \frac{1}{\pi}\mathbb{E}\left(\int_{-\pi m}^{\pi m} |\hat{h}^*(u) - \mathbb{E}\hat{h}^*(u)|^2\,du\right) \\
&\quad + \frac{1}{\pi}\int_{-\pi m}^{\pi m} |\psi_\Delta^2(u) - 1|^2 |h^*(u)|^2\,du \\
&\le \frac{1}{\pi}\left(\int_{-\pi m}^{\pi m} \mathrm{Var}(\hat{h}^*(u))\,du\right) \\
&\quad + \frac{4\Delta^2}{\pi}\int_{-\pi m}^{\pi m} u^2 c^2(u)|h^*(u)|^2\,du
\end{aligned}$$

TABLE 2
Estimation of (b, σ), $b_0 = 1$, the true value of b in parenthesis, $\sigma = 0.5$, $K = 200$ replications

Model	**(n, Δ)**	**$(5.10^4, 0.05)$**	**$(5.10^4, 0.01)$**	**$(5.10^4, 10^{-3})$**	**$(10^4, 10^{-3})$**
Poisson	$\hat{b}$ $(b = 1)$	1.000 (0.02)	0.997 (0.04)	0.995 (0.123)	1.001 (0.280)
Gaussian	$\hat{\sigma}(1/2)$	0.602 (0.03)	0.527 (0.002)	0.504(0.002)	0.504 (0.005)
	$\hat{\sigma}(1/4)$	0.589 (0.03)	0.521 (0.002)	0.503 (0.002)	0.503 (0.002)
Poisson	$\hat{b}$ $(b = 1.5)$	1.502 (0.05)	1.502 (0.051)	1.494 (0.142)	1.461 (0.359)
Exp(1)	$\hat{\sigma}(1/2)$	0.611 (0.003)	0.530 (0.003)	0.505 (0.002)	0.505 (0.005)
	$\hat{\sigma}(1/4)$	0.594 (0.003)	0.522 (0.003)	0.503 (0.002)	0.503 (0.005)
Gamma	$\hat{b}$ $(b = 2)$	2.001 (0.02)	2.000 (0.05)	1.998 (0.177)	2.018 (0.335)
(1, 1)	$\hat{\sigma}(1/2)$	0.705 (0.004)	0.562 (0.003)	0.512 (0.002)	0.513 (0.005)
	$\hat{\sigma}(1/4)$	0.677 (0.004)	0.548 (0.003)	0.508 (0.002)	0.508 (0.005)
Bilateral	$\hat{b}$ $(b = 1.4286)$	1.426 (0.035)	1.4286 (0.076)	1.4493 (0.264)	1.405 (0.619)
Gamma	$\hat{\sigma}(1/2)$	0.862 (0.005)	0.628 (0.004)	0.526 (0.003)	0.526 (0.006)
(0.7, 1), (1.1)	$\hat{\sigma}(1/4)$	0.798 (0.004)	0.593 (0.003)	0.516 (0.002)	0.515 (0.006)

TABLE 3
Estimation of (b, σ), $b_0 = 1$, the true value of b in parenthesis, $\sigma = 1$, power variation method for estimation of σ, $K = 200$ replications

Model	**(n, Δ)**	**$(5.10^4, 0.05)$**	**$(5.10^4, 0.01)$**	**$(5.10^4, 10^{-3})$**	**$(10^4, 10^{-3})$**
Poisson	$\hat{b}$ (1)	0.999 (0.025)	1.005 (0.059)	0.998 (0.178)	1.025 (0.85)
Gaussian	$\hat{\sigma}(1/2)$	1.082 (0.005)	1.026 (0.004)	1.006 (0.004)	1.005 (0.009)
	$\hat{\sigma}(1/4)$	1.072 (0.005)	1.020 (0.005)	1.004 (0.004)	1.003 (0.01)
Poisson	$\hat{b}$ (1.5)	1.510 (0.026)	1.498 (0.06)	1.481 (0.190)	1.485 (0.442)
Exp(1)	$\hat{\sigma}(1/2)$	1.096 (0.005)	1.030 (0.004)	1.006 (0.004)	1.006 (0.009)
	$\hat{\sigma}(1/4)$	1.080 (0.005)	1.022 (0.004)	1.003 (0.004)	1.003 (0.010)
Gamma	$\hat{b}$ (2)	2.00 (0.026)	1.995 (0.068)	1.991 (0.196)	2.023 (0.195)
(1, 1)	$\hat{\sigma}(1/2)$	1.172 (0.005)	1.062 (0.005)	1.014 (0.004)	1.014 (0.004)
	$\hat{\sigma}(1/4)$	1.152 (0.005)	1.050 (0.005)	1.010 (0.005)	1.010 (0.004)
Bilateral	$\hat{b}$ (1.4286)	1.425 (0.04)	1.431 (0.10)	1.429 (0.28)	1.492 (0.63)
Gamma	$\hat{\sigma}(1/2)$	1.330 (0.006)	1.136 (0.005)	1.033 (0.005)	1.033 (0.01)
(0.7, 1), (1.1)	$\hat{\sigma}(1/4)$	1.284 (0.006)	1.105 (0.005)	1.022 (0.005)	1.022 (0.01)

TABLE 4
Values of n, Δ, $n\Delta$, $n\Delta^2$, $n\Delta^{2-r}$ for $r = 1/2$ and $r = 1/4$

(n, Δ)	**$(5.10^4, 0.05)$**	**$(5.10^4, 0.01)$**	**$(5.10^4, 10^{-3})$**	**$(10^4, 10^{-3})$**
$n\Delta$	2500	500	50	10
$n\Delta^2$	125	5	0.05	0.01
$n\Delta^{2-1/2}$	559	50	1.6	0.3
$n\Delta^{2-1/4}$	264	16	0.3	0.06

(see Lemma 2.2 for the upper bound of $|\psi_\Delta(u) - 1|$ and note that $|\psi_\Delta(u)| \leq 1$). Now, we use the decomposition

$$\begin{aligned}
&\Delta\big(\hat{h}^*(u) - \mathbb{E}(\hat{h}^*(u))\big) \\
&\quad = \big(\hat{\psi}^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u)\big)\big(\hat{\psi}^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u)\big) \\
&\qquad + \big(\hat{\psi}^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u)\big)\psi'_\Delta(u) + \big(\hat{\psi}^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u)\big)\psi'_\Delta(u) \\
&\qquad - \big(\hat{\psi}^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u)\big)\big(\hat{\psi}^{(0)}_{\Delta,2}(u) - \psi_\Delta(u)\big) \\
&\qquad - \big(\hat{\psi}^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u)\big)\psi_\Delta(u) - \big(\hat{\psi}^{(0)}_{\Delta,2}(u) - \psi_\Delta(u)\big)\psi''_\Delta(u).
\end{aligned} \tag{8.1}$$

Considering each term consecutively and exploiting the independence of the samples, we obtain

$$\begin{aligned}
\operatorname{Var}(\hat{h}^*(u)) &\leq \frac{6}{\Delta^2}\left(\frac{\mathbb{E}^2(Z_1^2)}{n^2} + 2\frac{\mathbb{E}^2(Z_1^2)}{n} + \frac{\mathbb{E}(Z_1^4)}{n^2} + 2\frac{\mathbb{E}(Z_1^4)}{n}\right) \\
&\leq 36\frac{\mathbb{E}(Z_1^4/\Delta)}{n\Delta}.
\end{aligned} \tag{8.2}$$

Thus, the first risk bound (3.6) is proved. Analogously, we have

$$\begin{aligned}
\mathbb{E}(\|\bar{h}_m - h\|^2) \leq \|h_m - h\|^2 &+ \frac{1}{\pi}\int_{-\pi m}^{\pi m} |\mathbb{E}\bar{h}^*(u) - h^*(u)|^2\, du \\
&+ \frac{1}{\pi}\int_{-\pi m}^{\pi m} \operatorname{Var}(\bar{h}^*(u))\, du.
\end{aligned}$$

For the variance of $\bar{h}^*(u)$, we use: $\bar{h}^*(u) - \mathbb{E}\bar{h}^*(u) = -\Delta^{-1}(\hat{\psi}^{(2)}_\Delta(u) - \psi''_\Delta(u))$. Thus,

$$\operatorname{Var}(\bar{h}^*(u)) \leq \frac{1}{2n\Delta}\mathbb{E}(Z_1^4/\Delta).$$

Next, for the bias of $\bar{h}^*(u)$, we use [see first (3.4) and then (2.1)]

$$|\mathbb{E}\bar{h}^*(u) - h^*(u)|^2 \leq 2|h^*(u)|^2||\psi_\Delta(u) - 1|^2 + 2\Delta^2|\phi^4(u)|.$$

Hence, there is an additional term in the risk bound equal to

$$\frac{2}{\pi}\Delta^2 \int_{-\pi m}^{\pi m} |\phi^4(u)|\, du = \Delta^2 B_m. \tag{8.3}$$

If h^* is integrable, $|\phi(u)| \leq C$ and $B_m = O(m)$. Otherwise, $|\phi^4(u)| \leq C|u|^4$ and $B_m = O(m^5)$.

8.2. *Proof of Proposition* 3.2. As $\|h-h_m\|^2=(1/\pi)\int_{|u|\geq\pi m}|h^*(u)|^2\,du$, the definition of $\mathcal{C}(a,L)$ implies clearly that $\|h-h_m\|^2\leq(L/2\pi)(\pi m)^{-2a}$. The compromise between this term and the variance term of order $m/(n\Delta)$ is standard: it leads to choose $m=O((n\Delta)^{1/(2a+1)})$ and yields the order $O((n\Delta)^{-2a/(2a+1)})$.

For $a>1/2$, we have

$$\left|\int_0^u |h^*(v)|\,dv\right|\leq\sqrt{L\int(1+v^2)^{-a}\,dv}<+\infty.$$

Therefore, h^* is integrable and $|\phi(u)|\leq|b|+|h^*|_1$.

The last term in the risk bound (3.6) is less than

$$K\Delta^2\int_{-\pi m}^{\pi m}u^2|h^*(u)|^2\,du\leq L\Delta^2(\pi m)^{2(1-a)_+}.$$

If $a\geq 1$ and $n\Delta^3\leq 1$, we have $\Delta^2(\pi m)^{2(1-a)_+}=\Delta^2\leq(n\Delta)^{-1}$.

If $a\in(1/2,1)$, the inequality $\Delta^2m^{2(1-a)}\leq m^{-2a}$ is equivalent to $\Delta^2m^2\leq 1$. As $m\leq n\Delta$, $\Delta^2m^2\leq 1$ holds if $n\Delta^2\leq 1$.

For the additional bias term appearing in the risk bound of $\bar{h}_m$, we have $B_m=O(m)$. Thus, $m\Delta^2\leq m^{-2a}$ holds, for $m=O((n\Delta)^{1/(2a+1)})$, if $m^{1+2a}\Delta^2=(n\Delta)\Delta^2\leq 1$ which in turn holds if $n\Delta^3\leq 1$.

8.3. *Proof of Theorem* 3.1. We only study $\hat{h}_{\hat{m}}$ as the result for $\bar{h}_{\bar{m}}$ can be proved analogously (and is even simpler).

The proof is given in two steps. We define, for some ϱ, $0<\varrho<1$,

$$\Omega_\varrho:=\left\{\left|\frac{[(1/n\Delta)\sum_{k=1}^n Z_k^2][(1/n\Delta)\sum_{k=n+1}^{2n} Z_k^2]}{(\mathbb{E}(Z_1^2/\Delta))^2}-1\right|\leq\varrho/2\right\}$$
$$\cap\left\{\left|\frac{[(1/n\Delta)\sum_{k=1}^n Z_k^4]}{(\mathbb{E}(Z_1^4/\Delta))}-1\right|\leq\varrho/2\right\},$$

so that $\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2)=\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2\mathbb{1}_{\Omega_\varrho})+\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2\mathbb{1}_{\Omega_\varrho^c})$.

Step 1. For the study of $\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2\mathbb{1}_{\Omega_\varrho^c})$, we refer to the analogous proof given in Comte and Genon-Catalot (2009) (see Section A4 therein). Using that $\mathbb{E}(Z_1^{16})<+\infty$, we can prove $\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2\mathbb{1}_{\Omega_\varrho^c})\leq C/(n\Delta)$. For this, we make use of the Rosenthal inequality [see Hall and Heyde (1980)].

Step 2. Study of $\mathbb{E}(\|\hat{h}_{\hat{m}}-h\|^2\mathbb{1}_{\Omega_\varrho})$.

The proof relies on the following decomposition of γ_n:

$$\begin{aligned}\gamma_n(t)-\gamma_n(s)&=\|t-h\|^2-\|s-h\|^2+2\langle t-s,h\rangle-\frac{1}{\pi}\langle\hat{h}^*,t^*-s^*\rangle\\&=\|t-h\|^2-\|s-h\|^2-2\nu_n(t-s)-2R_n(t-s),\end{aligned}$$

where

$$\nu_n(t) = \frac{1}{2\pi}\langle \hat{h}^* - \mathbb{E}(\hat{h}^*), t^*\rangle, \qquad R_n(t) = \frac{1}{2\pi}\langle \mathbb{E}(\hat{h}^*) - h^*, t^*\rangle.$$

As $\gamma_n(\hat{h}_m) = -\|\hat{h}_m\|^2$, we deduce from (3.10) that, for all $m \in \mathcal{M}_n$,

$$\gamma_n(\hat{h}_{\hat{m}}) + \mathrm{pen}(\hat{m}) \le \gamma_n(h_m) + \mathrm{pen}(m).$$

This yields

$$\|\hat{h}_{\hat{m}} - h\|^2 \le \|h - h_m\|^2 + \mathrm{pen}(m) - \mathrm{pen}(\hat{m}) + 2\nu_n(\hat{h}_{\hat{m}} - h_m) + 2R_n(\hat{h}_{\hat{m}} - h_m).$$

Then, for $\phi_n = \nu_n, R_n$, we use the inequality

$$\begin{aligned} 2\phi_n(\hat{h}_{\hat{m}} - h_m) &\le 2\|\hat{h}_{\hat{m}} - h_m\| \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\phi_n(t)| \\ &\le \frac{1}{8}\|\hat{h}_{\hat{m}} - h_m\|^2 + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\phi_n(t)|^2. \end{aligned}$$

Using that $\|\hat{h}_{\hat{m}} - h_m\|^2 \le 2\|\hat{h}_{\hat{m}} - h\|^2 + 2\|\hat{h}_m - h\|^2$ and some algebra, we find

$$\begin{aligned} \frac{1}{4}\|\hat{h}_{\hat{m}} - h\|^2 &\le \frac{7}{4}\|h - h_m\|^2 + \mathrm{pen}(m) - \mathrm{pen}(\hat{m}) \\ &\quad + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |R_n(t)|^2 + 8 \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\nu_n(t)|^2. \end{aligned} \tag{8.4}$$

We have to study the terms containing a supremum, which are of different nature. First, for $R_n(t)$, we have the following.

LEMMA 8.1. *We have*: $\sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |R_n(t)|^2 \le C\Delta^2 \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2\, du$.

PROOF. We have $R_n(t) = \frac{1}{2\pi}\langle t^*, (1 - \psi_\Delta^2) h^*\rangle$. By using Lemma 2.2, we find

$$\begin{aligned} \sup_{t \in S_m + S_{\hat{m}}, \|t\|=1} |\langle t^*, (1 - \psi_\Delta^2) h^*\rangle|^2 &\le \sup_{t \in S_{m_n}, \|t\|=1} |\langle t^*, (1 - \psi_\Delta^2) h^*\rangle|^2 \\ &\le 2\pi \left\| (1 - \psi_\Delta^2) h^* \mathbb{1}_{[-\pi m_n, \pi m_n]} \right\|^2 \\ &\le C\Delta^2 \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2\, du. \end{aligned}$$

□

On the other hand, ν_n is decomposed: $\nu_n(t) = \sum_{j=1}^4 \nu_{n,j}(t) + r_n(t)$ with

$$\begin{aligned} r_n(t) &= \frac{1}{2\pi\Delta}\big\langle t^*, \big(\hat{\psi}_{\Delta,1}^{(1)}(u) - \psi'_\Delta(u)\big)\big(\hat{\psi}_{\Delta,2}^{(1)}(u) - \psi'_\Delta(u)\big)\big\rangle \\ &\quad - \frac{1}{2\pi\Delta}\big\langle t^*, \big(\hat{\psi}_{\Delta,1}^{(2)}(u) - \psi''_\Delta(u)\big)\big(\hat{\psi}_{\Delta,2}^{(0)}(u) - \psi_\Delta(u)\big)\big\rangle, \end{aligned} \tag{8.5}$$

and

$$\nu_{n,1}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\psi''_\Delta - \hat{\psi}^{(2)}_{\Delta,1})\psi_\Delta\rangle, \qquad \nu_{n,2}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\psi_\Delta - \hat{\psi}^{(0)}_{\Delta,2})\psi''_\Delta\rangle.$$

$$\nu_{n,3}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\hat{\psi}^{(1)}_{\Delta,1} - \psi'_\Delta)\psi'_\Delta\rangle, \qquad \nu_{n,4}(t) = \frac{1}{2\pi\Delta}\langle t^*, (\hat{\psi}^{(1)}_{\Delta,2} - \psi'_\Delta)\psi'_\Delta\rangle.$$

LEMMA 8.2. *We have*: $\mathbb{E}(\sup_{t\in S_m+S_{\hat{m}},\|t\|=1}|r_n(t)|^2) \le \frac{C}{n}$.

PROOF. Using the independence of the subsamples, we can write

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat{m}},\|t\|=1}|r_n(t)|^2\Big)\\
&\quad\le \mathbb{E}\Big(\sup_{t\in S_{m_n},\|t\|=1}|r_n(t)|^2\Big)\\
&\quad\le \frac{1}{2\pi^2\Delta^2}\mathbb{E}\big[\big\|(\hat{\psi}^{(1)}_{\Delta,1} - \psi'_\Delta)(\hat{\psi}^{(1)}_{\Delta,2} - \psi'_\Delta)\mathbb{1}_{[-\pi m_n,\pi m_n]}\big\|^2\\
&\qquad\qquad + \big\|(\hat{\psi}^{(2)}_{\Delta,1} - \psi''_\Delta)(\hat{\psi}^{(0)}_{\Delta,2} - \psi_\Delta)\mathbb{1}_{[-\pi m_n,\pi m_n]}\big\|^2\big] \qquad (8.6)\\
&\quad\le \frac{1}{2\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}\mathbb{E}\big[|\hat{\psi}^{(1)}_{\Delta,1}(u) - \psi'_\Delta(u)|^2\big]\mathbb{E}\big[|\hat{\psi}^{(1)}_{\Delta,2}(u) - \psi'_\Delta(u)|^2\big]\,du\\
&\qquad + \frac{1}{2\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}\mathbb{E}\big[|\hat{\psi}^{(2)}_{\Delta,1}(u) - \psi''_\Delta(u)|^2\big]\mathbb{E}\big[|\hat{\psi}^{(0)}_{\Delta,2}(u) - \psi_\Delta(u)|^2\big]\,du\\
&\quad\le \frac{m_n}{\pi\Delta^2}\bigg(\frac{[\mathbb{E}(Z_1^2)]^2}{n^2} + \frac{\mathbb{E}(Z_1^4)}{n^2}\bigg) \le \frac{C}{n}
\end{aligned}$$

because $m_n \le n\Delta$ and $\mathbb{E}(Z_1^2)$ and $\mathbb{E}(Z_1^4)$ have order Δ. $\square$

Now, the study of the $\nu_{n,j}$'s relies on Lemma A.1. Let us first study the process $\nu_{n,1}$. We must split $Z_k^2 = Z_k^2\mathbb{1}_{Z_k^2\le k_n\sqrt{\Delta}} + Z_k^2\mathbb{1}_{Z_k^2>k_n\sqrt{\Delta}}$ with k_n to be defined later. This implies that $\nu_{n,1}(t) = \nu^P_{n,1}(t) + \nu^R_{n,1}(t)$ (P for Principal, R for residual) with

$$\begin{aligned}
&\nu^P_{n,1}(t) = \frac{1}{n}\sum_{k=1}^n[f_t(Z_k) - \mathbb{E}(f_t(Z_k))]\\
&\qquad\qquad\text{with } f_t(z) = \frac{1}{2\pi\Delta}z^2\mathbb{1}_{z^2\le k_n\sqrt{\Delta}}\langle t^*, e^{iz\cdot}\psi_\Delta\rangle,
\end{aligned} \tag{8.7}$$

and $\nu^R_{n,1}(t) = \nu_{n,1}(t) - \nu^P_{n,1}(t)$. We prove the following results for $\nu_{n,1}$ and $\nu_{n,2}$.

PROPOSITION 8.1. *Under the assumptions of Theorem* 3.1, *choose* $k_n = C\frac{\sqrt{n}}{\ln(n\Delta)}$ *and*

$$p(m,m') = 4\mathbb{E}(Z_1^4/\Delta)\frac{m\vee m'}{\Delta}, \tag{8.8}$$

then

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1}[\nu_{n,1}^P(t)]^2-p(m,\hat m)\Big)_+ +\mathbb{E}\Big[\sup_{t\in S_{m_n},\|t\|=1}\big|\nu_{n,1}^{(R)}(t)\big|^2\Big]$$
$$\le C\frac{\ln^2(n\Delta)}{n\Delta},$$

where C is a constant.

PROPOSITION 8.2. *Under the assumptions of Theorem 3.1,*

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1}[\nu_{n,2}(t)]^2-p(m,\hat m)\Big)_+\le\frac{C}{n\Delta},$$

where C is a constant.

For both $\nu_{n,3}$ and $\nu_{n,4}$, which are similar, we have to split again $Z_k = Z_k\mathbb{1}_{|Z_k|\le k_n\sqrt{\Delta}}+Z_k\mathbb{1}_{|Z_k|>k_n\sqrt{\Delta}}$ with the same k_n as above. We define $\nu_{n,j}(t)=\nu_{n,j}^P(t)+\nu_{n,j}^R(t)$ as previously, for $j=3,4$.

PROPOSITION 8.3. *Under the assumptions of Theorem 3.1, define for $j=3,4$*

$$q(m,m')=4\mathbb{E}^2(Z_1^2/\Delta)\frac{m\vee m'}{\Delta}, \tag{8.9}$$

then

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1}[\nu_{n,j}^P(t)]^2-q(m,\hat m)\Big)_+ +\mathbb{E}\Big[\sup_{t\in S_{m_n},\|t\|=1}\big|\nu_{n,j}^{(R)}(t)\big|^2\Big]$$
$$\le C\frac{\ln^2(n\Delta)}{n\Delta},$$

where C is a constant.

Now, on Ω_ϱ, the following inequality holds (by bounding the indicator by 1), for any choice of κ:

$$(1-\varrho)\,\mathrm{pen}_{th}(m)\le\mathrm{pen}(m)\le(1+\varrho)\,\mathrm{pen}_{th}(m),$$

where $\mathrm{pen}_{th}(m)=\mathbb{E}(\mathrm{pen}(m))$. It follows from (8.4) that

$$\begin{aligned}\frac{1}{4}\mathbb{E}(\|\hat h_{\hat m}-h\|^2\mathbb{1}_{\Omega_\varrho})\le{}&\frac{7}{4}\|h-h_m\|^2+\mathrm{pen}_{th}(m)-\mathbb{E}(\mathrm{pen}(\hat m)\mathbb{1}_{\Omega_\varrho})\\&+C\Delta^2\int_{-\pi m_n}^{\pi m_n}u^2|h^*(u)|^2\,du\\&+8\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1}|\nu_n(t)|^2\mathbb{1}_{\Omega_\varrho}\Big).\end{aligned}\tag{8.10}$$

Recalling that

$$\nu_n(t) = r_n(t) + \nu_{n,1}^P(t) + \nu_{n,1}^R(t) + \nu_{n,2}(t) + \nu_{n,3}^P(t) + \nu_{n,3}^R(t) + \nu_{n,4}^P(t) + \nu_{n,4}^R(t),$$

we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1} |\nu_n(t)|^2 \mathbb{1}_{\Omega_\varrho}\Big) \\
&\quad\le 8\bigg(\frac{C}{n\Delta} + \sum_{j\in\{1,3,4\}} \mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1} |\nu_{n,j}^P(t)|^2 \mathbb{1}_{\Omega_\varrho}\Big) \\
&\qquad\qquad + \mathbb{E}\Big(\sup_{t\in S_m+S_{\hat m},\|t\|=1} |\nu_{n,2}(t)|^2 \mathbb{1}_{\Omega_\varrho}\Big)\bigg) \qquad (8.11)\\
&\quad\le 8\bigg(\frac{C'}{n\Delta} + 2\mathbb{E}\big[\big(p(m,\hat m) + q(m,\hat m)\big)\mathbb{1}_{\Omega_\varrho}\big]\bigg).
\end{aligned}$$

We note that $p(m,m') + q(m,m') = \frac{1}{4\kappa}(\mathrm{pen}_{th}(m) + \mathrm{pen}_{th}(m'))$. Thus,

$$\begin{aligned}
&\mathrm{pen}_{th}(m) - \mathbb{E}(\mathrm{pen}(\hat m)\mathbb{1}_{\Omega_\varrho}) + 128\mathbb{E}\big[\big(p(m,\hat m) + q(m,\hat m)\big)\mathbb{1}_{\Omega_\varrho}\big] \\
&\quad\le \mathrm{pen}_{th}(m) - (1-\varrho)\mathbb{E}(\mathrm{pen}_{th}(\hat m)\mathbb{1}_{\Omega_\varrho}) + \frac{32}{\kappa}\mathbb{E}\big[\big(\mathrm{pen}_{th}(m) + \mathrm{pen}_{th}(\hat m)\big)\mathbb{1}_{\Omega_\varrho}\big] \\
&\quad\le \left(1 + \frac{32}{\kappa}\right)\mathrm{pen}_{th}(m) + \left(\frac{32}{\kappa} - (1-\varrho)\right)\mathbb{E}[\mathrm{pen}_{th}(\hat m)\mathbb{1}_{\Omega_\varrho}].
\end{aligned}$$

Therefore, we choose κ such that $(32/\kappa - (1-\varrho)) \le 0$, that is $\kappa \ge 32/(1-\varrho)$. This together with (8.10) and (8.11) yields

$$\begin{aligned}
\frac{1}{4}\mathbb{E}(\|\hat h_{\hat m} - h\|^2 \mathbb{1}_{\Omega_\varrho}) &\le \frac{7}{4}\|h - h_m\|^2 + (2-\varrho)\,\mathrm{pen}_{th}(m) \\
&\quad + C\Delta^2 \int_{-\pi m_n}^{\pi m_n} u^2 |h^*(u)|^2\,du + \frac{C''}{n\Delta}.
\end{aligned}$$

8.4. *Proof of Propositions* 8.1–8.3.

PROOF OF PROPOSITION 8.1. Let $m'' = m \vee m'$, and note that $S_m + S_{m'} = S_{m''}$. We evaluate the constants M, H, v to apply Lemma A.1 to $\nu_{n,1}^P(t)$ [see (8.7)]:

$$\begin{aligned}
\sup_{z\in\mathbb{R}} |f_t(z)| &\le \frac{k_n}{2\pi\sqrt{\Delta}} \sup_{z\in\mathbb{R}} \left| \int_{-\pi m''}^{\pi m''} t^*(-u) e^{iuz} \psi_\Delta(u)\,du \right| \\
&\le \frac{k_n}{2\pi\sqrt{\Delta}} \int_{-\pi m''}^{\pi m''} |t^*(u)|\,du \le \frac{k_n}{2\pi\sqrt{\Delta}} \left(2\pi m'' \int_{-\pi m''}^{\pi m''} |t^*(u)|^2\,du\right)^{1/2} \\
&= \frac{k_n}{\sqrt{\Delta}} (m'')^{1/2} \|t\| = \frac{k_n\sqrt{m''}}{\sqrt{\Delta}} := M.
\end{aligned}$$

Moreover,

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{m'},\|t\|=1}\big[\nu_{n,1}^P(t)\big]^2\Big) \le \frac{1}{2\pi n\Delta^2}\int_{-\pi m''}^{\pi m''}\mathbb{E}(Z_1^4)\psi_\Delta^2(u)\,du$$

$$\le \frac{m''\mathbb{E}(Z_1^4/\Delta)}{n\Delta} := H^2.$$

The most delicate term is v:

$$\begin{aligned}
\operatorname{Var}(f_t(Z_1)) &= \frac{1}{4\pi^2\Delta^2}\mathbb{E}\bigg(Z_1^4\mathbb{1}_{Z_1^2\le k_n\sqrt{\Delta}}\bigg|\iint e^{ixZ_1}t^*(-x)\psi_\Delta(x)\,dx\bigg|^2\bigg)\\
&\le \frac{1}{4\pi^2\Delta^2}\mathbb{E}\bigg(Z_1^4\iint e^{i(x-y)Z_1}t^*(-x)t^*(y)\psi_\Delta(x)\psi_\Delta(-y)\,dx\,dy\bigg)\\
&= \frac{1}{4\pi^2\Delta^2}\iint \psi_\Delta^{(4)}(x-y)t^*(-x)t^*(y)\psi_\Delta(x)\psi_\Delta(-y)\,dx\,dy,
\end{aligned}$$

where we recall that $\psi_\Delta^{(4)}(x)=\mathbb{E}(Z_1^4e^{ixZ_1})$. Making use of the basis $(\varphi_{m'',j},\, j\in\mathbb{Z})$ of $S_{m''}$, we have $t=\sum_{j\in\mathbb{Z}}t_j\varphi_{m'',j}$ with $\|t\|^2=\sum_{j\in\mathbb{Z}}t_j^2=1$,

$$\begin{aligned}
\operatorname{Var}(f_t(Z_1)) &\le \frac{1}{4\pi^2\Delta^2}\sum_{j,k\in\mathbb{Z}}t_jt_k\iint \psi_\Delta^{(4)}(x-y)\varphi_{m'',j}^*(-x)\varphi_{m'',k}^*(y)\\
&\qquad\qquad\times\psi_\Delta(x)\psi_\Delta(-y)\,dx\,dy\\
&\le \frac{1}{4\pi^2\Delta^2}\bigg(\sum_{j,k\in\mathbb{Z}}\bigg|\iint \psi_\Delta^{(4)}(x-y)\varphi_{m'',j}^*(-x)\varphi_{m'',k}^*(y)\\
&\qquad\qquad\times\psi_\Delta(x)\psi_\Delta(-y)\,dx\,dy\bigg|^2\bigg)^{1/2}\\
&= \frac{1}{4\pi^2\Delta^2}\bigg(\iint_{[-\pi m'',\pi m'']^2}|\psi_\Delta^{(4)}(x-y)|^2|\psi_\Delta(x)|^2\\
&\qquad\qquad\times|\psi_\Delta(-y)|^2\,dx\,dy\bigg)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(f_t(Z_1)) &\le \frac{1}{4\pi^2\Delta^2}\bigg(\iint_{[-\pi m'',\pi m'']^2}|\psi_\Delta^{(4)}(x-y)|^2\,dx\,dy\bigg)^{1/2}\\
&\le \frac{\sqrt{2\pi m''}}{4\pi^2\Delta^2}\bigg(\int_{[-2\pi m'',2\pi m'']}|\psi_\Delta^{(4)}(z)|^2\,dz\bigg)^{1/2}.
\end{aligned}\tag{8.12}$$

Therefore, we need to study $\int_{[-2\pi m'',2\pi m'']}|\psi_\Delta^{(4)}(z)|^2\,dz$. Recall that $\phi(u)=ib-\int_0^u h^*(v)\,dv$. We have

$$\psi_\Delta^{(4)}=\Delta\big[\phi^{(3)}+\Delta\big(4\phi\phi''+3(\phi')^2\big)+6\Delta^2\phi'\phi^2+\Delta^3\phi^4\big]\psi_\Delta,$$

where

$$\phi'(u) = -h^*(u), \qquad \phi''(u) = -i\int e^{iux}x^3 n(x)\,dx,$$
$$\phi^{(3)}(u) = \int e^{iux}x^4 n(x)\,dx$$

satisfy: $\int |\phi'(u)|^2\,du = \|h\|^2$, $|\phi'(u)| \le |h|_1$ and thanks to (H4), the Parseval equality yields

$$\int |\phi''(u)|^2\,du = \int x^6 n^2(x)\,dx = \int x^2 h^2(x)\,dx,$$
$$\int |\phi^{(3)}(u)|^2\,du = \int x^8 n^2(x)\,dx = \int x^4 h^2(x)\,dx.$$

By assumption, h^* is in $\mathbb{L}_1(\mathbb{R})$, thus, $|\phi(u)| \le |b| + |h^*|_1 := M_\phi$. Therefore,

$$|\psi_\Delta^{(4)}|^2 \le C\Delta^2\bigl(|\phi^{(3)}|^2 + \Delta^2\bigl((\phi'')^2 + (\phi')^4\bigr) + \Delta^4(\phi')^2 + \Delta^6\bigr),$$

where C is a constant depending on M_ϕ and $|h|_1$. Therefore,

$$\begin{aligned}\int_{-2\pi m''}^{2\pi m''} |\psi_\Delta^{(4)}(u)|^2\,du &\le C\Delta^2\biggl[\int x^4 h^2(x)\,dx + \Delta^2\biggl(\int x^2 h^2(x)\,dx + 4\pi m''|h|_1^4\biggr)\\ &\qquad + \Delta^4\|h\|^2 + 4\pi m''\Delta^6\biggr]\\ &\le C_1\Delta^2\biggl[\int x^4 h^2(x)\,dx + \Delta^2\int x^2 h^2(x)\,dx + \Delta^4\|h\|^2\biggr]\\ &\quad + C_2 m''\Delta^4.\end{aligned}$$

Thus, using Assumptions (H1), (H3), (H4),

$$\int_{[-2\pi m'', 2\pi m'']} |\psi_\Delta^{(4)}(u)|^2\,du \le K(\Delta^2 + m''\Delta^4).$$

As $m''\Delta^4 \le n\Delta^5$ and $n\Delta^3 \le 1$ we get $\int_{[-2\pi m'', 2\pi m'']} |\psi_\Delta^{(4)}(u)|^2\,du \le 2K\Delta^2$. This together with (8.12) yields $v = c\sqrt{m''}/\Delta$ where c is a constant.

Applying Lemma A.1 yields, for $\epsilon^2 = 1/2$ and $p(m, m')$ given by (8.8) yields

$$\begin{aligned}&\mathbb{E}\Bigl(\sup_{t\in S_m+S_{m'}, \|t\|=1} [\nu_{n,1}^P(t)]^2 - p(m,m')\Bigr)_+\\ &\qquad \le C_1\biggl(\frac{\sqrt{m''}}{n\Delta} e^{-C_2\sqrt{m''}} + \frac{k_n^2 m''}{n^2\Delta} e^{-C_3\sqrt{n}/k_n}\biggr)\end{aligned}$$

as $p(m, m') = 4H^2$. We choose

$$k_n = \frac{C_3}{4}\frac{\sqrt{n}}{\ln(n\Delta)},$$

and as $m \le n\Delta$, we get

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\Big)_+$$
$$\le C_1'\bigg(\frac{\sqrt{m''}}{n\Delta}e^{-C_2\sqrt{m''}} + \frac{1}{(\Delta n)^4\ln^2(n\Delta)}\bigg).$$

As $C_2xe^{-C_2x}$ is decreasing for $x \ge 1/C_2$, and its maximum is $1/(eC_2)$, we get

$$\sum_{m'=1}^{m_n}\sqrt{m''}e^{-C_2\sqrt{m''}} \le \sum_{\sqrt{m'}\le 1/C_2}(eC_2)^{-1} + \sum_{\sqrt{m'}\ge 1/C_2}\sqrt{m'}e^{-C_2\sqrt{m'}}$$
$$\le \frac{1}{eC_2^3} + \sum_{m'=1}^{\infty}\sqrt{m'}e^{-C_2\sqrt{m'}} < +\infty.$$

It follows that

$$\sum_{m'=1}^{m_n}\mathbb{E}\Big(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,1}^P(t)]^2 - p(m,m')\Big)_+ \le \frac{C}{n\Delta}.$$

Let us now study the second term $\nu_{n,j}^{(R)}(t)$ in the decomposition of $\nu_{n,j}(t)$. The cases $j = 3, 4$ being similar, we consider only $\nu_{n,j}^{(R)}(t)$ for $j = 1$:

$$\mathbb{E}\Big[\sup_{t\in S_{m_n},\|t\|=1}\big|\nu_{n,1}^{(R)}(t)\big|^2\Big]$$
$$\le \frac{1}{4\pi^2\Delta^2}\mathbb{E}\bigg(\int_{-\pi m_n}^{\pi m_n}\bigg|\frac{1}{n}\sum_{k=1}^{n}(Z_k^2\mathbb{1}_{Z_k^2>k_n\sqrt{\Delta}}e^{iuZ_k} - \mathbb{E}(Z_k^2\mathbb{1}_{Z_k^2>k_n\sqrt{\Delta}}e^{iuZ_k}))\bigg|^2$$
$$\times |\psi_\Delta^2(u)|^2\,du\bigg)$$
$$\le \frac{\mathbb{E}(Z_1^4\mathbb{1}_{Z_1^2>k_n\sqrt{\Delta}})}{4n\pi^2\Delta^2}\int_{-\pi m_n}^{\pi m_n}du \le \frac{m_n\mathbb{E}(Z_1^{4+2p})}{2\pi n\Delta^2(k_n\sqrt{\Delta})^p}$$
$$\le K\frac{\mathbb{E}(Z_1^{4+2p}/\Delta)\ln^p(n\Delta)}{2\pi(n\Delta)^{p/2}},$$

using $m_n \le n\Delta$ and recalling that $k_n = (C_3/4)(\sqrt{n}/\ln(n\Delta))$. Taking $p = 2$, which is possible because $\mathbb{E}(Z_1^8) < +\infty$, gives a bound of order $\ln^2(n\Delta)/(n\Delta)$.

Proposition 8.1 is proved. $\square$

PROOF OF PROPOSITION 8.2. For $\nu_{n,2}$, the variables are bounded without splitting, and the function f_t is replaced by $\tilde{f}_t(z) = (2\pi\Delta)^{-1}\langle t^*, e^{iz\cdot}\psi_\Delta''\rangle$. We just check the orders of M, H^2 and v for the application of Lemma A.1. For $t \in S_{m''} = S_m + S_{m'}$ and $\|t\| \le 1$, we have

$$\sup_{z\in\mathbb{R}} |\tilde{f}_t(z)| \le \frac{1}{2\pi\Delta}\sqrt{\int_{-\pi m''}^{\pi m''} |t^*(-u)|^2\,du \int_{-\pi m''}^{\pi m''} |\psi''_\Delta(u)|^2\,du}$$
$$\le \sqrt{m''}\frac{\mathbb{E}(Z_1^2)}{\Delta} \le C\sqrt{m''} := M.$$

Next,

$$\mathbb{E}\Big(\sup_{t\in S_m+S_{m'},\|t\|=1}[\nu_{n,2}(t)]^2\Big) \le \frac{1}{2\pi n\Delta^2}\int_{-\pi m''}^{\pi m''} |\psi''_\Delta(u)|^2\,du$$
$$\le \frac{m''\mathbb{E}^2(Z_1^2/\Delta)}{n\Delta} := H^2.$$

Following the same line as previously for v, we get

$$\operatorname{Var}(\tilde{f}_t(Z_1)) \le \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2} |\psi_\Delta(u-v)|^2|\psi''_\Delta(u)|^2|\psi''_\Delta(-v)|^2\,du\,dv\right)^{1/2}.$$

As $\psi''_\Delta = \Delta(\phi' + \Delta\phi^2]\psi_\Delta$, we get (recall that $M_\phi = |b| + |h^*|_1$ is the upper bound of $|\phi(u)|$)

$$\operatorname{Var}(\tilde{f}_t(Z_1)) \le \frac{1}{4\pi^2\Delta^2}\int_{-\pi m''}^{\pi m''} |\psi''_\Delta(x)|^2\,dx \le \frac{2\Delta^2(\|h^*\|^2 + 2\pi m''\Delta^2 M_\phi^2)}{4\pi^2\Delta^2}$$
$$\le \frac{1}{\pi}(\|h\|^2 + M_\phi^2 m_n\Delta^2) \le \frac{\|h\|^2 + M_\phi^2}{\pi} := v$$

as $m_n\Delta^2 \le n\Delta^3 \le 1$. □

PROOF OF PROPOSITION 8.3. Here, f_t is replaced by $\breve{f}_t(z) = z\mathbb{1}_{|z|\le k'_n\sqrt{\Delta}}\langle t^*, e^{iz\cdot}\psi'_\Delta\rangle$. Using now that $|\psi'_\Delta(u)| \le \mathbb{E}(|Z_1|) \le \sqrt{\mathbb{E}(Z_1^2)}$, we obtain here that $M = k'_n\sqrt{m''}\sqrt{\mathbb{E}(Z_1^2/\Delta)}$. On the other hand, we find $H^2 = m''\mathbb{E}^2(Z_1^2)/(n\Delta^2)$. Last, we find

$$\operatorname{Var}(\breve{f}_t(Z_1)) \le \frac{1}{4\pi^2\Delta^2}\left(\iint_{[-\pi m'',\pi m'']^2}\big|\psi_\Delta^{(2)}(u-v)\big|^2|\psi'_\Delta(u)|^2|\psi'_\Delta(-v)|^2\,du\,dv\right)^{1/2}.$$

With the bounds for $|\psi'_\Delta|$ and $\int_{-2\pi m''}^{2\pi m''} |\psi''_\Delta(z)|^2\,dz$, we obtain $v = c\mathbb{E}(Z_1^2/\Delta)\sqrt{m''}$. □

8.5. *Proof of Proposition* 4.2. Let us take $m = O((n\Delta)^{1/(2a+1)})$. When $p \in \mathcal{C}(a, L)$, the first two terms of (4.4) are of order $O((n\Delta)^{-2a/(2a+1)})$. The third term is $O(\Delta^2 m^{2(2-a)_+})$. If $a \geq 2$, its order is Δ^2 and is less than $1/(n\Delta)$ if $n\Delta^3 \leq 1$.

If $a \in (0, 2)$, $\Delta^2 m^{2(2-a)} = O(\Delta^2 (n\Delta)^{2(2-a)/(1+2a)})$ which has lower rate than $O((n\Delta)^{-2a/(2a+1)})$ if $\Delta^2(n\Delta)^{4/(1+2a)} \leq O(1)$, that is $n\Delta^{1+(1+2a)/2} = n\Delta^{3/2+a} \leq O(1)$. We must consider in addition the terms $\Delta^2 m^3$ and $\Delta^4 m^7$. As previously, $\Delta^2 m^3 \leq (n\Delta)^{-2a/(2a+1)}$ if $n\Delta^{(6a+5)/(2a+3)} \leq O(1)$ that is $n\Delta^{5/3} \leq 1$ if $a > 0$ and $n\Delta^2$ if $a \geq 1/2$. Moreover, $\Delta^4 m^7 \leq (n\Delta)^{-2a/(2a+1)}$ if $n\Delta^{(10a+11)/(2a+7)} \leq 1$ that is $n\Delta^{11/7} \leq 1$ if $a > 0$ and $n\Delta^2 \leq 1$ if $a \geq 1/2$.

8.6. *Proof of Proposition* 4.1. As previously, $\|\bar{p}_m - p\|^2 = \frac{1}{2\pi}(\|p^* - p_m^*\|^2 + \|p_m^* - \bar{p}_m^*\|^2)$. The variance of $\bar{p}_m$ satisfies

$$\begin{aligned}
\mathbb{E}(\|\bar{p}_m - p_m\|^2) &= \frac{1}{2\pi}\mathbb{E}(\|\bar{p}_m^* - p_m^*\|^2) \\
&= \frac{1}{2\pi}\int_{-\pi m}^{\pi m} \bigl(\mathrm{Var}(\bar{p}^*(u)) + |\mathbb{E}(\bar{p}^*(u)) - p^*(u)|^2\bigr)\,du,
\end{aligned}$$

where

$$\mathrm{Var}(\bar{p}^*(u)) \leq \frac{\mathbb{E}(Z_1^6)}{n\Delta^2} = \frac{\mathbb{E}(Z_1^6/\Delta)}{n\Delta}.$$

We have $|h^*(u)| \leq |h|_1$. By Lemma 2.2, $|\tilde{\phi}(u)| \leq |b| + |u|(|h|_1 + \sigma^2) \leq C(1 + |u|)$. Inserting these bounds in (4.3) implies

$$\begin{aligned}
|\mathbb{E}(\bar{p}^*(u)) - p^*(u)| &\leq C\Delta|p^*(u)||u|(1+|u|) \\
&\quad + C'\Delta(1+|u|) + C''\Delta^2(1+|u|)^3.
\end{aligned} \tag{8.13}$$

Gathering the terms gives the announced bound for the risk of $\bar{p}_m$. This ends the proof of Proposition 4.1.

8.7. *Proof of Theorem* 4.1. The proof follows the same lines as for the adaptive estimator of h. We introduce, for $0 < \varrho < 1$,

$$\Omega_b := \left\{\left|\frac{[(1/(n\Delta))\sum_{k=1}^n Z_k^6]}{(\mathbb{E}(Z_1^6/\Delta))} - 1\right| \leq \varrho\right\}.$$

Provided that $\mathbb{E}(Z_1^{24}) < \infty$, we can make use of the Rosenthal inequality to obtain:

$$\mathbb{E}(\|\bar{p}_{\tilde{m}} - p\|^2 \mathbb{1}_{\Omega_\varrho^c}) \leq C/n\Delta.$$

For the study of $\mathbb{E}(\|\bar{p}_{\tilde{m}} - p\|^2 \mathbb{1}_{\Omega_\varrho})$, the decomposition is similar to the previous case [see (8.4)] where $\hat{h}_{\hat{m}}, h$ are now replaced by $\bar{p}_{\tilde{m}}, p$. The processes $R_n(t)$ and $\nu_n(t)$ are given by

$$\nu_n(t) = \frac{1}{2\pi}\langle \bar{p}^* - \mathbb{E}(\bar{p}^*), t^*\rangle, \qquad R_n(t) = \frac{1}{2\pi}\langle \mathbb{E}(\bar{p}^*) - p^*, t^*\rangle.$$

The term $R_n(t)$ is dealt using (8.13). For the term containing $\nu_n(t)$, we need apply Lemma A.1. So, ν_n is split into the sum of a principal and a residual term, respectively denoted by ν_n^P and ν_n^R with

$$\nu_n^P(t) = \frac{1}{n}\sum_{k=1}^{n}[f_t(Z_k) - \mathbb{E}(f_t(Z_k))] \tag{8.14}$$
$$\text{with } f_t(z) = \frac{1}{2\pi\Delta} z^3 \mathbb{1}_{|z|^3 \le k_n\sqrt{\Delta}} \langle t^*, e^{iz\cdot}\rangle,$$

and $\nu_n^R(t) = \nu_n(t) - \nu_n^P(t)$. Everything is analogous. The difference is that, for applying Lemma A.1, we have to bound $\int_{-2\pi m''}^{2\pi m''} |\psi_\Delta^{(6)}(u)|^2\, du$ (instead of $\int_{-2\pi m''}^{2\pi m''} |\psi_\Delta^{(4)}(u)|^2\, du$ previously). Using $\psi_\Delta' = \Delta\tilde{\phi}\psi_\Delta$ [see (2.1)–(4.2)], we find

$$\begin{aligned}\psi_\Delta^{(6)} &= \Delta\psi_\Delta\phi^{(5)} + \Delta^2\psi_\Delta[6\tilde{\phi}\phi^{(4)} + 15\phi^{(3)}(\phi'(u) - \sigma^2)] \\ &\quad + \Delta^3\psi_\Delta[15\phi^{(3)}\tilde{\phi}^2 + 60\phi''(\phi'(u) - \sigma^2)\tilde{\phi} + 15(\phi'(u) - \sigma^2)^3] \\ &\quad + \Delta^4\psi_\Delta[17\phi''\tilde{\phi}^{(3)} + 36\tilde{\phi}^{(2)}(\phi'(u) - \sigma^2)^2] \\ &\quad + 12\Delta^5\psi_\Delta\tilde{\phi}^4(\phi'(u) - \sigma^2) + \Delta^6\psi_\Delta\tilde{\phi}^6.\end{aligned}$$

Now, $\tilde{\phi}(u) \le C(1 + |u|)$ and all the derivatives of $\tilde{\phi}, \phi$ are bounded. Moreover, under (H6), $\int |\phi^{(5)}(u)|^2\, du = \int x^6 |p(x)|^2\, dx < +\infty$. Thus, we find the following bound:

$$\int_{-2\pi m''}^{2\pi m''} |\psi_\Delta^{(6)}|^2 \le C\Delta^2(1 + \Delta^2 m^3 + \Delta^4 m^5 + \Delta^6 m^7 + \Delta^8 m^9 + \Delta^{10} m^{13}) = O(\Delta^2),$$

as $m \le \sqrt{n\Delta}$. The proof may then be completed as for $\hat{h}_{\hat{m}}$.

8.8. *Proof of Proposition* 5.1. Proof of (i). The assumptions and the fact that $r \le 1$ imply

$$|\Gamma_\Delta|^r = \left|\sum_{s\le\Delta} \Gamma_s - \Gamma_{s_-}\right|^r \le \sum_{s\le\Delta} |\Gamma_s - \Gamma_{s_-}|^r.$$

Taking expectations yields $\mathbb{E}|\Gamma_\Delta|^r \le \Delta \int |\gamma|^r n(\gamma)\, d\gamma$.

Proof of (ii). Consider f a nonnegative function such that $f(0) = 0$. We have

$$\mathbb{E}\sum_{s\le t} f(X_s - X_{s_-}) = \mathbb{E}\sum_{s\le t} f(B_{\Gamma_s} - B_{\Gamma_{s_-}}).$$

Then, $\sum_{s\le t} \mathbb{E} f(B_{\Gamma_s} - B_{\Gamma_{s_-}}) = \sum_{s\le t} \int_{\mathbb{R}} f(x)(\mathbb{E} e^{(-x^2/2(\Gamma_s - \Gamma_{s_-}))} \frac{1}{\sqrt{2\pi(\Gamma_s - \Gamma_{s_-})}})\, dx$. Since, for all x,

$$\mathbb{E}\sum_{s\le t} e^{(-x^2/2(\Gamma_s - \Gamma_{s_-}))} \frac{1}{\sqrt{2\pi(\Gamma_s - \Gamma_{s_-})}} = t\int_0^{+\infty} e^{-x^2/2\gamma} \frac{1}{\sqrt{2\pi\gamma}} n_\Gamma(\gamma)\, d\gamma,$$

we get the formula for n_X. Setting $m_\alpha = \mathbb{E}|X|^\alpha$, for X a standard Gaussian variable, yields

$$\int_{\mathbb{R}} |x|^\alpha n_X(x)\,dx = m_\alpha \int_0^{+\infty} \gamma^{\alpha/2} n_\Gamma(\gamma)\,d\gamma.$$

Thus, $\mathbb{E}|X_\Delta|^r = m_r \mathbb{E}(\Gamma_\Delta^{r/2})$. As $r/2 \le 1$, $\Gamma_\Delta^{r/2} = (\sum_{s\le\Delta} \Gamma_s - \Gamma_{s_-})^{r/2} \le \sum_{s\le\Delta}(\Gamma_s - \Gamma_{s_-})^{r/2}$. Taking expectation gives the result.

Proof of (iii). The result is proved, for example, in Barndorff-Nielsen, Shephard and Winkel [(2006), Theorem 1, page 804] [see also Aït-Sahalia and Jacod (2007)].

8.9. *Proof of Proposition* 5.2. We have $\mathbb{E}(Z_k) = \Delta b$ and, for $\ell \ge 2$, $\mathbb{E}(Z_k^\ell) = \Delta c_\ell + o(\Delta)$. Therefore, $\hat{b}$ is an unbiased estimator of b and, for $\ell \ge 2$, $\sqrt{n\Delta}|\mathbb{E}\hat{c}_\ell - c_\ell| = \sqrt{n\Delta}\,O(\Delta)$. Hence, the additional condition $n\Delta^3 = o(1)$ to erase the bias.

Setting $c_1 = b$, $\hat{c}_1 = \hat{b}$, as $\operatorname{Var} Z_k^\ell = \Delta c_{2\ell} + o(\Delta)$ for $\ell \ge 1$, we have $n\Delta \operatorname{Var} \hat{c}_\ell = c_{2\ell} + O(\Delta)$. Writing $\sqrt{n\Delta}(\hat{c}_\ell - \mathbb{E}\hat{c}_\ell) = (n\Delta)^{-1/2}\sum_{k=1}^n (Z_k^\ell - \mathbb{E}Z_k^\ell) = \sum_{k=1}^n \chi_{k,n}$, it is now enough to prove that $\sum_{k=1}^n \mathbb{E}|\chi_{k,n}|^{2+\varepsilon}$ tends to 0. Under the assumption, we have

$$\sum_{k=1}^n \mathbb{E}|\chi_{k,n}|^{2+\varepsilon} \le \frac{C}{n^{\varepsilon/2}\Delta^{1+\varepsilon/2}}\bigl(\mathbb{E}|Z_k|^{\ell(2+\varepsilon)} + |\mathbb{E}(Z_k^\ell)|^{2+\varepsilon}\bigr) \le \frac{C}{(n\Delta)^{\varepsilon/2}},$$

which gives the result.

8.10. *Proof of Proposition* 5.3. The study of (5.3) relies on the following result which is standard for $r = 2$.

LEMMA 8.3. *Let $Y_t = \theta t + \sigma W_t$ for θ a constant and consider $\tilde{\sigma}_n^{(r)} = \frac{1}{m_r n \Delta^{r/2}} \sum_{k=1}^n |Y_{k\Delta} - Y_{(k-1)\Delta}|^r$.*

Then, for all r, $\sqrt{n}(\tilde{\sigma}_n^{(r)} - \sigma^r)$ converges in distribution to a centered Gaussian distribution with variance $\sigma^{2r}(m_{2r}/m_r^2 - 1)$ as n tends to infinity, Δ tends to 0, *$n\Delta$ tends to infinity, and $n\Delta^2$ tends to* 0.

PROOF. We have $\mathbb{E}\tilde{\sigma}_n^{(r)} = \frac{1}{m_r}\mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^r$, for X a standard Gaussian variable. Thus,

$$\begin{aligned}\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r &= \sigma^r(e^{-\theta^2\Delta/2\sigma^2} - 1)\\ &\qquad + \frac{1}{m_r} e^{-\theta^2\Delta/2\sigma^2} \int |u|^r (e^{\theta u\sqrt{\Delta}/\sigma^2} - 1) e^{-u^2/(2\sigma^2)} \frac{du}{\sigma\sqrt{2\pi}}.\end{aligned}$$

Noting that $e^{\theta u\sqrt{\Delta}/\sigma^2} - 1 = \theta u\sqrt{\Delta}/\sigma^2 + \Delta\sum_{n\ge 2}\frac{1}{n!}(u\theta/\sigma^2)^n \Delta^{n/2-1}$ and that $\int |u|^r u e^{-u^2/(2\sigma^2)}\,du/(\sigma\sqrt{2\pi}) = 0$, we easily obtain

$$\bigl|\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r\bigr| \le c\Delta.$$

Thus, $\sqrt{n}|\mathbb{E}\tilde{\sigma}_n^{(r)} - \sigma^r| = o(1)$ if $\sqrt{n}\Delta = (n\Delta^2)^{1/2} = o(1)$. Noting that $\mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^k$ converges to $\sigma^k m_k$ as Δ tends to 0, we get $n\operatorname{Var}\tilde{\sigma}_n^{(r)} \to \sigma^{2r}(m_{2r}/m_r^2 - 1)$.

Finally, we look at $\chi_{k,n} = n^{-1}(|\theta\sqrt{\Delta} + \sigma(W_{k\Delta} - W_{(k-1)\Delta})/\sqrt{\Delta}|^r - \mathbb{E}|\theta\sqrt{\Delta} + \sigma X|^r)$, which satisfies $n\mathbb{E}\chi_{k,n}^4 \le c/n^3$. Hence, $\sqrt{n}(\tilde{\sigma}_n^{(r)} - \mathbb{E}\tilde{\sigma}_n^{(r)})$ converges in distribution to the centered Gaussian with the announced variance which completes the proof. $\square$

PROOF OF (i). As noted above, $L_t = b_0 t + \sigma W_t + \Gamma_t$ with $b_0 = b - \int xn(x)\,dx$. Using that, for $r \le 1$, $||\sum a_i + b_i|^r - |\sum a_i|^r| \le \sum |b_i|^r$, we get $|\hat{\sigma}_n^{(r)} - \tilde{\sigma}_n^{(r)}| \le \frac{1}{m_r n\Delta^{r/2}}\sum_{k=1}^n |\Gamma_{k\Delta} - \Gamma_{(k-1)\Delta}|^r$, where $\tilde{\sigma}_n^{(r)}$ is built with $Y_t = b_0 t + \sigma W_t$ as in the previous lemma. Thus, applying Proposition 5.1(i),

$$\mathbb{E}\sqrt{n}|\hat{\sigma}_n^{(r)} - \tilde{\sigma}_n^{(r)}| \le \frac{1}{m_r}\sqrt{n}\Delta^{1-r/2}\int |x|^r n(x)\,dx.$$

Since $r < 1$, the constraint $n\Delta^{2-r} = o(1)$ can be fulfilled and implies $n\Delta^2 = o(1)$. Hence, the result follows from the previous proposition.

PROOF OF (ii). The proof is analogous to the previous one [using Proposition 5.1(ii)] and is omitted. As $\sigma(r) = [\hat{\sigma}_n^{(r)}]^{1/r}$, we conclude for $\hat{\sigma}(r)$ by using the delta-method. $\square$

APPENDIX: THE TALAGRAND INEQUALITY

The following result follows from the Talagrand concentration inequality given in Klein and Rio (2005) and arguments in Birgé and Massart (1998) (see the proof of their Corollary 2, page 354).

LEMMA A.1 (Talagrand inequality). *Let $Y_1, \dots, Y_n$ be independent random variables, let $\nu_{n,Y}(f) = (1/n)\sum_{i=1}^n [f(Y_i) - \mathbb{E}(f(Y_i))]$ and let $\mathcal{F}$ be a countable class of uniformly bounded measurable functions. Then for $\epsilon^2 > 0$*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|^2 - 2(1+2\epsilon^2)H^2\Big]_+ \le \frac{4}{K_1}\left(\frac{v}{n}e^{-K_1\epsilon^2 nH^2/v} + \frac{98M^2}{K_1 n^2 C^2(\epsilon^2)}e^{-2K_1 C(\epsilon^2)\epsilon/(7\sqrt{2})nH/M}\right),$$

with $C(\epsilon^2) = \sqrt{1+\epsilon^2} - 1$, $K_1 = 1/6$ and

$$\sup_{f\in\mathcal{F}} \|f\|_\infty \le M, \qquad \mathbb{E}\Big[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|\Big] \le H, \qquad \sup_{f\in\mathcal{F}} \frac{1}{n}\sum_{k=1}^n \operatorname{Var}(f(Y_k)) \le v.$$

By standard density arguments, this result can be extended to the case where $\mathcal{F}$ is a unit ball of a linear normed space, after checking that $f \mapsto \nu_n(f)$ is continuous and $\mathcal{F}$ contains a countable dense family.

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