

QUOTIENT CORRELATION: A SAMPLE BASED ALTERNATIVE TO PEARSON'S CORRELATION¹

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The quotient correlation is defined here as an alternative to Pearson's correlation that is more intuitive and flexible in cases where the tail behavior of data is important. It measures nonlinear dependence where the regular correlation coefficient is generally not applicable. One of its most useful features is a test statistic that has high power when testing nonlinear dependence in cases where the Fisher's Z -transformation test may fail to reach a right conclusion. Unlike most asymptotic test statistics, which are either normal or χ^2 , this test statistic has a limiting gamma distribution (henceforth, *the gamma test statistic*). More than the common usages of correlation, the quotient correlation can easily and intuitively be adjusted to values at tails. This adjustment generates two new concepts—the tail quotient correlation and the tail independence test statistics, which are also gamma statistics. Due to the fact that there is no analogue of the correlation coefficient in extreme value theory, and there does not exist an efficient tail independence test statistic, these two new concepts may open up a new field of study. In addition, an alternative to Spearman's rank correlation, a rank based quotient correlation, is also defined. The advantages of using these new concepts are illustrated with simulated data and a real data analysis of internet traffic.

1. Introduction. In measuring the linear relation, Pearson's correlation, based on the product of z -scores, is extremely widely used. Applications related to correlation coefficients are thriving in many areas. However, there does not coexist a term with Pearson's correlation based on the quotient as far as we know, and furthermore, there is no analogue of the correlation coefficient in extreme value theory [de Haan and de Ronde (1998)]. In this paper the quotient is chosen for its intuitive appeal.

This paper introduces a class of quotient correlations which can be used as an alternative to Pearson's correlation, and a class of rank based quotient correlations which can be used as an alternative to Spearman's rank correlation. Like both Pearson's correlation coefficients and Spearman's rank correlation coefficients, the quotient correlation coefficients and the rank based quotient correlation coefficients measure the level of agreement between two random variables.

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In testing linear (nonlinear) independencies, although much work has been done, there is no public consensus on which method is the best. In the literature, Fisher (1921) introduces the extremely useful Z-transformation function, and Hotelling (1953) states that the best present-day usage in dealing with correlation coefficients is based on Fisher (1915, 1921). Recently, Freeman and Modarres (2005) showed that after Box–Cox transformation to approximate normal scales, Fisher’s Z-transformation test is more efficient when testing independence between two positive random variables. Besides Fisher’s Z-transformation test, many other testing procedures have been developed. These procedures are summarized in books by Cromwell, Labys and Terraza (1994), Cromwell, Hannan, Labys and Terraza (1994), Kanji (1999) and Thode (2002), among others.

We deliver a new test statistic, which we call the gamma test for independence, based on the quotient correlation concept. Considering that almost all existing asymptotic tests are normal or χ^2 tests, our test is innovative because it has an asymptotic gamma distribution. More importantly, we show a theorem which assures that the limiting distribution of the test is still a gamma distribution even if one uses parametric estimated distribution functions or nonparametric empirical distribution functions to transform the data into unit Fréchet scales which are assumed in the definition of the test statistic, that is, parameter estimations and marginal transformations won’t affect the limiting distribution of the test statistic. We shall show its efficiency and the inefficiency of Fisher’s Z-transformation test when nonlinear dependence occurs.

There is a growing body of literature on modeling and testing tail dependence, also known as extremal dependence or asymptotic dependence, between the components of a two dimensional random vector, which refers to the concurrence of extreme values in each component. Examples dealing with asymptotically independent random variables can be found in Watson (1954), Sibuya (1960), de Haan and Resnick (1977), Chow and Teugels (1978), Anderson and Turkman (1991), Joe (1997) and Grady (2000). A significant step is due to Ledford and Tawn (1996, 1997). They introduced a sub-model which can be used to model tail dependence and nearly tail independence. Several papers, such as Peng (1999), Draisma, Drees, Ferreira and de Haan (2004), Ferreira and de Haan (2004) and others, have focused on estimation and rare events modeling, particularly of Dutch coast wind and water level extremes. However, it is not clear whether the data are tail dependent or not due to different conclusions. In the meantime, Hsing, Kluppelberg and Kuhn (2003) develop visualization methods to check tail dependence, and Falk and Michel (2004) study a different sub-model and graphic checking of tail dependence. These models are certainly useful and important in extreme value theory. However, constructing a test statistic for tail independence is still an interesting open problem [Campos, Marron, Resnick, Park and Jeffay (2003)]. Longin and Solnik (2001) and Campos et al. (2003) point out that in the study of extremal dependence, misleading results, even conflicting results, are often reported in the literature.

The quotient correlations can easily and intuitively be adjusted to values at tails. This adjustment enables us to construct tail dependence measures and to construct efficient test statistics for tail independence. This adjustment generates two new concepts—the tail quotient correlation and tail independence test statistics, which are also gamma statistics and effectively detect tail independencies or tail dependencies in all simulation examples to be presented here.

The paper is organized as follows: in Section 2 we introduce the quotient correlation coefficient and the rank based quotient correlation coefficient. We then in Section 3 develop the gamma test statistic for testing independence. We compare the gamma test with the Fisher’s Z-transformation test. Section 4 introduces the tail quotient correlation concept. In Section 5 we obtain our gamma test statistics for testing tail independence. A necessary condition for two random variables being tail independent is presented here. In Section 6 simulation examples are analyzed. In Section 7 a real data example of internet traffic is tested. Concluding remarks are given in Section 8. Technically involved proofs are deferred to Section 9.

2. The quotient. Suppose X and Y are identically distributed positive random variables satisfying $P(X \geq Y) > 0$, $P(X \leq Y) > 0$. Then the quotients between X and Y are Y/X and X/Y . Suppose $\{(X_i, Y_i), i = 1, \dots, n\}$ is a bivariate random sample of (X, Y) . Then we will have $2n$ quotients.

2.1. *The quotient correlation coefficient.* It is easy to see that both $\max_{i \leq n} \{X_i/Y_i\} - 1$ and $\max_{i \leq n} \{Y_i/X_i\} - 1$ are asymptotically nonnegative. When these two values are both close to zero for a sufficiently large sample size n , one can say that the changing magnitudes of X and Y are very close to each other. The same is true for $\max_{i \leq n} \{X_i/Y_i\} \max_{i \leq n} \{Y_i/X_i\} - 1$ being close to zero.

We now use the above three quantities to define a new statistical coefficient, that is, the quotient correlation coefficient:

$$(2.1) \quad q_n = \frac{\max_{i \leq n} \{Y_i/X_i\} + \max_{i \leq n} \{X_i/Y_i\} - 2}{\max_{i \leq n} \{Y_i/X_i\} \times \max_{i \leq n} \{X_i/Y_i\} - 1}$$

Some functional properties of q_n are illustrated by looking at the following function:

$$f(x, y) = \frac{x + y - 2}{xy - 1} \quad \text{for } x \geq 1, y \geq 1, x + y > 2.$$

The case of $x + y = 2$ is trivial since it implies X_i and Y_i are equal. It is easy to see $0 \leq f(x, y) \leq 1$, $f(1, y) = 1$, $f(x, 1) = 1$, and $\frac{\partial f}{\partial x} = -\frac{(y-1)^2}{(xy-1)^2} < 0$, $\frac{\partial f}{\partial y} = -\frac{(x-1)^2}{(xy-1)^2} < 0$, $\frac{\partial^2 f}{\partial x \partial y} = -2\frac{(x-1)(y-1)}{(xy-1)^3} < 0$, which imply $f(x_1, y_1) \leq f(x_2, y_2)$, where $x_1 \geq x_2, y_1 \geq y_2$.

The properties of the function $f(x, y)$ defined above suggest that, asymptotically, q_n can take values ranging from 0 to 1. The monotonicity of $f(x, y)$ suggests that, for a fixed sample size n , the larger the $\max_{i \leq n} \{X_i/Y_i\}$ or the $\max_{i \leq n} \{Y_i/X_i\}$, the smaller the q_n , and hence the lower the agreement of changing magnitudes. The boundary properties of $f(x, y)$, that is, $f(x, 1) = 1$, $f(1, y) = 1$, suggest that the value of q_n is largely determined by the smaller value of $\max_{i \leq n} \{X_i/Y_i\}$ and $\max_{i \leq n} \{Y_i/X_i\}$. This property is very useful, especially when there exist outliers in one of the two sequences.

The computed q_n value can suggest the dependence magnitude between X and Y . For example, when q_n is close to 1, there are co-movements between X_i and Y_i , and hence, X and Y are nearly completely dependent.

Notice that in the definition of q_n the values of X_i and Y_i are required to be positive. In practice, this may not be the case. A variable transformation can resolve the issue. Sometimes a simple shift of the data to the right works too. For example, we can have $Y'_i = c - Y_i$, $i = 1, \dots, n$, being positive values, where c is chosen such that both $\max_{i \leq n} \{Y'_i/X_i\}$ and $\max_{i \leq n} \{X_i/Y'_i\}$ are greater than 1. Then we can use Y'_i and X_i to define q_n as we did in (2.1). When q_n is close to 1, the movement of X_i is opposite to that of Y_i .

We now give an example that shows even if the supports of X and Y are $(0, \infty)$, X/Y and Y/X are bounded, and hence, q_n is greater than 0 regardless of sample size.

EXAMPLE 2.1. Suppose Z_l , $l = 1, \dots, L$, are independent unit Fréchet random variables with distribution function $F(x) = \exp(-1/x)$, $x > 0$. Let $X = \max_{1 \leq l \leq L} \alpha_{l1} Z_l$, $Y = \max_{1 \leq l \leq L} \alpha_{l2} Z_l$, where $\alpha_{li} > 0$, $\sum_l \alpha_{li} = 1$, $i = 1, 2$. Let $c_1 = \max_{l \leq L} (\alpha_{l1}/\alpha_{l2})$ and $c_2 = \max_{l \leq L} (\alpha_{l2}/\alpha_{l1})$, then $P(X/Y \leq c_1) = P(\max_l \alpha_{l1} Z_l < \max_l c_1 \alpha_{l2} Z_l) = 1$, and $P(Y/X \leq c_2) = P(\max_l \alpha_{l2} Z_l < \max_l c_2 \alpha_{l1} Z_l) = 1$. Therefore, $\max_{i \leq n} \{Y_i/X_i\} \xrightarrow{\text{a.s.}} c_1$, $\max_{i \leq n} \{X_i/Y_i\} \xrightarrow{\text{a.s.}} c_2$, and $q_n \xrightarrow{\text{a.s.}} \frac{c_1+c_2-2}{c_1*c_2-1}$, as $n \rightarrow \infty$.

This is a special case of multivariate maxima of moving maxima processes (or M4 processes) which have been relatively extensively studied in Smith and Weissman (1996), Zhang (2004, 2006), Zhang and Shinki (2006) and Zhang and Smith (2004a). This example will be used in Section 6.

Remark 2.2 stresses the importance of having the data used to compute q_n in a unified scale.

REMARK 2.2. Suppose $\{(X_i^*, Y_i^*, U_i^*), i = 1, \dots, n\}$ is a trivariate random sample of (X^*, Y^*, U^*) , where X^* , Y^* , U^* are independent. Let $X_i = U_i^* X_i^*$, $Y_i = U_i^* Y_i^*$, then X_i and Y_i are dependent. However, $X_i/Y_i = X_i^*/Y_i^*$ which suggests that the value of q_n computed based on X_i and Y_i and the value of q_n computed based on X_i^* and Y_i^* are the same. At first glance the quotient correlation

seemed unusable. Notice that X_i and X_i^* do not have the same distribution. When both X_i and X_i^* are converted into unit Fréchet scales, the quotients computed from the resulting variables will no longer give the same q_n value. In other words, the quotient correlation coefficient is applicable.

It is natural to use unit Fréchet scores to define the quotient measure since the limiting distributions of maximum of quotients are a unit Fréchet distribution when each quotient is derived from two independent unit Fréchet (or unit exponential) random variables. When marginals are not unit Fréchet, marginal transformation is needed. Nonparametric methods of converting the data into unit Fréchet scales will be discussed in Section 2.2. For parametric method details, we refer to Smith (2003), Zhang (2005) and Zhang and Smith (2004b).

The quotient correlation q_n measures the nonlinear dependence since the quotient itself is a nonlinear bivariate function. One can see that the value of q_n mainly depends on the tail observations of two random variables X and Y . If X and Y are tail independent, the value of q_n turns out to be negligibly small. In this case, we propose to replace the maximal quotients by the k th largest quotients (or the p th percentiles) in their corresponding quotient series. In practice, k should be small (1, 2, 3, etc.) in order to make the calculated dependence measures meaningful. Similarly, the choice of p should be meaningful. These new statistics will be studied in a different project. We focus on (2.1) in this work.

2.2. The rank based quotient correlation coefficient. Suppose now that $X_{1n} \leq X_{2n} \leq \dots \leq X_{nn}$ are the ordered sample of X_1, X_2, \dots, X_n ; $Y_{1n} \leq Y_{2n} \leq \dots \leq Y_{nn}$ are the ordered sample of Y_1, Y_2, \dots, Y_n . Define the empirical distribution function of the X sample by $\widehat{F}_X(x) = 1/n \sum_{i=1}^n I_{(X_i \leq x)}$. Then $\widehat{F}_X(X_{kn}) = k/n$. Similarly, define the empirical distribution function of the Y sample by $\widehat{F}_Y(y) = 1/n \sum_{i=1}^n I_{(Y_i \leq y)}$. Then $\widehat{F}_Y(Y_{kn}) = k/n$. Therefore, when we transform the data into the unit Fréchet scale based on the empirical distribution functions and ranks, we get the same unit Fréchet scaled values for two identical ranks in both X and Y samples.

The transformation can be done with the function $-1/[\epsilon + \log\{\widehat{F}_X(X_{kn})\}] = -1/[\epsilon + \log(k/n)]$, $k = 1, 2, \dots, n$, where ϵ is a small number to avoid division by 0. In practice, however, this small number is sensitive to the transformed values depending on the sample size n . Similar discussions hold for the Y sample too. Here, we introduce a different procedure to do data transformation.

Using the above arguments, we generate a unit Fréchet sample of size n and denote $Z_{(1)} < Z_{(2)} < \dots < Z_{(n)}$ as the ordered sample. The rank based quotient correlation is defined as

$$(2.2) \quad q_n^{\mathfrak{R}} = \frac{\max_{i \leq n} \{Z_{(\text{rank}[Y_i])} / Z_{(\text{rank}[X_i])}\} + \max_{i \leq n} \{Z_{(\text{rank}[X_i])} / Z_{(\text{rank}[Y_i])}\} - 2}{\max_{i \leq n} \{Z_{(\text{rank}[Y_i])} / Z_{(\text{rank}[X_i])}\} \times \max_{i \leq n} \{Z_{(\text{rank}[X_i])} / Z_{(\text{rank}[Y_i])}\} - 1}.$$

One immediate advantage of this definition is that the transformation does not depend on any correction term. It is obvious that the definition is also based on a

simulated random sample of a unit Fréchet random variable. We have found this form is easy to implement and the simulation results are very close to q_n , which is based on a parametric transformation. In practice, we suggest simulating 10 or more unit Fréchet samples of size n , computing and reporting the mean or median value of $q_n^{\mathfrak{R}}$.

After establishing quotient correlation coefficients, in the next section we present one of the important application features of quotient correlations, that is, their uses in testing independence between random variables.

3. The gamma test statistic for testing independence.

3.1. *The hypothesis and the test statistic.* It is well known that any absolutely continuous random variable can be converted into a random variable with desired properties by marginal transformation. Without loss of generality, suppose now $\{(X_i, Y_i), i = 1, \dots, n\}$ is a bivariate random sample of (X, Y) with unit Fréchet margins. Then we can formulate the following hypothesis test of independence:

$$(3.1) \quad H_0^c : X \text{ and } Y \text{ are independent} \quad \text{versus} \quad H_1^c : X \text{ and } Y \text{ are dependent.}$$

Here the superscript c means that the complete data are used in (3.1). Later we shall give cases where only tail values are used.

The following theorem shows the properties of a random sample under H_0^c of (3.1).

THEOREM 3.1. *If X and Y are independent and have unit Fréchet margins, and $(X_i, Y_i), i = 1, \dots, n$, is a random sample from (X, Y) , then $\lim_{n \rightarrow \infty} P\{n^{-1} \times (\max_{i \leq n} Y_i/X_i + 1) \leq x\} = e^{-1/x}$, and random variables $\max_{i \leq n} Y_i/X_i$ and $\min_{i \leq n} Y_i/X_i$ are asymptotically independent—that is,*

$$(3.2) \quad \lim_{n \rightarrow \infty} P \left\{ n^{-1} \left(\max_{i \leq n} Y_i/X_i + 1 \right) \leq x, \right. \\ \left. n^{-1} \left(\max_{i \leq n} X_i/Y_i + 1 \right) \leq y \right\} = e^{-1/x-1/y}.$$

Furthermore, as $n \rightarrow \infty$, the random variable q_n is asymptotically gamma distributed—that is,

$$(3.3) \quad nq_n \xrightarrow{\mathcal{L}} \zeta,$$

where ζ is a gamma(2, 1) random variable.

A proof of Theorem 3.1 is deferred to Section 9.

The limiting distribution of nq_n leads to the following testing procedure (the gamma test): if $nq_n > \zeta_\alpha$, where ζ_α is the upper α th percentile of the gamma(2, 1) distribution, H_0^c of (3.1) is rejected; otherwise, it is retained.

As mentioned earlier, we can transform the underlying random variables to any other scales of interest. This gives the power from a probabilistic point of view one may assume without loss of generality that the marginal distributions are known. Under the null hypothesis, although we do not have any parameters that associate one population with another, we do have parameters in each individual population. Estimators for the marginal distribution may influence the asymptotic behavior of tests. In a statistical problem X_i could be a unit Fréchet random variable resulting from theoretical variable transformation, while $\tilde{X}_i(n)$ could be a unit Fréchet scaled value resulting from a variable transformation using an estimated distribution function. In most applications $\tilde{X}_i(n)$ are used in statistical inferences including testing tail independencies.

In the following Theorem 3.2, we introduce a sufficient condition under which the marginal effects from the estimated parameter values do not have any effects on the asymptotic behavior of tests.

THEOREM 3.2. *Suppose $\theta, \theta_i, i = 1, \dots, n$, are absolutely continuous random variables with distribution function $G_\theta(x)$, and $\eta, \eta_i, i = 1, \dots, n$, are absolutely continuous random variables with distribution function $G_\eta(x)$. Denote any estimated distribution function for θ by $\hat{G}_\theta(x)$, and any estimated distribution function for η by $\hat{G}_\eta(x)$. Define $\tilde{X}_i = F^{-1}(\hat{G}_\theta(\theta_i))$, $\tilde{Y}_i = F^{-1}(\hat{G}_\eta(\eta_i))$, $i = 1, \dots, n$, where $F(x)$ is the unit Fréchet distribution function. Suppose $\hat{G}_\theta(\theta_i) \xrightarrow{\text{a.s.}} G_\theta(\theta_i)$, $\hat{G}_\eta(\eta_i) \xrightarrow{\text{a.s.}} G_\eta(\eta_i)$, for all i , as $n \rightarrow \infty$. Then random variables $\max_{i \leq n} \tilde{Y}_i / \tilde{X}_i$ and $\max_{i \leq n} \tilde{X}_i / \tilde{Y}_i$ are each asymptotically gamma distributed, and they are asymptotically independent—that is,*

$$(3.4) \quad \lim_{n \rightarrow \infty} P \left\{ n^{-1} \left(\max_{i \leq n} \tilde{Y}_i / \tilde{X}_i + 1 \right) \leq x, n^{-1} \left(\max_{i \leq n} \tilde{X}_i / \tilde{Y}_i + 1 \right) \leq y \right\} = e^{-1/x-1/y}.$$

Furthermore, as $n \rightarrow \infty$, the random variable,

$$(3.5) \quad \tilde{q}_n = \frac{\max_{i \leq n} \{\tilde{Y}_i / \tilde{X}_i\} + \max_{i \leq n} \{\tilde{X}_i / \tilde{Y}_i\} - 2}{\max_{i \leq n} \{\tilde{Y}_i / \tilde{X}_i\} \times \max_{i \leq n} \{\tilde{X}_i / \tilde{Y}_i\} - 1}$$

is asymptotically gamma distributed—that is, $n\tilde{q}_n \xrightarrow{\mathcal{L}} \zeta$, where ζ is a gamma(2, 1) random variable.

A proof of Theorem 3.2 is deferred to Section 9.

An immediate practical application is to apply the empirical distribution function in $\hat{G}_\theta(\cdot)$ and $\hat{G}_\eta(\cdot)$, and we have the following corollary.

COROLLARY 3.3. *Suppose X_1, \dots, X_n are observed variables with absolutely continuous distribution $F_X(x)$, and Y_1, \dots, Y_n are observed variables*

with absolutely continuous distribution $F_Y(y)$. Define

$$\widehat{F}_X(x) = \frac{1}{n + 1/n} \sum_{i=1}^n I_{(X_i < x)}, \quad \widehat{F}_Y(y) = \frac{1}{n + 1/n} \sum_{i=1}^n I_{(Y_i < y)}.$$

Then, as $n \rightarrow \infty$ the empirical distribution transformation based quotient correlation

$$\begin{aligned} q_n^e = & \left(\max_{i \leq n} \{ \log(\widehat{F}_X(X_i)) / \log(\widehat{F}_Y(Y_i)) \} \right. \\ & \left. + \max_{i \leq n} \{ \log(\widehat{F}_Y(Y_i)) / \log(\widehat{F}_X(X_i)) \} - 2 \right) \\ (3.6) \quad & \times \left(\max_{i \leq n} \{ \log(\widehat{F}_X(X_i)) / \log(\widehat{F}_Y(Y_i)) \} \right. \\ & \left. \times \max_{i \leq n} \{ \log(\widehat{F}_Y(Y_i)) / \log(\widehat{F}_X(X_i)) \} - 1 \right)^{-1} \end{aligned}$$

is asymptotically gamma distributed—that is, $nq_n^e \xrightarrow{\mathcal{L}} \zeta$, where ζ is a gamma(2, 1) random variable.

Notice that q_n^e and $q_n^{\mathfrak{R}}$ are asymptotically equal in distribution, and we also have $nq_n^{\mathfrak{R}} \xrightarrow{\mathcal{L}} \zeta$, as $n \rightarrow \infty$. We apply $q_n^{\mathfrak{R}}$ in this work.

3.2. *Power comparisons between the gamma test and Fisher’s Z-transformation.* We now assume X and Y are normal random variables for the moment. Suppose r_{XY} is the sample correlation coefficient of a bivariate random sample of size n . Then Fisher’s Z -transformation is $W = \frac{1}{2} \log\left(\frac{1+r_{XY}}{1-r_{XY}}\right)$, which satisfies $E(W) \cong \frac{1}{2} \log\left(\frac{1+\rho_{XY}}{1-\rho_{XY}}\right)$, $\text{Var}(W) \cong \frac{1}{n-3}$, and $Z = \frac{W-E(W)}{\sqrt{\text{Var}(W)}} \xrightarrow{\mathcal{L}} N(0, 1)$, which in turn allows us to test different hypotheses about the correlation—see details in papers by Fisher (1915, 1921), Hotelling (1953), Freeman and Modarres (2005) and a book by Härdle and Simar (2003). In our applications we test the null hypothesis of $\rho_{XY} = 0$ versus the alternative hypothesis of $\rho_{XY} \neq 0$.

When observed variables are actually independent, we have found that both Fisher’s Z -transformation test and the gamma test have low Type I error rate (less than 5%), which, suggests that both tests are practically efficient. We now show the efficiency of the gamma test when variables are nonlinear dependent.

EXAMPLE 3.1. In this example, the correlation coefficient between two random variables is zero, while the two random variables are actually dependent. For example, suppose X is standard normal, and let $Y = X^2$. It is easy to show $\text{cov}(X, Y) = 0$. Like the previous example, we simulate 100 bivariate random samples of size $n, n = 25, \dots, 100$. For each random sample, we conduct both Fisher’s

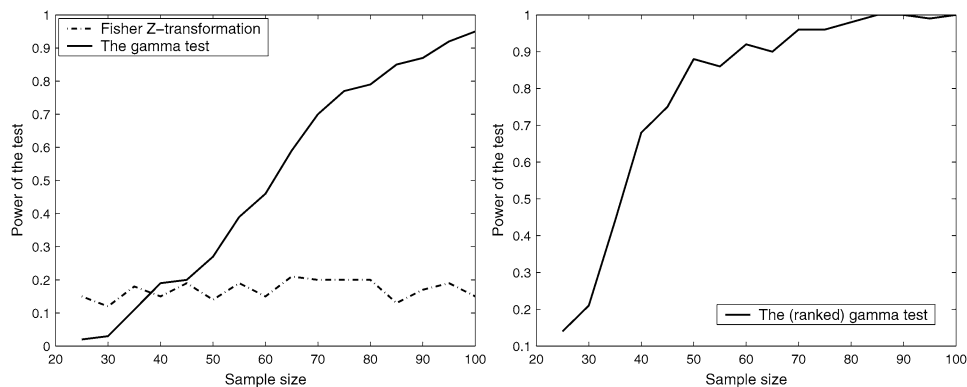


FIG. 1. Comparison between Fisher’s Z-transformation test and the gamma test. Left panel is based on q_n and Fisher’s Z-transformation function. Right panel is rank based test.

Z-transformation test and the gamma test with significance level $\alpha = 0.05$. At each sample size n , we compute the empirical power of each test. The performances are illustrated in the left panel of Figure 1. Clearly, one can see that when sample size is larger than 37, the gamma test is superior, but Fisher’s Z-transformation test is not applicable in this instance.

Needless to say, we are not proposing a test statistic to replace Fisher’s Z-transformation test. In our point of view, these two tests can be used as two primary tests for testing independence since in most cases they will agree with each other and both tests are easy to implement. However, if the gamma test fails to suggest independence, while Fisher’s Z-transformation test does suggest independence, the data may be actually drawn from two nonlinearly dependent random variables.

As a comparison, we performed the rank based quotient tests (replaced q_n by q_n^{rk}) to the example above. The results are plotted in the right panel in Figure 1. One can see that its performance is just as good as (or slightly better than) the quotient correlation.

Notice that the power curve is not very smooth, and they may not be very accurate. For the purpose of demonstrating that the gamma test outperforms the Fisher’s Z-transformation test in cases when nonlinear dependencies occur, we argue that Figure 1 may be sufficiently enough. To estimate the power curve much more accurate, one can use monotone smoothing which uses the strengths from neighboring points; see He and Shi (1998), for example, we will apply this method in a different study.

4. The tail quotient correlation. In the literature the characterization of tail dependence between random variables is through a tail dependence index [Sibuya (1960) and Embrechts, McNeil and Straumann (2002)]. We present the definition next.

DEFINITION 4.1. Suppose random variables X and Y are identically distributed with $x_F = \sup\{x \in \mathbb{R} : P(X \leq x) < 1\}$. The quantity (if exists): $\lambda = \lim_{u \rightarrow x_F} P(X > u | Y > u)$ is called the bivariate tail dependence index, or the tail dependence index, between X and Y . It quantifies the amount of dependence of the bivariate upper tails of X and Y . If $\lambda > 0$, then X and Y are tail dependent, otherwise the two random variables are tail independent.

This definition has been extended to cross-sectional tail dependencies in multivariate time series; see Zhang and Huang (2006) for example. Heffernan, Tawn and Zhang (2007) showed that X and Y are not necessarily identically distributed when one studies tail dependencies. However, we restrict our study to identically distributed random variables in this work.

To calculate λ , the joint distribution of X and Y must be known. In practice, when data is available, the data can be used to estimate the tail dependence index based on the empirical distribution functions and a very large threshold value u . We seek a new alternative quotient based statistical coefficient that measures the magnitude of tail dependence between two variables.

In general, when we study tail values, on one hand, we should use as much tail information as possible; on the other hand, we should use as little data from the center and below the threshold as possible. It is natural to use values above a threshold. However, this idea raises questions: how do we deal with the values below the threshold, and how do we get the quotient of the co-responses. Our choice is to use the threshold value for all those values below the threshold.

Using the arguments above, we define the tail quotient correlation coefficient as

$$(4.1) \quad q_{u,n} = \frac{\max_{i \leq n} \{(u + W_i)/(u + V_i)\} + \max_{i \leq n} \{(u + V_i)/(u + W_i)\} - 2}{\max_{i \leq n} \{(u + W_i)/(u + V_i)\} \times \max_{i \leq n} \{(u + V_i)/(u + W_i)\} - 1},$$

where u is a positive threshold, and W_i and V_i are exceedance values over the threshold u of positive random variables X_i and Y_i respectively. Here we assume X_i and Y_i are identically distributed.

Similarly, we can define a rank based tail quotient correlation coefficient. With the established notation in (2.2), a rank based tail quotient correlation is defined by

$$(4.2) \quad q_{u,n}^{\mathcal{R}} = \left(\max_{i \leq n} \left\{ \frac{u \mathbf{1}_{(W_i=0)} + Z(\text{rank}[W_i]) \mathbf{1}_{(W_i>0)}}{u \mathbf{1}_{(V_i=0)} + Z(\text{rank}[V_i]) \mathbf{1}_{(V_i>0)}} \right\} + \max_{i \leq n} \left\{ \frac{u \mathbf{1}_{(V_i=0)} + Z(\text{rank}[V_i]) \mathbf{1}_{(V_i>0)}}{u \mathbf{1}_{(W_i=0)} + Z(\text{rank}[W_i]) \mathbf{1}_{(W_i>0)}} \right\} - 2 \right) \times \left(\max_{i \leq n} \left\{ \frac{u \mathbf{1}_{(W_i=0)} + Z(\text{rank}[W_i]) \mathbf{1}_{(W_i>0)}}{u \mathbf{1}_{(V_i=0)} + Z(\text{rank}[V_i]) \mathbf{1}_{(V_i>0)}} \right\} \times \max_{i \leq n} \left\{ \frac{u \mathbf{1}_{(V_i=0)} + Z(\text{rank}[V_i]) \mathbf{1}_{(V_i>0)}}{u \mathbf{1}_{(W_i=0)} + Z(\text{rank}[W_i]) \mathbf{1}_{(W_i>0)}} \right\} - 1 \right)^{-1},$$

where u is a global threshold value from a unit Fréchet distribution function and $\mathbf{1}_{(\cdot)}$ is an indicator function. We will use this rank based tail quotient correlation to conduct the tail independence test in Section 7.

One of the important uses of the tail quotient correlation is in the testing of tail independence which we study in the next section. Simulation examples are illustrated in Section 6.

5. The gamma test statistics for testing tail independence. The following theorem gives a necessary condition that two random variables are tail independent.

THEOREM 5.1 (Necessary condition). *In Definition 4.1 suppose that X and Y are unit Fréchet random variables, and $\lambda = 0$, then $q_n \xrightarrow{\text{a.s.}} 0$, $q_{un} \xrightarrow{\text{a.s.}} 0$ as $n \rightarrow \infty$.*

A proof of Theorem 5.1 is deferred to Section 9.

It is clear from Theorem 5.1 that a large tail quotient correlation would suggest tail dependence, and a zero value of tail quotient correlation would suggest that tail dependence is very weak. In most cases, however, the computed tail quotient correlation coefficient may never be zero. A statistical inference problem then arises: How small does a tail quotient correlation coefficient need to be for us to conclude that the two random variables are tail independent? This turns out to be a statistical hypothesis inference problem:

$$(5.1) \quad H_0^t : (X, Y) \text{ is tail independent} \quad \Leftrightarrow \quad H_1^t : (X, Y) \text{ is tail dependent.}$$

Here the superscript t indicates that the test is for tail (in)dependence only.

Notice that (2.1) and (3.3) together test the null hypothesis of independence, not a null hypothesis of tail independence. We argue that the tail quotient correlation leads to a true tail independent test statistic under H_0^t next.

Let

$$(5.2) \quad \begin{pmatrix} X_1, & X_2, & \dots, & X_n \\ Y_1, & Y_2, & \dots, & Y_n \end{pmatrix}$$

be an independent array of unit Fréchet random variables. Now let (U_i, Q_i) , $i = 1, \dots, n$ be a bivariate random sequence, where both U_i and Q_i are correlated and have support over $(0, u]$ for a high threshold value u . Let $X_{ui} = X_i I_{\{X_i > u\}} + U_i I_{\{X_i \leq u\}}$, $Y_{ui} = Y_i I_{\{Y_i > u\}} + Q_i I_{\{Y_i \leq u\}}$, $i = 1, \dots, n$. Then

$$(5.3) \quad \begin{pmatrix} X_{u1} \\ Y_{u1} \end{pmatrix}, \begin{pmatrix} X_{u2} \\ Y_{u2} \end{pmatrix}, \dots, \begin{pmatrix} X_{un} \\ Y_{un} \end{pmatrix}$$

is a bivariate random sequence drawn from two dependent random variables X_{ui} and Y_{ui} . Notice that $X_{ui} I_{\{X_{ui} > u\}} (= X_i I_{\{X_i > u\}})$ and $Y_{ui} I_{\{Y_{ui} > u\}} (= Y_i I_{\{Y_i > u\}})$ are

independent, but $X_{ui}I_{\{X_{ui} \leq u\}}$ ($= U_i I_{\{X_i \leq u\}}$) and $Y_{ui}I_{\{Y_{ui} \leq u\}}$ ($= Q_i I_{\{Y_i \leq u\}}$) are dependent. In fact, one can easily construct arbitrarily dependent structure for values below the threshold u . Consequently, if only tail values are concerned, we can assume the tail values are drawn from (5.2) under the null hypothesis of tail independence. We have the following theorem.

THEOREM 5.2. *Suppose V_i and W_i are exceedance values (above u) in (5.3), and U_i and Q_i have the distribution $e^{1/u} e^{-1/x}$, $0 < x < u$. Then*

$$(5.4) \quad P\left(\frac{u + W_i}{u + V_i} \leq t\right) = \begin{cases} \frac{t}{1+t} - \frac{t}{1+t} e^{-(1+t)/u}, & \text{if } 0 < t < 1, \\ \frac{t}{1+t} + \frac{1}{1+t} e^{-(1+t)/u}, & \text{if } t \geq 1, \end{cases}$$

$$(5.5) \quad \lim_{n \rightarrow \infty} P\left\{n^{-1} \left[\max_{i \leq n} (u + W_i)/(u + V_i) + 1 \right] \leq x\right\} = e^{-(1-e^{-1/u})/x}.$$

Moreover,

$$(5.6) \quad \lim_{n \rightarrow \infty} P\left\{n \left[\min_{i \leq n} (u + W_i)/(u + V_i) \right] \leq x\right\} = 1 - e^{-(1-e^{-1/u})x}.$$

The random variables $\max_{i \leq n} (u + W_i)/(u + V_i)$ and $\max_{i \leq n} (u + V_i)/(u + W_i)$ are tail independent—that is,

$$(5.7) \quad \lim_{n \rightarrow \infty} P\left\{n^{-1} \left[\max_{i \leq n} \frac{(u + W_i)}{(u + V_i)} + 1 \right] \leq x, n^{-1} \left[\max_{i \leq n} \frac{(u + V_i)}{(u + W_i)} + 1 \right] \leq y\right\} = e^{-(1-e^{-1/u})/x - (1-e^{-1/u})/y}.$$

Furthermore, as $n \rightarrow \infty$, the random variable $q_{u,n}$ is asymptotically gamma distributed, that is,

$$(5.8) \quad nq_{u,n} \xrightarrow{\mathcal{L}} \zeta,$$

where ζ is gamma(2, $1 - e^{-1/u}$) distributed.

A proof of Theorem 5.2 is deferred to Section 9. Here are some intuitive interpretations. If $\max\{(u + W_i)/(u + V_i)\}$ or $\max\{(u + V_i)/(u + W_i)\}$ is close to 1, q_{un} will have a larger value, which in turn implies that nq_{un} has a larger probability to reject H_0^t or a small p -value will be obtained. A larger q_{un} value implies that there exists tail dependence (extreme co-movements) between X_i and Y_i .

Equations (4.1) and (5.8) together establish a new test statistic—we call it the gamma test for the null hypothesis of tail independence—which can be used to determine whether there is tail dependence between two random variables. When $nq_{un} > \zeta_\alpha$, where ζ_α is the upper α th percentile of the gamma(2, $1 - e^{-1/u}$) distribution, H_0^t of (5.1) is rejected.

Considering the threshold value u as a parameter, we obtain a new family (4.1) and (5.8) of gamma test statistics. When u tends to zero, we obtain the special case of (2.1) and (3.3).

Notice that as u is increasing, the value of $nq_{u,n}$ is increasing since both $\max\{(u + W_i)/(u + V_i)\}$ and $\max\{(u + V_i)/(u + W_i)\}$ are decreasing. Intuitively, a larger value of $nq_{u,n}$ renders to a rejection of the null hypothesis more likely. However, as u is increasing, $1 - e^{-1/u}$ is decreasing, and a decreasing scale parameter $1 - e^{-1/u}$ leads to a larger cutoff value ζ_α . Therefore, a truncation (below the threshold value u) of data does not make a rejection of H_0^I of (5.1) more likely. In fact, the gamma test can effectively detect all tail independent random variables in our simulation examples.

REMARK 5.1. The tail quotient correlation coefficient q_{um} can be used as a tail dependence measure once the null hypothesis of tail independence is rejected. In the literature several tail dependence measures have been proposed, such as those by Coles, Heffernan and Tawn (1999), Poon, Rockinger and Tawn (2001), Campos et al. (2003) and others. These measures and others have broad applications as long as tail events are concerned.

Also, notice that (5.8) holds when exceedance values of V_i and W_i are replaced by exceedance values of $\tilde{V}_i(n)$ and $\tilde{W}_i(n)$ resulting from $\tilde{X}_i(n)$ and $\tilde{Y}_i(n)$ respectively in Theorem 3.2.

The following theorem tells that one can apply the gamma tests when data are drawn from strict bivariate stationary processes which satisfy conditions specified in the theorem.

THEOREM 5.3. Suppose $\{(X_i^*, Y_i^*), i = 1, \dots, n\}$ is a strict bivariate stationary processes with marginal distribution $F_{X^*}(\cdot)$ and $F_{Y^*}(\cdot)$ which are absolutely continuous. $F(\cdot)$ is a unit Fréchet distribution function. Let $X_i = F^{-1}(F_{X^*}(X_i^*))$, $Y_i = F^{-1}(F_{Y^*}(Y_i^*))$, $i = 1, \dots, n$. Suppose the random sequences $\{\xi_i = X_i/Y_i\}$, $\{\xi_i' = Y_i/X_i\}$ satisfy Leadbetter's (1974) $D(u_n)$ and $D'(u_n)$ conditions for $u_n = nx - 1$, and Davis' (1982) C1 and C2 conditions. Suppose $\hat{F}_{X^*}(\cdot)$ and $\hat{F}_{Y^*}(\cdot)$ are estimated distribution functions for X_i^* and Y_i^* respectively, and as $n \rightarrow \infty$, $\hat{F}_{X^*}(X_i^*) \rightarrow F_{X^*}(X_i^*)$, $\hat{F}_{Y^*}(Y_i^*) \rightarrow F_{Y^*}(Y_i^*)$, for all $i = 1, 2, \dots$, almost surely. Define $\tilde{X}_i = F^{-1}(\hat{F}_{X^*}(X_i^*))$ and $\tilde{Y}_i = F^{-1}(\hat{F}_{Y^*}(Y_i^*))$. Then as $n \rightarrow \infty$, both (3.3) and (5.8) hold, and $n\tilde{q}_n \xrightarrow{\mathcal{L}} \zeta$, where \tilde{q}_n is defined as the one in (3.5).

A proof of Theorem 5.3 is deferred to Section 9.

6. Simulation examples. We have done substantial computation to check the efficiency of the gamma test. We have found that the gamma test has effectively detected tail (in)dependent random variables in all examples (including the sum and the maximum of a sequence of random variables, and variables in M -dependent sequences). Some typical examples are reported in the next section.

6.1. *Bivariate random samples.* In this section we simulate bivariate random samples of sample size 500 from a bivariate random variable (X, Y) . The value of ρ (correlation coefficient) is used to measure the correlation (linear dependence) between the two random variables. The value of θ in Gumbel type copulas is determined by $\theta = \sqrt{1 - \rho}$.

Notice that the correlation coefficient between two random variables may not exist. If the correlation coefficient does not exist, ρ does not have a specific meaning.

These simulated bivariate sequences are the following:

(a) Bivariate random samples drawn from two independent unit Fréchet random variables.

(b) Bivariate random samples drawn from the Gumbel copula,

$$C_\theta(H(u_1), H(u_2)) = e^{-(\sum_{i=1}^2 [-\log\{H(u_i)\}]^{1/\theta})^\theta}, \quad u_1 > 0, u_2 > 0,$$

where $H(u)$ is a unit Fréchet distribution function. Here $\theta = 0.4472$. Notice that the Gumbel copula is the only Archimedean copula which is also an extreme value copula [page 87, Mari and Kotz (2001)].

(c) Bivariate random samples drawn from the survival Gumbel copula,

$$P(X_{11} > u_1, X_{21} > u_2) = e^{-(\sum_{i=1}^2 [-\log\{P\{X_{i1} > u_i\}\}]^{1/\theta})^\theta}, \\ u_1 > 0, u_2 > 0, \theta = 0.4472.$$

(d) Bivariate random samples drawn from Example 2.1, where $1 \leq l \leq 30$ and the coefficients are simulated values.

(e) Bivariate random samples drawn from $(1/U, 1/(1-U))$, where U is uniform $(0, 1)$. (Sid Resnick contributed this example.)

(f) Bivariate normal random samples with correlation $\rho = 0.8$.

(g) Bivariate random samples drawn from $X = Z_1 E_1, Y = Z_2 E_1$, where E_1 is exponential 1. Z_1 and Z_2 are unit Fréchet.

(h) Bivariate random samples drawn from two $t(4)$ (Student t with 4 degrees of freedom) random variables with correlation $\rho = 0.8$.

For these eight examples, (a), (c), (e), (f) and (g) are tail independent examples, while (b), (d) and (h) are tail dependent examples. When we conduct the gamma test, the null hypotheses would be expected to be retained when data are drawn from any one of (a), (c), (e), (f) and (g), that is, larger p -values would be obtained; the null hypotheses would be expected to be rejected when data are drawn from any one of (b), (d) and (h), that is, smaller p -values would be obtained.

In our simulation study we do not use a fixed u value. Instead, the threshold values are automatically chosen at the $100p$ th percentiles of the sample data, where $p = 0.80, 0.825, 0.875, 0.90, 0.925, 0.95, 0.975$, respectively. We take the smallest value of the two $100p$ th percentiles from two different sample sequences. Some

TABLE 1

This table reports p-values obtained when the gamma test was applied to data drawn from models (a), (b), (c), (d), (e), (f), (g) and (h). The first column shows how the threshold values u were chosen

Percentile	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
0.8000	0.2554	0.0000	0.0538	0.0000	0.6889	0.0538	0.0904	0.0000
0.8250	0.2448	0.0000	0.0512	0.0000	0.6860	0.0825	0.0906	0.0000
0.8500	0.2380	0.0000	0.0492	0.0000	0.6822	0.1226	0.0869	0.0000
0.8750	0.2340	0.0000	0.0473	0.0000	0.6769	0.1762	0.0834	0.0000
0.9000	0.2307	0.0000	0.0549	0.0000	0.6738	0.2228	0.1105	0.0000
0.9250	0.2277	0.0000	0.0745	0.0000	0.6668	0.2621	0.1962	0.0000
0.9500	0.2269	0.0001	0.0860	0.0000	0.6605	0.2624	0.1940	0.0000
0.9750	0.2308	0.0009	0.1010	0.0002	0.6596	0.2878	0.1975	0.0000

typical test results from a single gamma test are summarized in Table 1. One can immediately see that the gamma test does work.

We have repeated running the program 100 times and recorded the total number of rejections of the null hypothesis of tail independence by the gamma test. The power of the gamma test among all simulated examples is 88%, which is very high. It would be fair enough to say that the gamma test is very efficient for testing tail independence.

7. Application to internet traffic. In network design, especially in internet traffic control, where the tail independence or dependence between two dependent variables can help networking researchers identify the main cause of internet delay, and design a better network structure, there have been a series of studies on tail dependence between internet traffic variables, for instance by Campos et al. (2003), Maulik, Resnick and Rootzén (2002), Resnick (2001, 2002), Zhang, Breslau, Paxson and Shenker (2002) and many others. These studies have shown that some variables are tail dependent, while some variables are tail independent.

Our application study tests tail (in)dependence of the joint behavior of large values of three internet traffic variables: size of response, time duration of response, and throughput (rate = size/time), which are in the context of HTTP (web browsing) responses. The data set was gathered from the University of North Carolina main link during April of 2001. The data set used in this analysis was for “morning” block (8:00AM-12:01PM). We obtained 76325 subsets of which there are 500 data points in each subset.

We performed the gamma test (using the rank based tail quotient correlation coefficient) on each subset. We found tail dependence between size of response and throughput in more than 3 percent of the data subsets. However, we did not find any statistical evidence that there is tail dependence between time duration and throughput. In Figure 2 we plot the tail behaviors between the traffic variables. The pattern in the right panel is discovered by the tail quotient test statistic. The data

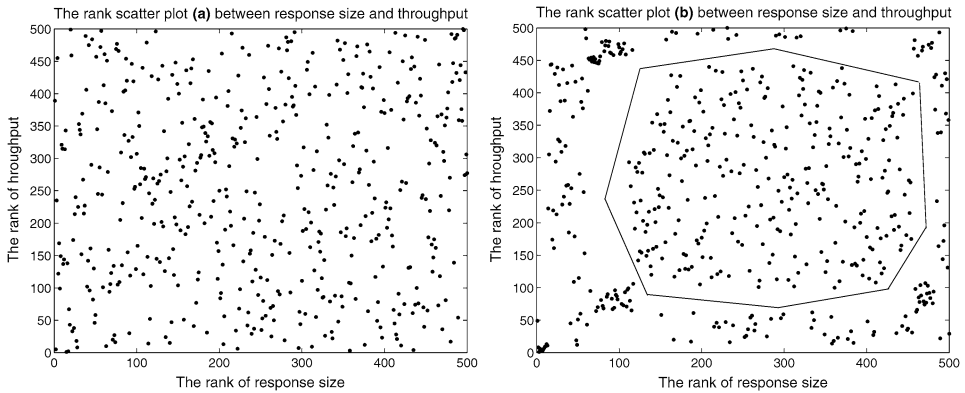


FIG. 2. Comparison between two different data sets collected at different time periods. The pattern in the right panel is discovered by the tail quotient test statistic. The data which generates the left panel is tested as tail independent. The data which generates the right panel is tested as tail dependent.

which generate the left panel is tested as tail independent. The data which generates the right panel is tested as tail dependent. As far as we know, such findings have not been reported in the literature.

These findings certainly encourage network researchers and network designers to better take into account the variable tail dependence in the network design and network traffic study. This is just a simple application of the gamma test in an internet traffic study. A detailed analysis of these traffic variables can be done, which will be a different research project. However, we restrict ourself to a traffic variable tail dependence test in this paper.

8. Concluding remarks. Resnick (1997) argues “Why non-linearities can ruin the heavy tailed modeler’s day.” What we hope to provide, in this paper, is an enabling methodology that will allow model builders to study tail behaviors of underlying multivariate time series on a scale previously not very attainable.

The new family of gamma test statistics can effectively lead to the right indications of tail (in)dependencies at high threshold levels, when the total sample size is about 500, as shown in our simulated examples. The gamma test has also given evidence of the tail independence of bivariate normal random variables, and of the tail independence of the sum and the maximum. The gamma test has outperformed Fisher’s Z-transformation test when data are nonlinearly dependent and the sample size ranges from 37 to 100.

The new concept of the gamma test for (tail) independence should enhance the statistical theory of hypothesis testing. At the least, it adds one more member to the previously existing family of test statistics.

The tests and models of tail dependencies are still far from perfect. There is much work yet to be done. We believe that the methodology developed in this

paper will be very useful in the analysis of financial data. Such research could produce more effective financial time series model and a clearer view of what has been missing in option pricing models.

In the literature, for exceedance problems, there have been existing test statistics derived based on empirical likelihood methods or empirical processes. Among those test statistics, the null hypothesis is tail dependencies which involve pre-specified model assumptions, and hence, they can be thought as testing a class of models. In our setup, the null hypothesis is tail independence. As a result, it is not practically useful to compare our test with existing tests, and it is not easy to carry out.

When the null hypothesis of (tail) independence is rejected, q_n (or q_{un}) can be used as a (tail) dependence measure or as an estimate of a tail dependence index. The properties of this new measure may need to be studied further. A generalization of q_{un} based on max-stable processes may be worth investigation.

The gamma tests and the (tail) dependence measures can be applied to many areas, for instance, financial study, internet traffic analysis (as mentioned above), environmental science, geophysics, microarray data analysis and psychological behavior study are just a few out of many areas.

9. Technical arguments.

PROOF OF THEOREM 3.1. First we have $P\{Y_i/X_i \leq t\} = t/(1+t)$, which implies

$$\lim_{n \rightarrow \infty} P\left\{n^{-1}\left(\max_{i \leq n} Y_i/X_i + 1\right) \leq x\right\} = e^{-1/x}.$$

To prove (3.2), we compute the following probabilities:

$$\begin{aligned} &P\left\{n^{-1}\left(\max_{i \leq n} Y_i/X_i + 1\right) \leq x, n^{-1}\left(\max_{i \leq n} X_i/Y_i + 1\right) \leq y\right\} \\ &= \left[P\{X_i/(ny - 1) < Y_i < (nx - 1)X_i\}\right]^n \\ &= \left[\int_0^\infty \int_{z/(ny-1)}^{(nx-1)z} de^{-1/y} de^{-1/z}\right]^n = \left[1 - \frac{1}{nx} - \frac{1}{ny}\right]^n, \end{aligned}$$

which implies (3.2). For (2.1), we have

$$\begin{aligned} nq_n &= n \frac{\max\{Y_i/X_i\} + \max\{X_i/Y_i\} - 2}{\max\{Y_i/X_i\} \max\{X_i/Y_i\} - 1} \\ &= \left(n^{-1}\left(\max\left\{\frac{Y_i}{X_i}\right\} + 1\right) + n^{-1}\left(\max\left\{\frac{X_i}{Y_i}\right\} + 1\right) - \frac{4}{n}\right) \\ &\quad \times \left(n^{-1}\left(\max\left\{\frac{Y_i}{X_i}\right\} + 1\right) n^{-1}\left(\max\left\{\frac{X_i}{Y_i}\right\} + 1\right)\right) \end{aligned}$$

$$\begin{aligned}
 & -n^{-1} \left[n^{-1} \left(\max \left\{ \frac{Y_i}{X_i} \right\} + 1 \right) + n^{-1} \left(\max \left\{ \frac{X_i}{Y_i} \right\} + 1 \right) \right]^{-1} \\
 &= \frac{[n^{-1}(\max\{Y_i/X_i\} + 1) + n^{-1}(\max\{X_i/Y_i\} + 1)][1 + o_p(1)]}{[n^{-1}(\max\{Y_i/X_i\} + 1)n^{-1}(\max\{X_i/Y_i\} + 1)][1 + o_p(1)]},
 \end{aligned}$$

which gives (2.1) by the Cramér–Wold device and Slutsky’s theorem. \square

LEMMA 9.1. *Suppose X, X_1, X_2, \dots , are positive random variables. Then $X_n \xrightarrow{\text{a.s.}} X$ if and only if there are two sequences of random variables $\xi_1(n), \xi_2(n)$ such that $\xi_1(n) < X_n/X < \xi_2(n), n = 1, 2, \dots$, and $\xi_1(n) \xrightarrow{\text{a.s.}} 1, \xi_2(n) \xrightarrow{\text{a.s.}} 1$, as $n \rightarrow \infty$.*

PROOF. The sufficient condition is obvious. The necessary condition can be shown by simply defining $\xi_1(n) = X_n/X * (1 - 1/n)$, and $\xi_2(n) = X_n/X * (1 + 1/n)$. \square

PROOF OF THEOREM 3.2. Denote $X_i = F^{-1}(G_\theta(\theta_i)), Y_i = F^{-1}(G_\eta(\eta_i))$, then $\tilde{X}_i \xrightarrow{\text{a.s.}} X_i, \tilde{Y}_i \xrightarrow{\text{a.s.}} Y_i$, as $n \rightarrow \infty$ and, hence, by Lemma 9.1, there exist $\xi_j(n) > 0, \xi_j(n) \xrightarrow{\text{a.s.}} 1, j = 1, 2, 3, 4$, as $n \rightarrow \infty$, and

$$\xi_1(n) < \frac{\tilde{X}_i}{X_i} < \xi_2(n), \quad \xi_3(n) < \frac{\tilde{Y}_i}{Y_i} < \xi_4(n), \quad i = 1, \dots, n,$$

which implies

$$\begin{aligned}
 & \mathbb{P} \left\{ \max_{i \leq n} \frac{Y_i \xi_4(n)}{X_i \xi_1(n)} \leq nx - 1, \max_{i \leq n} \frac{X_i \xi_2(n)}{Y_i \xi_3(n)} \leq ny - 1 \right\} \\
 & \leq \mathbb{P} \left\{ \max_{i \leq n} \frac{\tilde{Y}_i}{\tilde{X}_i} \leq nx - 1, \max_{i \leq n} \frac{\tilde{X}_i}{\tilde{Y}_i} \leq ny - 1 \right\} \\
 & \leq \mathbb{P} \left\{ \max_{i \leq n} \frac{Y_i \xi_3(n)}{X_i \xi_2(n)} \leq nx - 1, \max_{i \leq n} \frac{X_i \xi_1(n)}{Y_i \xi_4(n)} \leq ny - 1 \right\}.
 \end{aligned}$$

Since $\xi_j(n)/\xi_k(n)$ converges to 1 almost surely for all j and k , and $\max_{i \leq n} \frac{Y_i}{X_i}$ and $\max_{i \leq n} \frac{X_i}{Y_i}$ are asymptotically independent, so by Slutsky’s theorems, we have both the first probability and the last probability in the above inequalities converging to $e^{-1/x-1/y}$ as $n \rightarrow \infty$, and hence, (3.4) is true. The proof of the asymptotic distribution of $n\tilde{q}_n$ is similar to the proof of the asymptotic distribution nq_n in Theorem 3.1. \square

PROOF OF THEOREM 5.1. We prove $q_n \xrightarrow{\text{a.s.}} 0$, as $n \rightarrow \infty$ only. The proof of $q_{un} \xrightarrow{\text{a.s.}} 0$ is similar.

Suppose without loss of generality that there is a finite number $z, 1 < z < \infty$, such that $P(X/Y < z) = 1$. We have

$$\begin{aligned} \frac{P(X > u, Y > u)}{P(Y > u)} &= \frac{P(zY > X > u, zY > zu)}{P(X > u)} \\ &= \frac{P(zY > X > zu, zY > zu)}{P(X > u)} \\ &\quad + \frac{P(zY > X > u, zY > zu, u < X < zu)}{P(X > u)} \\ &\geq \frac{P(X > zu)}{P(X > u)} \rightarrow \frac{1}{z} > 0, \quad \text{as } u \rightarrow \infty, \end{aligned}$$

which contradicts with the zero limit on the left-hand side. So both ratios X/Y and Y/X have support over $(0, \infty)$. So $\max_{i \leq n} \frac{Y_i}{X_i} \xrightarrow{\text{a.s.}} \infty, \max_{i \leq n} \frac{X_i}{Y_i} \xrightarrow{\text{a.s.}} \infty$, as $n \rightarrow \infty$, and hence, $q_n \xrightarrow{\text{a.s.}} 0$. \square

PROOF OF THEOREM 5.2. We first prove (5.4). For $t < 1$, we have

$$\begin{aligned} &P\left\{\frac{u + W_i}{u + V_i} < t\right\} \\ &= P\left\{\frac{u + W_1}{u + V_1} < t\right\} = P\left\{\frac{u + (Z_1 - u)I_{Z_1 > u}}{u + (Z_2 - u)I_{Z_2 > u}} < t\right\} \\ &= P\left\{\frac{Z_1}{Z_2} < t, Z_1 > u, Z_2 > u\right\} + P\left\{\frac{u}{Z_2} < t, Z_1 < u, Z_2 > u\right\} \\ &\quad + P\left\{\frac{Z_1}{u} < t, Z_1 > u, Z_2 < u\right\} + P\{1 < t, Z_1 < u, Z_2 < u\} \\ &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \end{aligned}$$

where $\Delta_3 = 0, \Delta_4 = 0$ and

$$\begin{aligned} \Delta_1 &= P\{u < Z_1 < tZ_2\} \\ &= \int_u^\infty \int_{z_1/t}^\infty de^{-1/z_2} de^{-1/z_1} \\ &= \int_u^\infty (1 - e^{-t/z_1}) \frac{1}{z_1^2} e^{-1/z_1} dz_1 \\ &= 1 - e^{-1/u} - \int_u^\infty \frac{1}{z_1^2} e^{-(1+t)/z_1} dz_1 = 1 - e^{-1/u} - \frac{1}{1+t} [1 - e^{-(1+t)/u}], \end{aligned}$$

$$\Delta_2 = P\{Z_1 < u, Z_2 > u/t\} = e^{-1/u} (1 - e^{-t/u}) = e^{-1/u} - e^{-(1+t)/u}.$$

Adding Δ_1 and Δ_2 , we get (5.4) for $0 < t < 1$.

For $t \geq 1$, we have

$$\begin{aligned} \Delta_1 &= \mathbb{P}\left\{\frac{u}{t} < \frac{Z_1}{t} < Z_2, Z_2 > u\right\} \\ &= \int_u^{tu} \int_u^\infty de^{-1/z_2} de^{-1/z_1} + \int_{tu}^\infty \int_{z_1/t}^\infty de^{-1/z_2} de^{-1/z_1} \\ &= \frac{t}{1+t} + e^{-2/u} - e^{-1/u} - \frac{t}{1+t} e^{-(1+t)/(tu)}, \\ \Delta_2 &= \mathbb{P}\{Z_1 < u, Z_2 > u\} = e^{-1/u}(1 - e^{-1/u}) = e^{-1/u} - e^{-2/u}, \\ \Delta_3 &= \mathbb{P}\{u < Z_1 < tu, Z_2 < u\} = e^{-1/u}(e^{-1/(tu)} - e^{-1/u}) = e^{-(1+t)/(tu)} - e^{-2/u}, \\ \Delta_4 &= \mathbb{P}\{Z_1 < u, Z_2 < u\} = e^{-2/u}. \end{aligned}$$

Adding the above terms together, we thus obtain (5.4).

We now prove (5.5). Since

$$\begin{aligned} &\mathbb{P}\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1\right\} \\ &= \mathbb{P}\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1, X_1 > u, Y_1 > u\right\} \\ &\quad + \mathbb{P}\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1, X_1 > u, Y_1 < u\right\} \\ &\quad + \mathbb{P}\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1, X_1 < u, Y_1 > u\right\} \\ &\quad + \mathbb{P}\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1, X_1 < u, Y_1 < u\right\} \\ &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \mathbb{P}\left\{\frac{Y_1}{X_1} < nx - 1, \frac{X_1}{Y_1} < ny - 1, X_1 > u, Y_1 > u\right\} \\ &= \int_u^\infty \int_{\max\{x_1/(ny-1), u\}}^{(nx-1)x_1} de^{-1/y_1} de^{-1/x_1} \\ &= \int_u^{(ny-1)u} \int_u^{(nx-1)x_1} de^{-1/y_1} de^{-1/x_1} + \int_{(ny-1)u}^\infty \int_{x_1/(ny-1)}^{(nx-1)x_1} de^{-1/y_1} de^{-1/x_1} \\ &= \frac{nx-1}{nx} - \frac{nx-1}{nx} e^{-(nx)/(nx-1)u} + e^{-2/u} - \frac{1}{ny} - \frac{ny-1}{ny} e^{-(ny)/(ny-1)u}, \\ \Delta_2 &= \mathbb{P}\left\{\frac{u}{X_1} < nx - 1, \frac{X_1}{u} < ny - 1, X_1 > u, Y_1 < u\right\} \end{aligned}$$

$$\begin{aligned}
 &= P\{u < X_1 < (ny - 1)u, Y_1 < u\} = e^{-ny/(ny-1)u} - e^{-2/u}, \\
 \Delta_3 &= P\left\{\frac{Y_1}{u} < nx - 1, \frac{u}{Y_1} < ny - 1, X_1 < u, Y_1 > u\right\} = e^{-nx/(nx-1)u} - e^{-2/u},
 \end{aligned}$$

and $\Delta_4 = e^{-2/u}$. Therefore,

$$\begin{aligned}
 &P\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1\right\} \\
 &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \\
 &= 1 - \frac{1}{nx} - \frac{1}{ny} + \frac{1}{nx}e^{-nx/(nx-1)u} + \frac{1}{ny}e^{-ny/(ny-1)u}.
 \end{aligned}$$

Since

$$\begin{aligned}
 &P\left\{n^{-1}\left[\max_{i \leq n} \frac{(u + W_i)}{(u + V_i)} + 1\right] \leq x, n^{-1}\left[\max_{i \leq n} \frac{(u + V_i)}{(u + W_i)} + 1\right] \leq y\right\} \\
 &= P^n\left\{\frac{u + W_1}{u + V_1} < nx - 1, \frac{u + V_1}{u + W_1} < ny - 1\right\} \\
 &= \left[1 - \frac{1}{nx} - \frac{1}{ny} + \frac{1}{nx}e^{-nx/(nx-1)u} + \frac{1}{ny}e^{-ny/(ny-1)u}\right]^n \\
 &\rightarrow e^{-(1-e^{-1/u})/x - (1-e^{-1/u})/y} \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

which gives (5.5)–(5.7).

For (4.1) and (5.8), letting $\Upsilon_i = (u + W_i)/(u + V_i)$, $\Theta_i = (u + V_i)/(u + W_i)$, we have

$$\begin{aligned}
 nq_{un} &= n \frac{\max\{(u + W_i)/(u + V_i)\} + \max\{(u + V_i)/(u + W_i)\} - 2}{\max\{(u + W_i)/(u + V_i)\} \max\{(u + V_i)/(u + W_i)\} - 1} \\
 &= (n^{-1}(\max\{\Upsilon_i\} + 1) + n^{-1}(\max\{\Theta_i\} + 1) - 4/n) \\
 &\quad \times (n^{-1}(\max\{\Upsilon_i\} + 1)n^{-1}(\max\{\Theta_i\} + 1) \\
 &\quad \quad - n^{-1}[n^{-1}(\max\{\Upsilon_i\} + 1) + n^{-1}(\max\{\Theta_i\} + 1)])^{-1} \\
 &= \frac{[n^{-1}(\max\{\Upsilon_i\} + 1) + n^{-1}(\max\{\Theta_i\} + 1)][1 + o_p(1)]}{[n^{-1}(\max\{\Upsilon_i\} + 1)n^{-1}(\max\{\Theta_i\} + 1)][1 + o_p(1)]},
 \end{aligned}$$

which gives (4.1) by the Cramér–Wold device and Slutsky’s theorem. So the proof of the theorem is complete. \square

PROOF OF THEOREM 5.3. Using Theorem 3.5.2 of Leadbetter, Lindgren and Rootzén (1983) and Proposition 3.1 of Davis (1982), we immediately get $M_n = \max(\xi_1, \xi_2, \dots, \xi_n)$ and $W_n = \min(\xi_1, \xi_2, \dots, \xi_n)$ are asymptotically independent. Then the proof of (3.3) follows from Theorem 3.1, and the proof of (5.8) follows from Theorem 5.2. The last part follows from Theorem 3.2. \square

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