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SINGLE-GENERATOR GENERALIZED CYCLIC FACTORIAL DESIGNS AS PSEUDOFACTOR DESIGNS

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The class of single-generator generalized cyclic designs is shown to be degrees of freedom equivalent to a subclass of the class of prime-level pseudofactor designs.

1. Introduction. Voss and Dean (1987) compared the classes of single-replicate factorial designs with one blocking category given in the literature. Defining two such designs to be *df equivalent* if each design confounds the same number of degrees of freedom with respect to each factorial space, they: (i) showed the class of pseudofactor designs and the class of generalized cyclic designs of Dean and John (1975) and John and Dean (1975) to be quite rich and sometimes but not always equivalent, and (ii) gave an example of a pseudofactor design for which there does not exist a df equivalent generalized cyclic design. Giovagnoli (1977) and Bailey (1985) each indicate that the use of prime-level pseudofactors, and hence the structure of an elementary rather than nonelementary Abelian group, offers more possible confounding patterns.

It is my conjecture that, for each generalized cyclic design, there exists a df equivalent pseudofactor design. In Section 2, my conjecture is proven by construction for the case of single-generator generalized cyclic designs, that is, generalized cyclic designs for which the principal block is cyclic. In Section 3, the construction argument is shown by counterexample not to apply to multiple-generator generalized cyclic designs.

2. Generalized cyclic designs with principal block cyclic. Consider a single-replicate $s_1 \times s_2 \times \cdots \times s_n$ factorial experiment to be conducted in blocks of size k. For purposes of comparing the classes of generalized cyclic designs with and without pseudofactors, it is sufficient [see Bailey (1985)] to assume each s_i is a power of the same prime; say $s_i = p^{k_i}$, $i = 1, 2, \dots, n$. Denote the cyclic group of integers under addition modulo r by $C(r)$. Denote the set of treatments by $T = \{t\colon\ t = (t_1, t_2, \dots, t_n),\ t_i \in C(s_i)\}$, where t_i denotes the level of the ith factor F_i. Then $|T| = p^K$, for $K = \sum_{i=1}^n k_i$.

In this setting, a *design* is a partition of T into subsets of size k.

It is well known that useful designs may be constructed by imposing the structure of an Abelian group of order p^K on T, then using the cosets of a well-chosen subgroup as the partition [see Bailey (1985)]. One such group structure is the p-group G_t which is the direct sum of the cyclic groups $C(s_i)$, namely, $G_t = (T, +) = C(s_1) \oplus \cdots \oplus C(s_n)$, where $t_i \in C(s_i)$.

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A *generalized cyclic design* is a partition of T corresponding to the cosets of a specific subgroup $B \subset G_t$ [see Dean and John (1975) and John and Dean (1975)]. B is called the *principal block* or *initial block* of the design.

Bailey (1985) indicates for blocked single replicates that "... design problems for general finite Abelian groups may be essentially reduced to similar problems for elementary Abelian groups."

This reduction is known to be achievable by giving the same set T of treatments the structure of an elementary Abelian group G_x as follows. Replace the ith factor at $s_i = p^{k_i}$ levels by k_i pseudofactors each at p levels. Explicitly, replace t_i an integer (mod s_i) by $(x_{i1}, x_{i2}, \dots, x_{ik_i})$, where each x_{ij} is an integer (mod p) such that

$$t_i = p^{k_i - 1}x_{i1} + \cdots + px_{ik_i - 1} + x_{ik_i}. \tag{1}$$

Thus, for $X = \{x\colon\ x = (x_{11}, x_{12}, \dots, x_{1k_1}, x_{21}, \dots, x_{nk_n}),\ x_{ij} \in C(p)\}$, $G_x = (X, +) = C(p) \oplus \cdots \oplus C(p)$, where $x_{ij} \in C(p)$.

A bijection $\phi\colon G_t \to G_x$ is induced by (1), but the two groups are not isomorphic unless T is elementary Abelian.

A *pseudofactor design* is a partition of T induced by ϕ and the cosets of a specific subgroup $S_1 \subset G_x$. Explicitly; if $S_1, S_2, \dots, S_b$ are the different cosets of S_1, then the hth block is $B_h = \{t\colon t \in T,\ \phi(t) \in S_h\}$. $B = B_1$ is called the *initial block* of the design.

REMARK. The choice of bijection between $\{t_i\}$ and $\{(x_{i1}, \dots, x_{ik_i})\}$ utilized previously to induce ϕ is convenient for proving Lemma 2 but irrelevant to the number of degrees of freedom confounded in any factorial space.

The rest of this section contains a proof that, for any single-generator generalized cyclic design, there exists a df equivalent pseudofactor design.

For fixed r and $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, and for any $t \in T$ and subset $B \subset T$, define $\sigma(t) = (t_{i_1}, \dots, t_{i_r})$ a subtreatment and $B^* = \{\sigma(t)\colon t \in B\}$.

LEMMA 1. *Let $B \subset T$ be the initial block of either a generalized cyclic or a pseudofactor design. Then the number of degrees of freedom for main effects and interactions of factors $F_{i_1}, F_{i_2}, \dots, F_{i_r}$ which are confounded with blocks is* $(|T^*|/|B^*|) - 1$.

PROOF. Lemma 1 follows directly from Theorem 4.2 of Bailey (1977). □

LEMMA 2. *If $t \in T$ generates a subgroup $B \subset G_t$ of order p^q, then for* $x = \phi(t)$

(i) $x_{iq} \neq 0$ *for some i*;
(ii) $x_{ij} = 0$ *for* $j = q + 1, q + 2, \dots, k_i$, $i = 1, 2, \dots, n$.

PROOF. Assume $t = (t_1, t_2, \dots, t_n)$ generates a subgroup of G_t of order p^q. Then for each i, t_i generates a cyclic subgroup $A_i \subset C(p^{k_i})$ such that $|A_i| \leq p^q$, with equality holding for some i. If $|A_i| = p^q$, then t_i is a multiple of p^{k_i-q} but not of p^{k_i-q+1}, so (i) follows from (1). If $|A_i| \leq p^q$, then t_i is a multiple of p^{k_i-q}, so (ii) follows from (1). □

Let B_c denote a cyclic subgroup of G_t. Then B_c determines a generalized cyclic design, $|B_c| = p^m$ for some integer m, and B_c is generated by a single element, $t_{(m)}$, say. Let $t_{(r)} = p^{m-r}t_{(m)}$, where $t_{(r)i} = p^{m-r}t_{(m)i} \pmod{s_i}$ for $t_{(q)} = t_{(q)1} \cdots t_{(q)n}$, $q = r, m$. Then $t_{(r)}$ generates a cyclic subgroup $T_r \subset B_c$ with $|T_r| = p^r$, $r = 1, 2, \dots, m$, and $T_1 \subset T_2 \subset \cdots \subset T_m = B_c$.

Now, let $x_{(r)} = \phi(t_{(r)})$, $r = 1, 2, \dots, m$. Let S be the subgroup of G_x generated by $\{x_{(1)}, x_{(2)}, \dots, x_{(m)}\}$ and X_r the subgroup of S generated by $x_{(r)}$.

LEMMA 3. *$S = X_1 + X_2 + \cdots + X_m$ is a direct sum and $|S| = |B_c| = p^m$.*

PROOF. Fix r, $1 \leq r \leq m$. Then $t_{(r)} \neq 0$ implies $x_{(r)} \neq 0$, so $|X_r| = p$. Also, $|T_r| > |T_i|$ for $i = 1, \dots, r-1$, so $x \in X_r$ implies $x \notin (X_1 + X_2 + \cdots + X_{r-1})$, by Lemma 2. Hence, by induction on r, $X_1 + \cdots + X_m$ is a direct sum, and the lemma follows. □

REMARK. X_1 and T_1 are isomorphic, but not X_r and T_r for any $r > 1$.

THEOREM 1. *The designs with initial blocks B_c and S are df equivalent.*

PROOF. Let $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$. For $t_{(m)} = (t_1, \dots, t_n)$ the generator of B_c, define $t^*_{(m)} = (t_{i_1}, \dots, t_{i_s})$. Then $t^*_{(m)}$ generates a cyclic subgroup T_q^* with $|T_q^*| = p^q$ for some $q \leq m$. Following the lines of the proof of Lemma 3: Define subgroups $T_1^* \subset T_2^* \subset \cdots \subset T_q^*$, where T_r^* is generated by $t^*_{(r)} = p^{q-r}t^*_{(m)}$. Consider S^* generated by $\{x^*_{(1)}, \dots, x^*_{(q)}\}$, where $x^*_{(r)} = (x_{i_1}, \dots, x_{i_s})$ for $x_{(r)} = (x_1, \dots, x_n) = \phi(t_{(r)})$. Then $x^*_{(1)}, \dots, x^*_{(q)}$ generate directly summable subgroups of order p with direct sum S^*, so $|S^*| = |T_q^*| = p^q$. Hence, df equivalence follows from Lemma 1. □

3. Generalized cyclic designs with noncyclic principal blocks. The construction used in this article to identify a pseudofactor design df equivalent to an arbitrary single-generator generalized cyclic design does not generalize to the case when the principal block is not a cyclic subgroup. If the generalized cyclic design subgroup $B \subset G_t$ is not cyclic, then it is the direct sum $B^1 \oplus \cdots \oplus B^u$ of u cyclic subgroups. By Theorem 1, for each subgroup $B^v \subset G_t$, a subgroup $S^v \subset G_x$ can be constructed as before such that B^v and $\{t: t \in T, \phi(t) \in S^v\}$ are the initial blocks of df equivalent designs; $v = 1, \dots, u$. Furthermore, one can show that $S^1, \dots, S^u$ are directly summable so that the generalized cyclic design with principal block B and the pseudofactor design with principal block $\{t: t \in T, \phi(t) \in S, S = S^1 \oplus \cdots \oplus S^u\}$ have blocks the same size, that is, $|B| = |S|$. However, direct summability is lost when attention is restricted to a subset of the factors. Hence, df equivalence of such designs does not follow in general, as illustrated in the following example.

EXAMPLE. Consider a 9^4 single replicate factorial experiment to be conducted in blocks of size 81. One such generalized cyclic design has subgroup $B = B^1 \oplus B^2$, where B^1 is generated by $t^1_{(2)} = 1212$ and B^2 by $t^2_{(2)} = 1224$. Then $t^1_{(1)} = 3 \times (1212) = 3636$ and $t^2_{(1)} = 3 \times (1224) = 3663$; $\phi_1(1212) = 01020102$ and $\phi_1(3636) = 10201020$, generating S^1; and $\phi_1(1224) = 01020211$ and $\phi_1(3663) = 10202010$, generating S^2. Also, S^1 and S^2 are directly summable, $S = S^1 \oplus S^2$ say, so $|B| = |S|$. However, while the generalized cyclic design with principal block B confounds 8 df for the interaction between the third and fourth factors, the pseudofactor design with initial block determined by S confounds only 2.

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REFERENCES

BAILEY, R. A. (1977). Patterns of confounding in factorial designs. *Biometrika* **64** 597–603.

BAILEY, R. A. (1985). Factorial design and Abelian groups. *Linear Algebra Appl.* **70** 349–368.

DEAN, A. M. and JOHN, J. A. (1975). Single replicate factorial experiments in generalized cyclic designs. II. Asymmetrical arrangements. *J. Roy. Statist. Soc. Ser. B* **37** 72–76.

GIOVAGNOLI, A. (1977). On the construction of factorial designs using Abelian group theory. *Rend. Sem. Mat. Univ. Padova* **58** 195–206.

JOHN, J. A. and DEAN, A. M. (1975). Single replicate factorial experiments in generalized cyclic designs. I. Symmetrical arrangements. *J. Roy. Statist. Soc. Ser. B.* **37** 63–71.

VOSS, D. T. and DEAN, A. M. (1987). A comparison of classes of single replicate factorial designs. *Ann. Statist.* **15** 376–384.

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