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THE LIMITING DISTRIBUTION OF THE AUTOCORRELATION COEFFICIENT UNDER A UNIT ROOT

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The limiting distribution of the normalized autocorrelation coefficient in the case of a unit root is given in a closed form. It is found that high order transcendental functions such as the parabolic cylinder functions are indispensable to express this distribution, thus departing from the simple standard normal distribution that arises in the case of a stable root. Using the formulae derived in this paper, some numerical results available from previous studies are then extended and refined. Finally, the formulae are manipulated analytically to explain the unusual shape of the distribution.

1. Introduction. Consider the time series $\{y_t\}$ generated by the process

$$y_t = \alpha y_{t-1} + \varepsilon_t, \tag{1.1}$$

where $\varepsilon_t \sim \mathrm{NID}(0, \sigma^2)$, $y_0 = c$ (constant), and α is the unknown root. The least squares estimator of α is

$$\hat{\alpha} = \sum_t y_t y_{t-1} \Big/ \sum_t y_{t-1}^2, \tag{1.2}$$

where $\sum_t$ refers to the sum from $t = 1$ to T (sample size). White (1958, 1959) considered deriving density functions for the normalized $\hat{\alpha}$ when $\alpha \in \mathbb{R}$ and $T \to \infty$. He derived closed forms for the densities when $|\alpha| \neq 1$, but was unable to do so for $|\alpha| = 1$. Instead, he presented a limiting variate expressed in terms of a Wiener process. His result is (after a minor correction)

$$U \equiv (\hat{\alpha} - 1)T/\sqrt{2} \to_d \left(W(1)^2 - 1\right)\left[\sqrt{8}\int_0^1 W(t)^2\, dt\right]^{-1}, \qquad \alpha = 1, \tag{1.3}$$

where $\to_d$ refers to convergence in distribution, and $W(t)$ is the standard Wiener process on the interval $[0, 1]$. For a generalization of this result to multivariate series, see Phillips and Durlauf (1986).

Later, Rao (1978) derived an integral which lends itself to numerical calculations to give the limiting density function of U without recourse to generating $W(t)$. However, his formula was quite complex, thus restraining its practicality. Also, Dickey and Fuller (1979) used simulations to derive empirical densities for U; and Evans and Savin (1981) used numerical inversion techniques to obtain the exact densities associated with U. In a related development, Sargan and Bhargava (1983) used contour integration to get a

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simplified integral expression for the distribution of the Durbin–Watson type of statistics, which are close approximations for $-U\sqrt{8}/T$.

However, no closed form for the limiting density functions of U has been given so far. It is, therefore, the aim of this paper to do so.

2. The densities. Let

$$Z = \left(\frac{\hat{\alpha}}{\alpha} - 1\right)\frac{T}{\sqrt{2}} = \frac{R}{S}, \qquad |\alpha| = 1, \tag{2.1}$$

which has a nondegenerate limiting distribution [White (1958)] and where

$$R = \frac{\sqrt{2}}{T\sigma^2}\left(\frac{\hat{\alpha}}{\alpha} - 1\right)\sum_t y_{t-1}^2, \qquad S = \frac{2}{T^2\sigma^2}\sum_t y_{t-1}^2.$$

The joint limiting characteristic function of R and S is

$$e^{-iv/\sqrt{2}}\left[\cos(2\sqrt{iu}) - v\sqrt{\frac{i}{2u}}\sin(2\sqrt{iu})\right]^{-1/2},$$

where v corresponds to R and u to S [e.g., Evans and Savin (1981), page 758, or White (1958), pages 1192–1193]. Then, a theorem by Gurland [(1948), pages 229–230] gives the cumulative distribution function (cdf) of Z as

$$\begin{aligned} F(z) &= \frac{1}{2} - \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)e^{-iv/\sqrt{2}} \\ &\quad\times\left[\cos(2\sqrt{iu}) - v\sqrt{\frac{1}{2u}}\sin(2\sqrt{iu})\right]^{-1/2}\Bigg|_{u=-vz}\frac{dv}{v} \\ &= \frac{1}{2} - \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)e^{-iv/\sqrt{2}} \\ &\quad\times\left[\cos(2\sqrt{-ivz}) - \sqrt{\frac{iv}{-2z}}\sin(2\sqrt{-ivz})\right]^{-1/2}\frac{dv}{v}. \end{aligned} \tag{2.2}$$

When $z < 0$, a change of variable (new v = old $v/\sqrt{-2}$) in (2.2) leads to

$$\begin{aligned} F(z) &= \frac{1}{2} + \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_L e^v\left[\cos(2\sqrt{-vx})\right. \\ &\qquad\left. - \sqrt{\frac{-v}{x}}\sin(2\sqrt{-vx})\right]^{-1/2}\frac{dv}{v} \\ &= \frac{1}{2} + \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_L e^v\left[\cosh(2\sqrt{vx})\right. \\ &\qquad\left. + \sqrt{\frac{v}{x}}\sinh(2\sqrt{vx})\right]^{-1/2}\frac{dv}{v}, \end{aligned} \tag{2.3}$$

where $x = -z\sqrt{2}$, and L is a path of integration made up of two segments: from $-\infty i$ to $-\varepsilon i$, and from εi to ∞i (note: ε is arbitrary). The integrand has a simple pole at $v = 0$ so that, by Cauchy's integral formula,

$$(2.4)\qquad \frac{1}{2\pi i}\oint_C e^v\left[\cosh(2\sqrt{vx}) + \sqrt{\frac{v}{x}}\,\sinh(2\sqrt{vx})\right]^{-1/2}\frac{dv}{v} = 1,$$

where C is any closed curve encircling no singularity other than $v = 0$, in the positive (counterclockwise) direction. Suppose that this curve C is a circle of radius ε, then we can rewrite (2.3) as

$$(2.5)\qquad F(z) = \frac{1}{2\pi i}\int_P e^v\left[\cosh(2\sqrt{vx}) + \sqrt{\frac{v}{x}}\,\sinh(2\sqrt{vx})\right]^{-1/2}\frac{dv}{v},$$

where the new path of integration P is obtained by adding that half of the circle C for which $\mathrm{Re}(v) > 0$ to the original path L. Since

$$\begin{aligned}\cosh(2\sqrt{vx}) + \sqrt{\frac{v}{x}}\,\sinh(2\sqrt{vx}) &= \left[\cosh(2\sqrt{vx})\right]\left[1 + \sqrt{\frac{v}{x}}\,\tanh(2\sqrt{vx})\right]\\ &= \left[\prod_{j=0}^{\infty}\left(1 + \frac{16vx}{(2j+1)^2\pi^2}\right)\right]\\ &\quad\times\left[1 + 16v\sum_{j=0}^{\infty}\left[16vx + (2j+1)^2\pi^2\right]^{-1}\right],\end{aligned}$$

[Spiegel (1981), pages 175, 267] then $x > 0$ implies that there are no singularities for the integrand when $\mathrm{Re}(v) > 0$. We are now in a position to use the Laplace inversion formula:

$$\begin{aligned}F(z) &= \mathscr{L}^{-1}\left\{\left[\cosh(2\sqrt{vx}) + \sqrt{\frac{v}{x}}\,\sinh(2\sqrt{vx})\right]^{-1/2}\Big/ v\right\}\\ &= \sqrt{2}\,\mathscr{L}^{-1}\left\{\left[e^{2\sqrt{vx}} + e^{-2\sqrt{vx}} + \sqrt{\frac{v}{x}}\left(e^{2\sqrt{vx}} - e^{-2\sqrt{vx}}\right)\right]^{-1/2}\Big/ v\right\}\end{aligned}$$

$$(2.6)\qquad = x^{1/4}\sqrt{2}\sum_j\binom{j-1/2}{j}\mathscr{L}^{-1}\left\{\frac{e^{-2w\sqrt{vx}}}{v}\,\frac{(\sqrt{v}-\sqrt{x})^j}{(\sqrt{v}+\sqrt{x})^{j+1/2}}\right\},$$

where $j = 0, 1, \ldots, \infty$ and $w = 2j + \frac{1}{2}$

$$= \sqrt{2}\sum_j\binom{j-1/2}{j}\sum_l\binom{j}{l}(-2)^l\mathscr{L}^{-1}\left\{\frac{e^{-2w\sqrt{vx}}}{v}\left(\sqrt{\frac{v}{x}}+1\right)^{-l-1/2}\right\},$$

where $l = 0, 1, \ldots, j$.

The last two steps are allowed because of the linearity property of the $\mathscr{L}^{-1}\{\ \}$ operator. The probability density function (pdf) can be obtained by differentiating the above expression with respect to z:

$$
\begin{aligned}
f(z) &= \frac{d}{dz} F(z) \\
&= \frac{2}{x^{1/4}} \sum_j \binom{j-1/2}{j} \sum_l \binom{j}{l} \left(-2\sqrt{x}\right)^l \\
(2.7) \qquad &\times \mathscr{L}^{-1}\left\{ \frac{e^{-2w\sqrt{vx}}}{\sqrt{v}\left(\sqrt{v}+\sqrt{x}\right)^{l+1/2}} \left[w - \frac{l/2+1/4}{\sqrt{vx}+x} \right] \right\} \\
&= \sqrt{\frac{8}{\pi y}} \sum_j \binom{j-1/2}{j} e^{-w^2y^2/2} \sum_l \binom{j}{l} (-2y)^l \\
&\times \left[wK\left(-l-\frac{1}{2}, (w+1)y\right) - \frac{l+1/2}{y} K\left(-l-\frac{3}{2}, (w+1)y\right) \right],
\end{aligned}
$$

where

$$
\begin{aligned}
K(\nu, s) &\equiv e^{s^2/4} D_\nu(s) \\
&= 2^{\nu/2}\sqrt{\pi}\left[{}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{s^2}{2}\right) \Big/ \Gamma\left(\frac{1-\nu}{2}\right) - \frac{s\sqrt{2}}{\Gamma(-\nu/2)} {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{s^2}{2}\right) \right]
\end{aligned}
$$

is a parabolic cylinder function [see Erdélyi (1953), volume 2, pages 117, 122, 123 for $D_\nu(s)$], ${}_1F_1(\ )$ is Kummer's series [Erdélyi (1953), volume 1, pages 248, 278], and $y = \sqrt{2|x|} = 2^{3/4}\sqrt{|z|}$. The last expression of (2.7) is reached with the help of an inversion formula obtained by applying an asymptotic summation theorem [derived in Abadir (1991) as a generalization of Erdélyi's (1953), volume 3, page 263] to expression 2.5.94 of Oberhettinger and Badii [(1973), page 259]. One of the referees has kindly suggested the following (less involved but less general) alternative derivation of the inversion formula:

$$
\mathscr{L}^{-1}\left\{ \frac{e^{-2w\sqrt{vx}}}{\sqrt{v}\left(\sqrt{v}+\sqrt{x}\right)^{\nu}} \right\} = e^{(2w+1)x} \mathscr{L}^{-1}\left\{ u^{-\nu/2} \frac{e^{-2(w+1)\sqrt{ux}}}{\sqrt{u}} \right\}
$$

by a change of variable $\sqrt{u} = \sqrt{v} + \sqrt{x}$, and where both Laplace inverses have a unit parameter. Applying the convolution theorem and two inversion formulae [e.g., Oberhettinger and Badii (1973), pages 209, 237, 258] to the right-hand side of the above equality gives

$$\mathscr{L}^{-1}\left\{\frac{e^{-2w\sqrt{vx}}}{\sqrt{v}\left(\sqrt{v}+\sqrt{x}\right)^{\nu}}\right\} = \frac{e^{(2w+1)x}}{\Gamma(\nu/2)\sqrt{\pi}}\int_0^1 (1-v)^{\nu/2-1} e^{-(w+1)^2 x/v}\,\frac{dv}{\sqrt{v}}$$

$$= \frac{e^{-w^2 x}}{\Gamma(\nu/2)\sqrt{\pi}}\int_0^\infty u^{\nu/2-1}(1+u)^{-\nu/2-1/2} e^{-(w+1)^2 xu}\,du,$$

by a change of variable

$$= \frac{\left(\sqrt{2}\right)^{\nu}}{\sqrt{\pi}} e^{-w^2 x} K\left(-\nu, (w+1)\sqrt{2x}\right),$$

from Erdélyi [(1953), volume 2, page 119],

which is the inversion formula used in (2.7).

As for the cdf of Z, it can be obtained by expanding $1/v$ in the last expression of (2.6) before inverting the integrand:

$$F(z) = \sqrt{\frac{2}{x}}\sum_j \binom{j-1/2}{j}\sum_l \binom{j}{l}(-2)^l$$

$$\times \sum_k \mathscr{L}^{-1}\left\{\frac{e^{-2w\sqrt{vx}}}{\sqrt{v}}\left(\sqrt{\frac{v}{x}}+1\right)^{-l-k-3/2}\right\}$$

(2.8) where $k = 0, 1, \ldots, \infty$

$$= 2\sqrt{\frac{y}{\pi}}\sum_j \binom{j-1/2}{j} e^{-w^2 y^2/2}\sum_l \binom{j}{l}(-2y)^l$$

$$\times \sum_k y^k K\left(-l-k-\frac{3}{2}, (w+1)y\right).$$

Both (2.7) and (2.8) converge by Hardy's theorem. However, when $z \to -\infty$, it is computationally more efficient to use the following asymptotic expansion of (2.6):

$$
\begin{aligned}
F(z) &= 2^{3/2} \sum_j \binom{j-1/2}{j} \sum_l \binom{j}{l} (-2)^l \\
&\quad \times \sum_k \frac{\binom{-l-1/2}{k}}{(\sqrt{2x})^k} \mathscr{L}^{-1} \left\{ \frac{e^{-2w\sqrt{vx}}}{(\sqrt{2v})^{2-k}} \right\} \\
&= \frac{2}{\sqrt{\pi}} \sum_j \binom{j-1/2}{j} e^{-w^2y^2/2} \sum_l \binom{j}{l} (-2)^l \\
&\quad \times \sum_k \frac{\binom{-l-1/2}{k}}{y^k} K(k-1, wy) \\
&= \frac{2}{\sqrt{\pi}} \sum_j \binom{j-1/2}{j} e^{-w^2y^2/2} \sum_k \frac{\binom{-1/2}{k}}{y^k} \\
&\quad \times {}_2F_1\left(-j, k+\frac{1}{2}; \frac{1}{2}; 2\right) K(k-1, wy) \\
&= \frac{2}{\sqrt{\pi}} \sum_j \binom{-1/2}{j} e^{-w^2y^2/2} \sum_k \frac{\binom{-1/2}{k}}{y^k} \\
&\quad \times {}_2F_1\left(-j, -k; \frac{1}{2}; 2\right) K(k-1, wy),
\end{aligned}
\tag{2.9}
$$

where $K(-1, wy) \equiv \sqrt{2\pi}\, e^{w^2y^2/2}\Phi(-wy)$, $\Phi(\cdot)$ is the standard normal cdf, $K(k-1, wy) \equiv He_{k-1}(wy)$ are the Hermite (finite) polynomials when $k \in \mathbb{N}$, and ${}_2F_1(\ )$ is Gauss' hypergeometric function [a finite series of $1 + \min(j, k)$ terms on the last line of (2.9)]. The Laplace inverse on the second line of (2.9) is in Oberhettinger and Badii [(1973), page 259], and the transform of Gauss' series on the last line is given in Erdélyi [(1953), volume 1, page 64].

Most asymptotic expansions are nonconvergent, with the magnitude of successive terms tracking a J curve of initial decline followed by a steep rise. The sum in k of (2.9) fits this description, with the length of the initial decline phase varying positively with y. It should be truncated while still at the initial phase, and the leading term of the remainder will indicate the order of the approximation. In practice, a precision of up to n decimal places (dp) would require the first term of the remainder to be zero to n dp. For a general book on asymptotic expansions, see Erdélyi (1956).

When $\mathbf{z} > \mathbf{0}$, similarly to (2.4),

$$(2.4') \quad \frac{1}{2\pi i}\oint_C e^{-v}\left[\cosh(2\sqrt{-vx}) - \sqrt{\frac{v}{-x}}\sinh(2\sqrt{-vx})\right]^{-1/2}\frac{dv}{v} = 1$$

by Cauchy's integral formula. Using the same definitions (for x, y, L, P, etc.) as before, a change of variable (new v = old v times $i/\sqrt{2}$) and (2.4′) can be used to rewrite (2.2) as

$$\begin{aligned} F(z) &= \frac{1}{2} - \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_L e^{-v}\left[\cos(2\sqrt{vx}) - \sqrt{\frac{v}{x}}\sin(2\sqrt{vx})\right]^{-1/2}\frac{dv}{v} \\ &= \frac{1}{2} - \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_L e^{-v}\left[\cosh(2\sqrt{-vx}) - \sqrt{\frac{v}{-x}}\sinh(2\sqrt{-vx})\right]^{-1/2}\frac{dv}{v} \\ & \qquad \text{since } x < 0 \\ &= 1 - \frac{1}{2\pi i}\int_P e^{-v}\left[\cosh(2\sqrt{-vx}) - \sqrt{\frac{v}{-x}}\sinh(2\sqrt{-vx})\right]^{-1/2}\frac{dv}{v} \\ & \qquad \text{from (2.4}') \\ &= 1 - \operatorname{Re}\frac{1}{2\pi i}\int_P e^{-v}\left[\cosh(2\sqrt{-vx}) - \sqrt{\frac{v}{-x}}\sinh(2\sqrt{-vx})\right]^{-1/2}\frac{dv}{v}. \end{aligned} \tag{2.10}$$

For the last step, see for example, Kendall and Stuart [(1977), volume 1, pages 97–99, especially (4.14)–(4.15)]. The integral is real by definition of a cdf, but the expansions used below necessitate such a step. Expanding (2.10) along the lines described in (2.6) and (2.8), then effecting a change of variable (new $v = -$old v),

$$\begin{aligned} F(z) &= 1 + \operatorname{Re}\frac{i}{\pi\sqrt{2}}\sum_j \binom{j-1/2}{j}\sum_l \binom{j}{l}(-2)^l \\ &\quad \times \int_P e^{-v-2w\sqrt{-vx}}\left[1 - \sqrt{\frac{v}{-x}}\right]^{-l-1/2}\frac{dv}{v} \\ &= 1 + \operatorname{Re}\frac{i}{\pi\sqrt{-2x}}\sum_j \binom{j-1/2}{j}\sum_l \binom{j}{l}(-2)^l \\ &\quad \times \sum_k \int_P e^{-v-2w\sqrt{-vx}}\left[1 - \sqrt{\frac{v}{-x}}\right]^{k-l-1/2}\frac{dv}{\sqrt{v}} \\ &= 1 + \operatorname{Re}\frac{1}{\pi\sqrt{-2x}}\sum_j \binom{j-1/2}{j}\sum_l \binom{j}{l}(-2)^l \\ &\quad \times \sum_k \int_Q e^{v-2wi\sqrt{-vx}}\left[1 - i\sqrt{\frac{v}{-x}}\right]^{k-l-1/2}\frac{dv}{\sqrt{v}}, \end{aligned} \tag{2.11}$$

where Q is the reflection of path P across the imaginary axis for v, keeping in mind not to cross the branch line $(x, 0)$. Note that the expansions have created two branch points: one at $v = 0$ and the other at $v = x < 0$. The real part of the integral around $v = 0$, with $-\pi \le \arg v < \pi$, is zero. So,

$$F(z) = 1 + \operatorname{Re} \frac{1}{\pi\sqrt{-2x}} \sum_j \binom{j-1/2}{j} \sum_l \binom{j}{l} (-2)^l \times \sum_k \int_P e^{v - 2wi\sqrt{-vx}} \left[1 - i\sqrt{\frac{v}{-x}}\right]^{k-l-1/2} \frac{dv}{\sqrt{v}}$$

and with the two branch points to the left of the path of integration,

$$\begin{aligned} F(z) &= 1 + \operatorname{Re} i\sqrt{\frac{2}{-x}} \sum_j \binom{j-1/2}{j} \sum_l \binom{j}{l} (-2)^l \\ &\quad \times \sum_k \mathscr{L}^{-1}\left\{ \frac{e}{\sqrt{v}}^{-2wi\sqrt{-vx}} \left[1 - i\sqrt{\frac{v}{-x}}\right]^{k-l-1/2} \right\} \\ &= 1 - \operatorname{Re} \frac{2}{\sqrt{\pi i y}} \sum_j \binom{j-1/2}{j} e^{w^2y^2/2} \sum_l \binom{j}{l} (-2iy)^l \\ &\quad \times \sum_k (iy)^{-k} K\left(k - l - \frac{1}{2}, i(w+1)y\right) \qquad (2.12) \\ &= 1 - \frac{1}{\pi}\sqrt{\frac{2}{y}} \sum_j \binom{j-1/2}{j} e^{-(2j+1)y^2} \sum_l \binom{j}{l} (2y)^l \\ &\quad \times \sum_k \frac{\Gamma(k-l+1/2)}{(-y)^k} \left[K\left(l - k - \frac{1}{2}, (w+1)y\right) \right. \\ &\qquad \left. + \operatorname{Re}(-1)^{k-l-1/2} K\left(l - k - \frac{1}{2}, -(w+1)y\right) \right], \end{aligned}$$

where the last formula follows from

$$K(\nu, s) = \frac{e^{s^2/2}\Gamma(\nu+1)}{\sqrt{2\pi}} \left[i^{-\nu} K(-\nu-1, -is) + i^{\nu} K(-\nu-1, is) \right]$$

[e.g., Erdélyi (1953), volume 2, page 117]. The asymptotic expansion

$$\begin{aligned} K(\nu, \xi) &= \xi^{\nu}\, {}_2F_0\left(\frac{-\nu}{2}, \frac{1-\nu}{2}; -\frac{2}{\xi^2} \right) + \operatorname{sgn}(\max(0, -\xi)) \\ &\quad \times \frac{\sqrt{2\pi}}{\Gamma(-\nu)} (-\xi)^{-\nu-1} e^{\xi^2/2}\, {}_2F_0\left(\frac{\nu}{2}, \frac{1+\nu}{2}; \frac{2}{\xi^2} \right), \end{aligned} \qquad (2.13)$$

where $\xi \in \mathbb{R}$ and

$$ {}_2F_0(\beta,\gamma;\zeta) = [\Gamma(\beta)\Gamma(\gamma)]^{-1}\sum_{j=0}^{\infty}\Gamma(\beta+j)\Gamma(\gamma+j)\frac{\zeta^j}{j!} $$

[e.g., Erdélyi (1953), volume 2, pages 122–123 and volume 1, page 182] allows us to write

$$ \begin{aligned} \operatorname{Re}(-1)^{k-l-1/2}K\left(l-k-\frac{1}{2},-(w+1)y\right) &= (-1)^{k-l-1/2}[-(w+1)y]^{l-k-1/2} \\ &\quad\times {}_2F_0\left(\frac{k-l}{2}+\frac{1}{4},\frac{k-l}{2}+\frac{3}{4};\frac{-2}{(w+1)^2y^2}\right) \\ &= K\left(l-k-\frac{1}{2},(w+1)y\right), \end{aligned} $$

which reduces (2.12) to

$$ \begin{aligned} F(z) = 1 - \frac{2}{\pi}\sqrt{\frac{2}{y}}\sum_j \binom{j-1/2}{j} e^{-(2j+1)y^2}\sum_l \binom{j}{l}(2y)^l \\ \times \sum_k \frac{\Gamma(k-l+1/2)}{(-y)^k}K\left(l-k-\frac{1}{2},(w+1)y\right). \end{aligned} \tag{2.14} $$

From the first expression of (2.11), derivations similar to the ones above yield

$$ \begin{aligned} f(z) &= \frac{d}{dz}F(z) \\ &= \operatorname{Re}\frac{-i}{\pi\sqrt{-x}}\sum_j\binom{j-1/2}{j}\sum_l\binom{j}{l}(-2)^l\int_P e^{-v-2w\sqrt{-vx}} \\ &\quad\times\left[1-\sqrt{\frac{v}{-x}}\right]^{-l-1/2}\left[w-\frac{1}{x}\left(\frac{l}{2}+\frac{1}{4}\right)\left[1-\sqrt{\frac{v}{-x}}\right]^{-1}\right]\frac{dv}{\sqrt{v}} \\ &= \operatorname{Re}\frac{-1}{\pi\sqrt{-x}}\sum_j\binom{j-1/2}{j}\sum_l\binom{j}{l}(-2)^l\int_Q e^{v-2wi\sqrt{-vx}} \\ &\quad\times\left[1-i\sqrt{\frac{v}{-x}}\right]^{-l-1/2}\left[w-\frac{1}{x}\left(\frac{l}{2}+\frac{1}{4}\right)\left[1-i\sqrt{\frac{v}{-x}}\right]^{-1}\right]\frac{dv}{\sqrt{v}} \\ &= \operatorname{Re}\frac{-1}{\pi\sqrt{-x}}\sum_j\binom{j-1/2}{j}\sum_l\binom{j}{l}(-2)^l\int_P e^{v-2wi\sqrt{-vx}} \\ &\quad\times\left[1-i\sqrt{\frac{v}{-x}}\right]^{-l-1/2}\left[w-\frac{1}{x}\left(\frac{l}{2}+\frac{1}{4}\right)\left[1-i\sqrt{\frac{v}{-x}}\right]^{-1}\right]\frac{dv}{\sqrt{v}} \end{aligned} \tag{2.15} $$

$$= \operatorname{Re} \frac{-i\sqrt{8}}{\sqrt{-2x}} \sum_j \binom{j-1/2}{j} \sum_l \binom{j}{l} (-2)^l$$

$$\times \mathscr{L}^{-1}\left\{ e^{-2wi\sqrt{-vx}} \left[1 - i\sqrt{\frac{v}{-x}}\right]^{-l-1/2} \right.$$

$$\left. \times \left[w - \frac{1}{x}\left(\frac{l}{2} + \frac{1}{4}\right)\left[1 - i\sqrt{\frac{v}{-x}}\right]^{-1} \right] \Big/ \sqrt{v} \right\}$$

$$= \operatorname{Re}\left\{ \sqrt{\frac{8}{\pi i y}} \sum_j \binom{j-1/2}{j} e^{w^2y^2/2} \sum_l \binom{j}{l} (-2iy)^l \right.$$

$$\times \left[wK\left(-l - \frac{1}{2}, i(w+1)y\right) \right.$$

$$\left.\left. + \left(l + \frac{1}{2}\right)\frac{i}{y} K\left(-l - \frac{3}{2}, i(w+1)y\right) \right] \right\}$$

$$= \frac{4}{\pi\sqrt{y}} \sum_j \binom{j-1/2}{j} e^{-(2j+1)y^2} \sum_l \binom{j}{l} (2y)^l \Gamma\left(\frac{1}{2} - l\right)$$

$$\times \left[wK\left(l - \frac{1}{2}, (w+1)y\right) + \frac{1}{y} K\left(l + \frac{1}{2}, (w+1)y\right) \right]$$

$$= \frac{4}{\sqrt{y}} \sum_j \binom{j-1/2}{j} e^{-(2j+1)y^2} \sum_l \binom{j}{l} \frac{(-2y)^l}{\Gamma(l+1/2)}$$

$$\times \left[wK\left(l - \frac{1}{2}, (w+1)y\right) + \frac{1}{y} K\left(l + \frac{1}{2}, (w+1)y\right) \right]$$

with the last step following from Erdélyi [(1953), volume 1, page 3]. If y were to be replaced by $\sqrt{2x}$ on the last formula of (2.7) and the real part of that expression taken, then we would get the last three formulae of (2.15). In other words, for $z \neq 0$,

$$f(z) = \frac{2}{\sqrt{\pi}} \operatorname{Re}\left\{ \left[\frac{2}{x}\right]^{1/4} \sum_j \binom{j-1/2}{j} e^{-w^2x} \sum_l \binom{j}{l} \left(-2\sqrt{2x}\right)^l \right.$$

$$\times \left[wK\left(-l - \frac{1}{2}, (w+1)\sqrt{2x}\right) \right.$$

$$\left.\left. - \frac{l+1/2}{\sqrt{2x}} K\left(-l - \frac{3}{2}, (w+1)\sqrt{2x}\right) \right] \right\}.$$

TABLE 1
Quantiles of the distribution of Z

Quantiles (%)	**1**	**2.5**	**5**	**10**	**50**	**90**	**95**	**97.5**	**99**
z	−9.684	−7.382	−5.685	−4.040	−0.603	0.667	0.907	1.137	1.437

3. Computations. Formulae (2.7)–(2.9), (2.14) and (2.15) formed the basis of a numerical comparison with known results (tables and graphs) such as in Evans and Savin (1981). They were found to be highly accurate and easily programmable. Moreover, the sums given in these formulae converge very rapidly, especially for j (hence l as well). The exception is for small positive values of z when a high accuracy is required. For a precision of five digits and $0 < z < 0.5$, (2.15) is nonconvergent in j, and (2.14) is nonconvergent in both j and k. For z in this range, some accuracy had to be sacrificed by early truncation of the series.

Table 1 is obtained by a selective grid search of z up to 3 dp, so that the corresponding cdf is nearest to the required quantile. The table reflects a strong negative skew in the distribution of Z, a fact pointed out by Dickey and Fuller (1979) and Evans and Savin (1981). The one percent lower limit of the distribution is -9.684, which compares to -2.326 for the normal distribution. In fact, numerical efficiency aside, a major advantage of the formulae derived above is that they lend themselves to analytic comparisons. For example, since

$$K(\nu,\xi) = O(\xi^{\nu}), \qquad \xi \to \infty \tag{3.1}$$

for $\xi \in \mathbb{R}$ (see (2.13) or Erdélyi [(1953), volume 2, page 122]), then (2.7) implies that

$$f(z) = O\left[\frac{2^{1/4} e^{z/\sqrt{8}}}{\sqrt{-3\pi z}}\right], \qquad z \to -\infty \tag{3.2}$$

and (2.15) implies

$$f(z) = O\left[\frac{2^{11/4} e^{-z\sqrt{8}}}{\sqrt{3\pi z}}\right], \qquad z \to \infty \tag{3.3}$$

which reflects a much faster rate of decay for the upper tail because of the argument of the exponent. Moreover, comparing (3.2) to the standard normal pdf,

$$\phi(s) \equiv \frac{e^{-s^2/2}}{\sqrt{2\pi}}, \tag{3.4}$$

one can understand why the lower tail of Z takes longer to die out than in the case of the standard normal variate.

In addition to (3.2) and (3.3), the following can be derived from (2.8):

$$
\begin{aligned}
F(z) &= O\left[2\sqrt{\frac{y}{\pi}}\, e^{-y^2/8} \sum_k y^k K\left(-k - \frac{3}{2}, \frac{3}{2}y\right)\right] \\
&= O\left[\frac{4}{3y}\sqrt{\frac{2}{3\pi}}\, e^{-y^2/8} \sum_k \left(\frac{2}{3}\right)^k\right] \qquad (3.5) \\
&= O\left[\frac{2^{7/4} e^{z/\sqrt{8}}}{\sqrt{-3\pi z}}\right], \qquad z \to -\infty
\end{aligned}
$$

and from (2.14),

$$
1 - F(z) = O\left[\frac{2^{5/4} e^{-z\sqrt{8}}}{\sqrt{3\pi z}}\right], \qquad z \to \infty. \tag{3.6}
$$

Formulae (3.5) and (3.6) can be inverted to give approximate values of z for sizes δ or $1 - \delta$, where $\delta > 0$ and small.

4. Conclusion. Unlike the formula in Rao (1978), the ones given here did not require lengthy numerical integrations. Moreover, they are more parsimonious than Rao's [(1978), page 186] expression which is 13 lines long. Also, the formulae derived here are more efficient than simulation-based formulae [White (1958), Dickey and Fuller (1979)].

It is important to remember that functions made of sums of transcendental functions do not have unique forms. For example, (2.6) could have been expanded into

$$
\begin{aligned}
F(z) &= \sqrt{2}\, x^{1/4} \sum_j \binom{j - 1/2}{j} \sum_l \binom{j}{l} (-2\sqrt{x})^l \\
&\quad \times \sum_k \binom{-l - 1/2}{k} (\sqrt{x})^k \mathscr{L}^{-1}\left\{\frac{e^{-2w\sqrt{vx}}}{(\sqrt{v})^{k+l+5/2}}\right\} \qquad (4.1) \\
&= 2\sqrt{\frac{y}{\pi}} \sum_j \binom{j - 1/2}{j} e^{-w^2y^2/2} \sum_l \binom{j}{l} (-2y)^l \\
&\quad \times \sum_k \binom{-l - 1/2}{k} y^k K\left(-l - k - \frac{3}{2}, wy\right),
\end{aligned}
$$

which is equivalent to (2.8), though different in form.

Two extensions apply to the results derived here. First, the distribution functions obtained above are exactly the same for a wide range of more general distributions for ε_t. For a discussion, see Rao [(1978), page 190] or Phillips (1987). Second, these functions are also the same when, instead of (1.1), we have a vector autoregressive process (VAR) with one unit characteristic root (eigenvalue), and the remaining characteristic roots are stable [see Fountis and

Dickey (1989)]. In this case, $\hat{\alpha}/\alpha$ in (2.1) is replaced by the largest modulus of the roots of the characteristic equation arising from the estimated VAR.

Finally, due to the definition of Z employed in (2.1), the results derived in this paper are valid for $|\alpha| = 1$, thus extending (1.3) to $\alpha = -1$ as well:

$$\left(\frac{\hat{\alpha}}{\alpha} - 1\right)\frac{T}{\sqrt{2}} \to_d \frac{W(1)^2 - 1}{\sqrt{8}\int_0^1 W(t)^2\,dt}, \qquad |\alpha| = 1 \tag{4.2}$$

and the derived functions are the densities of the right-hand side variate which has a wider applicability than just unit root theory.

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