

## A GENERAL THEOREM ON DECISION THEORY FOR NONNEGATIVE FUNCTIONALS: WITH APPLICATIONS<sup>1</sup>

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A distribution-free inadmissibility theorem is proved for estimating under quadratic loss nonnegative functionals that are allowed to depend on the unknown c.d.f. as well as the data. It follows from the theorem that, *subject to finiteness of the risk*, some natural estimators of the eigenvalues of many commonly occurring matrices in multivariate problems are inadmissible, typically from dimension 2, when samples are drawn from any multivariate elliptically symmetric distribution. As an example, if  $X_1, \dots, X_{k+1}$  are i.i.d. observations from such a distribution with dispersion matrix  $\Sigma$ , then the eigenvalues of  $S/k$  are inadmissible for the eigenvalues of  $\Sigma$  if a certain conditions holds where  $S$  is the sample sum of squares and products matrix. It also follows from the theorem that in the general scale-parameter family, the best equivariant estimator of the scale-parameters is inadmissible for  $p \geq 2$ , and some natural estimators of the losses of the best equivariant estimators are inadmissible, usually for  $p \geq 2$ . The theorem also has certain applications to the Ferguson family of distributions, the multivariate  $F$  distribution and for unbiased estimators in some families of distributions.

**1. Introduction.** Stein (1956) showed that the usual estimator of the mean of a  $p$ -variate normal distribution with identity covariance matrix is inadmissible under squared-error loss if  $p \geq 3$ . Since this monumental work, researchers in this general area have proved that the same inadmissibility phenomenon holds in an incredibly large number of problems. Brown (1966) proved the existence of the "Stein-effect" in general location problems under very general losses, Berger (1980) gave a general theory for the continuous exponential family, with applications in the gamma case and Hwang (1982a) treated the same problem in the discrete exponential family. The problem of simultaneously estimating multiple gamma scale-parameters was readdressed in DasGupta (1986a) and the following result was proved: For ordinary squared-error loss, the best equivariant estimate, say  $\delta_0(X)$ , of the scale-parameters is inadmissible for every  $p \geq 2$  and a dominating estimator is  $\delta_0(X) + c \cdot (\prod_j X_j)^{1/p} \cdot 1$ , where  $1' = (1, 1, \dots, 1)$  and  $c > 0$  is a suitable constant. Only later was it discovered that this *same* result holds for every  $p \geq 2$  in the arbitrary scale-parameter family; see DasGupta (1984b). Interestingly, in all of these cases in which the idea of shifting by the geometric mean works very well for proving inadmissibility results, the parametric functions being estimated are always nonnegative. The purpose of this paper is to prove a very simple theorem, motivated by Brown's (1979) heuristics, in the

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context of simultaneously estimating  $p$  nonnegative functionals (which can depend on the unknown parameters *as well as* the random variables) under quadratic losses; roughly speaking, the theorem says that if  $X = (X_1, \dots, X_p)$  has a distribution  $F \in \mathcal{F}$ , where  $\mathcal{F}$  is some family of distributions on  $\mathbb{R}^p$ , and if one wants to estimate nonnegative functionals  $\mu_1(X, F), \mu_2(X, F), \dots, \mu_p(X, F)$ , then under a suitable technical condition (see Theorem 1 in the next section), an estimate  $\delta(x) = (\delta_1(x), \delta_2(x), \dots, \delta_p(x))$  is inadmissible and a better estimator is  $\delta(x) + c \cdot \lambda(x) \cdot 1$ , where  $c$  is a suitable real number and  $\lambda(z)$  is a suitable size statistic, *not* necessarily the geometric mean of the  $\delta_j$ 's. The examples in the next section show that a variety of size statistics, like  $(\prod_j \delta_j(x))^{1/p}$ ,  $\sum_j \delta_j(x)$ ,  $\min(\delta_1(x), \dots, \delta_p(x))$  etc., work in different problems. The critical dimension of inadmissibility is almost always 2, but sometimes is 3 (see examples in the next section). Besides the theoretically interesting facts that the theorem is nonparametric in nature and that it can handle estimation of functionals which depend on parameters as well as the data, the wide variety of problems to which the result applies is surprising. Some important problems in which the inadmissibility of natural estimators can be proved by directly using this theorem include the following: (i) estimation of the means in arbitrary scale-parameter families, (ii) estimation of the eigenvalues of the dispersion matrix  $\Sigma$  and the matrix  $\Sigma^{-1}$  in a general elliptically symmetric distribution, (iii) estimation of the eigenvalues of  $\Sigma_1^{-1}\Sigma_2$  where  $\Sigma_i$ ,  $i = 1, 2$ , are the dispersion matrices of two elliptically symmetric distributions and independent random samples are available from the two populations, (iv) estimation of the eigenvalues of the matrix  $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ , which arises in multivariate regression, (v) estimation of loss-functions of the best equivariant estimate in the general scale-parameter family, (vi) estimation of the means in some members of the Ferguson family of distributions [see Ferguson (1962)] and (vii) estimation of the means in the multivariate  $F$  distribution. The theorem also gives some interesting inadmissibility results for unbiased estimates in certain cases. Note that problem (ii) arises in principal components analysis and problem (iii) in MANOVA. Problem (ii) has received the attention of several authors, including Stein (1975). Problems (i), (v), (vi) and (vii) are interesting from the viewpoint of general decision theory. We believe that this theorem probably has potential applications in other problems involving estimation of nonnegative functionals not considered by us in this article.

In Section 2, we give the main theorem and its applications and relate the main result to Brown's (1979) technique. In Section 3 some concluding remarks about the amount of risk-improvement and the scope for choice of a dominating estimator are made. Before closing this section, we would like to point out that the results in this paper are valid *subject to finiteness of risk*.

## 2. The main result and applications.

**THEOREM 1.** *Let  $X = (X_1, X_2, \dots, X_p)$  have a joint distribution  $F$ , where  $F$  belongs to some class  $\mathcal{F}$  of distributions on  $\mathbb{R}^p$ . Let  $\mu_i(X, F)$ ,  $1 \leq i \leq p$ , be nonnegative functionals and let  $0 \leq \delta_i(X)$ ,  $1 \leq i \leq p$ , be any estimators of*

$\mu_i(X, F)$ . Let  $\lambda(X_1, \dots, X_p)$  be any nonnegative statistic. Define  $A(F) = [E\{\lambda \sum_{i=1}^p \delta_i(x)\}] / [E\{\lambda \sum_{i=1}^p \mu_i(x, F)\}]$ . If either

- (a)  $\inf_{F \in \mathcal{F}} A(F) = r > 1$  or
- (b)  $\sup_{F \in \mathcal{F}} A(F) = R < 1$  holds and
- (c)  $E(\lambda^2(\mathbf{x})) \leq A_0 E\{\lambda(\mathbf{x}) \sum \delta_i\} \forall F$  and some  $A_0 > 0$ ,

then  $\delta(X)$  is inadmissible for estimating  $\mu(X, F)$  under sum of squared-error losses  $\sum_{i=1}^p (\delta_i - \mu_i)^2$ .

Moreover, a dominating estimator is given as  $\delta_1(X) = \delta(X) + c\lambda \cdot 1$ , where  $c$  is a suitable negative number in case (a) and a suitable positive number in case (b) [see (2.4) below].

REMARK 1. In a large number of examples to be considered in this section, there is enough homogeneity that  $A(F)$  is actually a constant for all  $F$ . In such cases verification of the sufficient condition(s) of Theorem 1 will be somewhat simplified.

It is interesting that problems (i) through (vii) listed in the Introduction all have a "pseudoscale structure" in them. (We will not attempt to formally define a pseudoscale structure, but roughly speaking we mean that  $[\sum_i E(\delta_i(\mathbf{x}))] / [\sum_i E\mu_i(\mathbf{x}, F)]$  and  $[\sum_i E(\delta_i^2(\mathbf{x}))] / [\sum_i E(\mu_i^2(\mathbf{x}, F))]$  behave like constants uniformly over  $F \in \mathcal{F}$ .) Suppose now we consider the problem of estimating, under sum of squared-error losses,  $p$  positive parameters  $\theta_1, \theta_2, \dots, \theta_p$  by estimates  $aX_1, aX_2, \dots, aX_p$ , respectively, where  $a$  is a real constant. Let  $\mu_i(\theta_i) = E(X_i)$ ,  $V_i(\theta_i) = E(X_i - \theta_i)^2$  and  $r(\mathbf{x}) = (a - 1)\mathbf{x}$ . Let us assume that  $\sum \mu_i(\theta_i) = \sum \theta_i$ . Brown's (1979) heuristic analysis then suggests that we should expect  $\mathbf{x} + r(\mathbf{x})$  to be inadmissible if we can find a function  $\lambda: \mathbb{R}^p \rightarrow \mathbb{R}^1$  satisfying the nonlinear differential inequality

$$(2.1) \quad \begin{aligned} & -2(a - 1)\lambda(\theta) \sum \theta_i - 2 \sum V_i(\theta_i) \frac{\partial}{\partial \theta_i} \lambda(\theta) - p\lambda^2(\theta) \geq 0 \forall \theta \\ \Leftrightarrow & p\lambda^2(\theta) + 2(a - 1)\lambda(\theta) \sum \theta_i + 2 \sum V_i(\theta_i) \frac{\partial}{\partial \theta_i} \lambda(\theta) \leq 0 \forall \theta \end{aligned}$$

[this is just inequality (6) in Brown (1979), page 964]. Suppose now  $\mu_1 \leq [V_i(\theta_i)] / \theta_i^2 \leq \mu_2 < \infty$  (in some problems it is easier to check that  $\mu_1 \leq [\sum_i V_i(\theta_i)] / (\sum_i \theta_i^2) \leq \mu_2 < \infty$ ). If one now lets  $\lambda(\theta) = c(\prod_j \theta_j)^{1/p}$ , then (2.1) reduces to

$$(2.2) \quad pc^2 \left( \prod_j \theta_j \right)^{2/p} + 2c(a - 1) \left( \prod_j \theta_j \right)^{1/p} \sum_i \theta_i + 2 \frac{c}{p} \left( \prod_j \theta_j \right)^{1/p} \sum_i \frac{V_i(\theta_i)}{\theta_i} \leq 0.$$

Using the facts that  $[V_i(\theta_i)] / \theta_i^2 \leq \mu_2$  and  $\prod_j (\theta_j)^{1/p} \leq \sum \theta_j / p$ , it easily follows that (2.2) will hold if there exists a  $c > 0$  such that  $c + 2(a - 1) + 2(\mu_2/p) < 0$ . If  $a < 1$ , then clearly a  $c > 0$  exists for large  $p$  because  $\mu_2/p \rightarrow 0$  as  $p \rightarrow \infty$ . Brown's (1979) analysis then strongly indicates that in problems with a pseu-

doscale structure,  $\alpha \mathbf{X}$  should be uniformly dominated by  $\alpha \mathbf{X} + c(\prod_j X_j)^{1/p}$  for large enough  $p$ , where  $c > 0$  is a suitable real number. Note that in order to formally carry out Brown's (1979) analysis, one would have to first make a variance stabilizing transformation. Theorem 1, however, uses arguments different from Brown's, although it is motivated by Brown (1979). Brown's (1979) heuristic analysis can be used to generate similar probable inadmissibility results in many estimation problems. Following this theorem, we have given a large number of examples in which the function  $\lambda$  is different from the geometric mean and in all of these examples, Brown's (1979) heuristics do suggest that such other  $\lambda$ 's can be used to obtain uniform domination. In choosing an appropriate function  $\lambda$  for constructing dominating estimators, the following heuristic prescription also seems to work very well: Find a function  $\lambda(\mathbf{X})$  such that  $\sum X_i / [\lambda(\mathbf{X})]$  is independent of either  $\lambda(\mathbf{X})$  or  $\sum X_i$  when all  $\theta_i$ 's are equal. Then  $\delta(\mathbf{X}) + c\lambda(\mathbf{X})$  is a plausible estimator dominating  $\delta(\mathbf{X})$ . For example, let  $X_1, \dots, X_{k+1}$  be i.i.d.  $N_p(\mu, \Sigma)$ , where  $\mu$  and  $\Sigma$  are unknown and let  $l_1 \geq \dots \geq l_p > 0$  be the ordered eigenvalues of  $S = \sum (X_j - \bar{X})(X_j - \bar{X})'$ . Consider the problem of estimating  $(\lambda_1, \dots, \lambda_p)$ , the ordered eigenvalues of  $\Sigma$ . It is well-known that  $\sum l_i / (\prod_j l_j)^{1/p} = \text{tr} S / |S|^{1/p}$  is independent of  $\text{tr} S$  when all  $\lambda_i$ 's are equal to some  $\lambda$  (i.e.,  $\Sigma = \lambda I$ ); this follows from Basu's (1955) theorem because  $\text{tr} S$  is complete sufficient for  $\lambda$  and  $\text{tr} S / |S|^{1/p}$  is an ancillary statistic. Thus, in the eigenvalue problem described above,  $|S|^{1/p}$  is a plausible choice for  $\lambda(\mathbf{l})$ .

**PROOF OF THEOREM 1.** We will prove the theorem when case (a) holds. The other case is exactly similar. By definition,

$$\begin{aligned} \Delta(F) &= E \sum_{i=1}^p \{ \delta_i(X) + c\lambda - \mu_i(X, F) \}^2 - E \sum_{i=1}^p \{ \delta_i(X) - \mu_i(X, F) \}^2 \\ &= E \left\{ pc^2\lambda^2 + 2c\lambda \sum_{i=1}^p (\delta_i(x) - \mu_i(X, F)) \right\} \\ (2.3) \quad &\leq E \left\{ (pc^2A_0 + 2c)\lambda \sum_{i=1}^p \delta_i(X) - 2c\lambda \sum_{i=1}^p \mu_i(X, F) \right\} \\ &= E \left[ \lambda \sum_{i=1}^p \mu_i(X, F) \right] \{ (pc^2A_0 + 2c)A(F) - 2c \} \end{aligned}$$

$$(2.4) \quad < 0 \quad \text{if } 0 > c > \frac{2(1-r)}{prA_0}.$$

Hence it is proved.  $\square$

The permissible values of  $c$  in case (b) are  $0 < c < [2(1 - R)]/pRA_0$ . With slight modifications, the assertion of the above theorem holds when the loss is  $L_Q = (\delta - \mu)'Q(\delta - \mu)$  where  $Q$  is a p.d. matrix of known constants and when the unit vector  $\mathbf{1}$  is replaced by an  $\alpha > 0$  (i.e., each component is positive) in the dominating estimator. Note that while it follows from Shinozaki (1975) that if

$\delta(\mathbf{x})$  is inadmissible under squared-error loss, then it is also inadmissible under  $L_Q$ , it does not follow from Shinozaki (1975) that essentially the same type of estimator dominates  $\delta(\mathbf{x})$  under both losses.

**REMARK 2.** In some problems, it is easier to show that  $C(F) = -E\{\lambda \sum (\delta_i(\mathbf{x}) - \mu_i(\mathbf{x}, F))\} > 0$  and that  $[E(\lambda^2(\mathbf{x}))]/[C(F)]$  is uniformly bounded over  $F \in \mathcal{F}$ . The line preceding (2.3) shows that in such a case one can find a small enough  $c > 0$  such that  $\Delta(F) < 0 \forall F \in \mathcal{F}$ ; see Example 3 for an illustration in the Pareto distribution.

**REMARK 3.** If a family of statistics  $\{\lambda_\alpha, \alpha \in T\}$  satisfies conditions (a) and (c) [or (b) and (c)] of Theorem 1, then very often a convex mixture  $\lambda = \int_T \lambda_\alpha dP(\alpha)$  also satisfies conditions (a) and (c) [(b) and (c)] of Theorem 1. This fact is useful in generating new improved estimators by taking convex combinations of different estimators. Of course, that such convex combinations are also uniformly dominating estimators follows from the convexity of squared-error loss. What is interesting is that such convex combinations also satisfy the hypothesis of Theorem 1; see DasGupta (1986b) for details.

We now give applications of this theorem to various estimation problems. At this point, we like to remind the reader that in almost all the examples given below, the functional  $A(F)$  defined in the theorem is, in fact, a constant independent of  $F$ . For a related result in this direction, see Gleser (1986).

**EXAMPLE 1.** Estimation of arbitrary scale-parameters. Let  $X_i \sim_{\text{indep.}} 1/(\theta_i) f_i(X_i/\theta_i)$ ,  $X_i, \theta_i > 0$ ,  $1 \leq i \leq p$ . The standard estimate of  $\theta_i$  is the best equivariant estimate  $\delta_i(X_i) = a_i X_i$ , where  $a_i = [E_{\theta_i=1}(X_i)]/[E_{\theta_i=1}(X_i^2)]$ . Take  $\lambda(\mathbf{X}) = (\prod_j X_j)^{1/p}$  in Theorem 1.

In this case,

$$\begin{aligned} A(\theta) &= \frac{E_\theta \left[ \sum_{i=1}^p a_i X_i (\prod_{j=1}^p X_j)^{1/p} \right]}{\sum_{i=1}^p \theta_i \cdot E_\theta (\prod_{j=1}^p X_j)^{1/p}} \\ (2.5) \quad &= \frac{E_\theta \left[ \sum_{i=1}^p a_i X_i^{1+1/p} (\prod_{j \neq i} X_j)^{1/p} \right]}{\sum_{i=1}^p \theta_i \cdot E_\theta (\prod_{j=1}^p X_j)^{1/p}}. \end{aligned}$$

Now observe

$$\begin{aligned} (2.6) \quad a_i E_{\theta_i} (X_i^{1+1/p}) &= \frac{E_{\theta_i=1}(X_i)}{E_{\theta_i=1}(X_i^2)} E_{\theta_i=1}(X_i^{1+1/p}) \theta_i^{1+1/p} \\ &< E_{\theta_i=1}(X_i^{1/p}) \theta_i^{1+1/p} \\ &= E_{\theta_i}(X_i^{1/p}) \theta_i \end{aligned}$$

[the inequality follows from Liapounov's inequality which implies that

$E(X^{1+1/p})E(X) < E(X^2)E(X^{1/p})$  for any positive random variable  $X$ ; see DasGupta (1984b)].

The mutual independence of the  $X_i$ 's, (2.5) and (2.6) now imply that  $\sup_{\theta} A(\theta) < 1$  and hence the best equivariant estimate is inadmissible for every  $p \geq 2$ . This example thus establishes inadmissibility of the standard estimates of the means (and in fact, *all* moments) in the gamma, Pareto, lognormal, rectangular,  $F$ , Weibull and the half-normal distributions with unknown scale-parameters. The class of improved estimators in DasGupta (1986a) are thus proved to be robust in the sense that in the entire scale-parameter family, they dominate the best scale-equivariant estimator of the scale-parameter  $\theta$ . Recently, Shinozaki (1984) has proved that the James–Stein estimators are also fairly robust and give uniform domination for many location parameter distributions.

**EXAMPLE 2.** Inadmissibility of unbiased estimates. The result of Theorem 1 can also be used to prove inadmissibility of unbiased estimates in certain families of distributions. Let  $0 \leq X_i, 1 \leq i \leq p$ , be independently distributed as  $F_1, F_2, \dots, F_p$ , respectively. Suppose we want to estimate  $\mu_i(F_i) = E_{F_i}(X_i), 1 \leq i \leq p$ . Identifying  $\delta_i(X)$  as  $X_i$  in Theorem 1, we will show that  $\inf_F A(F) > 1$  if  $\sqrt{X_i}$  has a coefficient of variation uniformly bounded away from zero (over  $\mathcal{F}_i$ ) and, hence,  $X$  is inadmissible for its expectation. We take  $\lambda(\mathbf{x}) = (\prod_j x_j)^{1/p}$ .

Toward this end, note that

$$\begin{aligned}
 E\left\{ \sum_i X_i \left( \prod_j X_j^{1/p} \right) \right\} &= \sum_i E\left\{ X_i^{1+1/p} \left( \prod_{j \neq i} X_j^{1/p} \right) \right\} \\
 (2.7) \qquad \qquad \qquad &\geq \sum_i \left\{ (EX_i)^{1+1/p} E\left( \prod_{j \neq i} X_j^{1/p} \right) \right\} \\
 &\qquad \qquad \qquad \text{(because of independence and Hölder's inequality)} \\
 &= \sum_i \left\{ \mu_i(F_i) (EX_i)^{1/p} E\left( \prod_{j \neq i} X_j^{1/p} \right) \right\}.
 \end{aligned}$$

Now by assumption

$$\begin{aligned}
 (2.8) \qquad \sup_{F_i} \frac{EX_i^{1/2}}{(EX_i)^{1/2}} &< 1 \\
 \Leftrightarrow \sup_{F_i} \frac{EX_i^r}{(EX_i)^r} &< 1 \quad \text{if } 0 < r < 1
 \end{aligned}$$

(as long as  $0 < r < 1$ , the values of  $r$  does not matter). Taking  $r = 1/p$  in (2.8), it now follows from (2.7) that

$$\inf_F \frac{E\left\{ \sum_i X_i \left( \prod_j X_j^{1/p} \right) \right\}}{E\left\{ \prod_j X_j^{1/p} \right\} \sum \mu_i(F_i)} > 1.$$

Distributions for which  $\sqrt{X_i}$  has a uniformly bounded (below) coefficient of variation have, in some sense, pseudoscale structures in them. It can be quite easily proved that scale mixtures of distributions uniformly satisfying (2.8) also satisfy (2.8); see DasGupta (1986b) for details.

One very simple example of this sort is  $h(x) = \varepsilon(1/\theta_1)f_1(x/\theta_1) + (1 - \varepsilon)(1/\theta_2)f_2(x/\theta_2)$ , where  $1 > \varepsilon > 0$  is known. Note that this is *not* a scale-parameter density.

**EXAMPLE 3.** Ferguson family of distributions. The heuristic prescription given in the paragraph preceding the proof of Theorem 1 for finding an appropriate function  $\lambda(\mathbf{x})$  in order to construct the improved estimator seems to work well for some well-known members of the Ferguson family of distributions [for the definition of the Ferguson family of distributions, see Ferguson (1962)]. An important feature of the Ferguson family of distributions is that if  $X_i \sim_{\text{indep.}} L(\theta_i, \sigma_i^2, \gamma_i)$  have distributions belonging to the Ferguson family, then one can find appropriate size functions  $G(\mathbf{x})$  [a statistic  $G$  is called a size function if  $G(a\mathbf{x}) = aG(\mathbf{x})$  for all  $a > 0$ ] such that  $\mathbf{x}/[G(\mathbf{X})]$  and  $G(\mathbf{X})$  are independently distributed. Note that  $G(\mathbf{x})$  can actually depend on the parameters also. For example, if  $X_i \sim_{\text{indep.}} \text{Pareto}(\theta_i)$  where  $\theta_i$ 's are the scale-parameters, then  $G(\mathbf{X}) = \min(X_1/\theta_1, \dots, X_p/\theta_p)$ . For the purpose of the following discussion, it will be understood that  $G(\mathbf{X})$  is such that  $X/[G(\mathbf{X})]$  and  $G(\mathbf{X})$  are independent under  $\theta = 1$ .

It turns out that in two important members of the Ferguson family of distributions, namely, the Pareto, and the lognormal, the natural estimators of the means (which are usually constant multiples of the  $X_j$ 's) can be uniformly dominated in risk by expanding them by a suitable positive multiple of  $G(\mathbf{x})$ .

For the lognormal case, the size  $G(\mathbf{X})$  is  $(\prod_j X_j)^{1/p}$  and thus, rather coincidentally, the result of Example 1 can be invoked to prove that uniform domination obtains. In the Pareto case, we will present the actual proof only for  $p = 2$ ; the calculations get somewhat complicated for larger  $p$ . Before we proceed further, we would like to point out that the appropriate size function  $G(\mathbf{x})$  such that  $\mathbf{x}/[G(\mathbf{x})]$  and  $G(\mathbf{x})$  are independent is usually found by equating  $G(\mathbf{x})$  to a sufficient statistic when all  $\theta_i$ 's are equal to some  $\theta$  [see James (1979), Lemma 4.1]. Now let  $X_i \sim_{\text{indep.}} \text{Pareto}(a, \theta_i)$ , where  $a > 2$  is the common shape-parameter, i.e.,

$$(2.9) \quad f\left(\frac{x_i}{\theta_i}\right) = \frac{a}{\theta_i} \left(\frac{x_i}{\theta_i}\right)^{-a-1}, \quad x_i > \theta_i, a > 2.$$

The best equivariant estimate of  $\theta_i$  is  $(a - 2)/(a - 1)x_i$ . Recall that  $G(\mathbf{x}) = \min(x_1, \dots, x_p)$  in this case. Now

$$(2.10) \quad \begin{aligned} & \sum_i E_\theta \left[ \frac{a-2}{a-1} x_i + cG(\mathbf{x}) - \theta_i \right]^2 - \sum_i E_\theta \left[ \frac{a-2}{a-1} x_i - \theta_i \right]^2 \\ &= E_\theta \left[ pc^2 G^2(\mathbf{x}) + 2c \frac{a-2}{a-1} G(\mathbf{x}) \sum_i X_{(i)} - 2c \sum_i \theta_i G(\mathbf{x}) \right], \end{aligned}$$

where  $X_{(1)} \leq \dots \leq X_{(p)}$  are the sample order statistics. Direct calculation (in the case  $p = 2$ ) yields that for  $\theta_1 \leq \theta_2$ , the risk-difference is equal to

$$(2.11) \quad \left[ pc^2 \frac{a}{a-2} f_1(r) - \frac{2ca}{a-1} f_2(r) \right] \theta_2^2,$$

where

$$(2.12) \quad r = \frac{\theta_1}{\theta_2} \leq 1,$$

$$f_1(r) = r^2 - \frac{r^a}{a-1},$$

$$f_2(r) = \frac{ar^a}{(a-1)(2a-1)} - \frac{r^{a+1}}{2a-1} + \frac{r}{(a-1)^2}.$$

A similar identity holds for  $\theta_2 \leq \theta_1$ .

It is clear that  $f_2(r) > 0 \forall r \in (0, 1]$ ; furthermore,  $[f_1(r)]/[f_2(r)] \rightarrow 0$  as  $r \rightarrow 0$  and  $[f_1(r)]/[f_2(r)]$  has no singularities on  $(0, 1]$ . Consequently,  $[f_1(r)]/[f_2(r)]$  is uniformly bounded on  $(0, 1]$ . Hence one can choose a small enough  $c > 0$  such that the risk-difference in (2.11) is uniformly negative. This proves that for two independent Pareto random variables, the best equivariant estimate can be uniformly dominated in risk by expanding it by a multiple of  $\min(X_1, X_2)$ . We believe that these calculations can be carried out for a general  $p$ ; the algebra, of necessity, gets more involved.

**EXAMPLE 4.** Estimation of the means in the multivariate  $F$  distribution. Let  $X_0, X_1, \dots, X_p$  be independently distributed as  $\Gamma(\alpha_i, \theta_i)$ ,  $\alpha_i > 0, \theta_i > 0$ . The joint distribution of  $Z_1, Z_2, \dots, Z_p$ , where  $Z_i = X_i/X_0$ , is called the multivariate  $F$  distribution.

In multifactor ANOVA problems,  $X_0$  plays the role of the error sum of squares, and  $X_1, X_2, \dots, X_p$  are the sums of squares due to the treatment effect, the block effect, etc. The joint distribution of the ratios  $Z_i = X_i/X_0$  is important for *simultaneously* testing that there is no treatment effect, no block effect, etc.

It is easy to verify that  $E(Z_i) = \alpha_i/(\alpha_0 - 1)\theta_0/\theta_i$ ; also, the best scale-equivariant estimate of  $\theta_0/\theta_i$  is  $(\alpha_0 - 2)/(\alpha_i + 1)Z_i$ . Note that  $\{Z_i\}$  are not independent. We will prove that for  $p \geq 2$ , the best scale-equivariant estimate is inadmissible and dominated uniformly in risk by  $(\alpha_0 - 2)/(\alpha_i + 1)Z_i + c(\prod_j Z_j)^{1/p}$ , for a suitable  $c > 0$ . In view of Theorem 1, we merely need to show that

$$(2.13) \quad \sup_{\theta} \frac{E_{\theta} \left[ \sum_i (\alpha_0 - 2)/(\alpha_i + 1) Z_i (\prod_j Z_j)^{1/p} \right]}{\theta_0 \sum_i \theta_i^{-1} E_{\theta} (\prod_j Z_j)^{1/p}} < 1.$$

Using the facts that  $X_0, X_1, \dots, X_p$  are independent, a direct calculation shows



that the numerator in (2.13) equals

$$1/(\alpha_0 - 1)(\sum_i\{(\alpha_i + 1/p)/(\alpha_i + 1)\})(\prod_j\{\Gamma(\alpha_j + 1/p)/\Gamma\alpha_j\})$$

and the denominator equals  $p/(\alpha_0 - 1)(\prod_j\{\Gamma(\alpha_j + 1/p)/\Gamma\alpha_j\})$  (the ratio is independent of  $\theta_0, \theta_1, \dots, \theta_p$ , and thus can as well be calculated under  $\theta = \mathbf{1}$ ). Thus (2.13) equals  $\sum_i\{(\alpha_i + 1/p)/(\alpha_i + 1)\}/p$ , which clearly is less than 1 for every  $p \geq 2$ . This proves the inadmissibility result.

**EXAMPLE 5.** Estimation of the eigenvalues of the dispersion matrix in multivariate normal distributions. Let  $X_1, X_2, \dots, X_{k+1}$  be i.i.d. observations from  $N_p(\mu, \Sigma)$ ,  $\mu$  and  $\Sigma$  (p.d.) both unknown. Let  $S = \sum_{j=1}^{k+1}(X_j - \bar{X})(X_j - \bar{X})'$ . It is well-known that  $S \sim W_p(k, \Sigma)$  and that  $S$  can be written as  $S = \sum_{j=1}^k U_j U_j'$ , where  $U_j \sim N_p(0, \Sigma)$ . Moreover, the unbiased estimate of  $\Sigma$  is  $S/k$  and the mle is  $S/(k + 1)$ . Consequently, if  $l_1 \geq l_2 \geq \dots \geq l_p > 0$  are the eigenvalues of  $S$ , then  $l_i/k$  as well as  $l_i/(k + 1)$  are in some sense "natural" estimates of the eigenvalues of  $\Sigma$ . It is known that  $l_i/(k + 1)$  are the mle's of the ordered eigenvalues of  $\Sigma$ , but  $l_i/k$  are not unbiased for them except in the case  $p = 1$ . Note that in this problem there is no best choice of the multiple of the  $\{l_i\}$ 's and motivated by the case  $p = 1$ , one may even consider the estimators  $l_i/(k + 2)$ .

In general, if one sets  $\delta_i(X) = al_i$ , then

$$(2.14) \quad A(\Sigma) = \frac{aE_{\Sigma}\{\text{tr } S|S|^{1/p}\}}{\text{tr } \Sigma E_{\Sigma}|S|^{1/p}} \quad [\text{we take } \lambda(\mathbf{1}) = |S|^{1/p}].$$

$A(\Sigma)$  is easily seen to be constant in  $\Sigma$ . Thus, if  $a = 1/k$ , then inadmissibility will follow if we can prove that

$$(2.15) \quad A(I) = \frac{E_I\{\text{tr } S|S|^{1/p}\}}{E_I(\text{tr } S)E_I(|S|^{1/p})} > 1$$

$$(2.16) \quad \Leftrightarrow \text{Cov}_I(\text{tr } S, |S|^{1/p}) > 0.$$

Direct calculation yields that  $\text{Cov}_I(\text{tr } S, |S|^{1/p}) = (k + 2/p)/k$  so that  $\{l_i/k\}$  are inadmissible for  $p \geq 2$ .

The mle's of the eigenvalues in this problem correspond to  $a = 1/(k + 1)$ . The ratio  $A(\Sigma)$  in this case is  $k/(k + 1) \cdot (k + 2/p)/k < 1$  for  $p \geq 3$  and hence it follows from Theorem 1 that the mle is inadmissible for  $p \geq 3$ . The Associate Editor has pointed out, using a different argument, that the mle is inadmissible even for  $p = 2$  and in fact for  $p = 2$  is dominated by  $(2k + 1)/[2k(k + 2)](l_1, l_2)$ . Note that for  $a = 1/k$  the constant  $c$  of the theorem is negative so that the improved estimators are shrinkage estimators, while for  $a = 1/(k + 1)$  the constant  $c$  is positive so that the improved estimators are now expanders. Theorem 1 indeed shows that for each  $a \neq (k + 2/p)^{-1}$ ,  $\{al_i\}$  are inadmissible, and  $a = (k + 2/p)^{-1}$  is *not* a natural choice except when  $p = 1$  or 2. We do not know if this choice of  $a$  implies admissibility of  $al_1, \dots, al_p$  or not.

**EXAMPLE 6.** Estimation of the eigenvalues of  $\Sigma$  in elliptically symmetric distributions. Let  $X_1, X_2, \dots, X_{k+1}$  be i.i.d. observations from a distribution with density  $|\Sigma|^{-1/2} \cdot f((x - \mu)' \Sigma^{-1}(x - \mu))$ , where  $\mu, \Sigma (> 0)$  are unknown. Once this sample is obtained, we define the matrix  $S$  in the usual fashion. It is easy to show that  $E(S/k) = \Sigma$ . Consider the problem of simultaneously estimating the eigenvalues of  $\Sigma$  by constant multiples of the eigenvalues of  $S$ . It turns out that  $\{l_i/k\}$  are inadmissible if  $f$  is of the form

$$f((x - \mu)' \Sigma^{-1}(x - \mu)) = \text{constant} \times \int \frac{e^{-1/2\tau^2(x - \mu)' \Sigma^{-1}(x - \mu)}}{\tau^p} dF(\tau)$$

where  $F$  is a probability measure on  $(0, \infty)$ . This can be seen by reasoning conditionally on  $\tau^2$  and then integrating over  $\tau^2$ ; see DasGupta (1986b) for details. Important densities which are such scale mixtures of normals are the spherically symmetric Cauchy,  $t$  and the double exponential densities.

The estimation of the eigenvalues of  $\Sigma$  when  $X_j$ 's are nonnormal is of importance in principal components analysis. As a dimension reduction technique, principal components analysis does not make any distributional assumptions whatsoever and the ordered eigenvalues of  $\Sigma$  are the variances of the principal components whether or not the  $X_j$ 's are normal.

**EXAMPLE 7.** A larger class of improved eigenvalue estimators when  $S \sim W_p(k, \Sigma)$ . Broader classes of improved estimators for the eigenvalues of  $\Sigma$  when  $S \sim W_p(k, \Sigma)$  are obtainable using Theorem 1.

The identities of Haff (1980) in conjunction with Theorem 1 imply that  $l_i/k + cr(t)$  dominates  $l_i/k$  for suitable  $c < 0$ , whenever

- (i)  $0 < \frac{r(t)}{t} \leq 1$ ,
- (ii)  $t^{-\epsilon}r(t)$  is nondecreasing in  $t$  for some  $\epsilon > 0$ ,

where  $t = |S|^{1/p}$ . Note that condition (i) is needed for hypothesis (c) of Theorem 1 to go through.

The identities of Haff (1980) can be used to generate improved estimators for the eigenvalues of  $\Sigma$  by shifting by suitable functions of  $\text{tr} S$  as well. We have been able to show that  $l_i/k + cr(\text{tr} S)$  dominates  $l_i/k$  for suitable  $c < 0$ , whenever

- (i) there exists an  $\epsilon > 0$  such that  $\epsilon \leq \frac{r(W)}{W} \leq 1 \forall W$ ,
- (ii)  $\frac{r(W)}{W}$  is nonincreasing.

A nice thing about these fairly wide classes of uniformly improved estimators is that one can now formally seek to select an improved estimator by using criteria like gamma-minimaxity or the restricted risk Bayes principle. For some

recent work on such selection problems, see Marazzi (1985), DasGupta and Berger (1986), DasGupta and Rubin (1987) and DasGupta and Bose (1987). Also see Brown's discussion on Berger (1983).

**REMARK 4.** Results similar in spirit to those in Example 5 have been obtained for estimation of the eigenvalues of the precision matrix  $\Sigma^{-1}$ , of  $\Sigma_1^{-1}\Sigma_2$  where  $S_i \sim_{\text{indep.}} W_p(k, \Sigma_i)$ ,  $i = 1, 2$ , and of the residual matrix  $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$  when  $\Sigma$  is partitioned as  $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ . These problems arise naturally in the context of testing problems in multivariate statistics and in multivariate regression; see DasGupta (1986b).

**EXAMPLE 8.** Estimation of losses in the scale-parameter family. We now give an application of Theorem 1 to a completely different type of problem. Let  $X_i \sim_{\text{indep.}} (1/\theta_i)f_i(x_i/\theta_i)$ . As mentioned in Example 1, the natural estimate of  $\theta_i$  under squared-error loss is the best equivariant estimate  $a_i X_i$ , where  $a_i = [E_{\theta_i-1}(X_i)]/[E_{\theta_i-1}(X_i^2)]$ . The loss in estimating  $\theta_i$  by  $a_i X_i$  is  $(a_i X_i - \theta_i)^2$ . An important decision-theoretic problem is that of estimating the actual loss incurred in estimating an unknown parameter; the idea here is that a statistician should report not only an estimate of the unknown parameter, but also a measure of accuracy of this estimate. An estimated loss serves this purpose. The related problem of estimating the risk function is also of some importance. Note that the loss incurred is a function of both the parameter and the data; we will use the squared-error as a criterion for choosing among different estimators of the loss. In the context of our scale-parameter problem, then, the problem under consideration is simultaneous estimation of the componentwise losses  $(a_i X_i - \theta_i)^2$ ,  $1 \leq i \leq p$ , under a second stage sum of squared-error loss. The problem of estimating the overall loss  $\sum_{i=1}^p (a_i X_i - \theta_i)^2$  could also be attempted; see Johnstone (1987). For ease in presentation, we will work out this example in the gamma case, but every step in the following analysis goes through in the arbitrary scale-parameter family.

Let  $X_i \sim_{\text{indep.}} \Gamma(\alpha, \theta_i)$ , with  $EX_i = \alpha/\theta_i$ ,  $1 \leq i \leq p$ . In this case, the best equivariant estimate of  $\theta_i^{-1}$  is  $a_i X_i$ , where  $a_i = 1/(\alpha + 1)$ . For estimating the loss  $(a_i X_i - \theta_i^{-1})^2$ , natural one-dimensional estimates are  $c_i X_i^2$ ; one reason for this is that the expected loss is proportional to  $\theta_i^{-2}$  and another reason is that unbiased estimators of the overall loss  $\sum_{i=1}^p (a_i X_i - \theta_i^{-1})^2$  are of the form  $\sum_{i=1}^p c_i X_i^2$  [i.e., for suitable constants  $c_i$ 's,  $E_\theta\{\sum_{i=1}^p c_i X_i^2 - \sum_{i=1}^p (a_i X_i - \theta_i^{-1})^2\} \equiv 0$ ].

Because of the scale-structure in the problem, there exists a  $c_i$  minimizing  $E[c_i X_i^2 - (X_i/(\alpha + 1) - \theta_i^{-1})^2]$ ; the optimum  $c_i$  satisfies

$$(2.19) \quad \left( c_i - \frac{1}{(\alpha + 1)^2} \right) EX_i^4 + \frac{2}{\alpha + 1} EX_i^3 - EX_i^2 = 0,$$

where  $E(\cdot)$  denotes expectation under  $\theta_i = 1$ .

By direct calculation now,  $c_i = 1/[(\alpha + 1)^2(\alpha + 2)]$ . Identifying  $\delta_i(X)$  as  $c_i X_i^2$  and  $\mu_i(X, \theta)$  as  $(X_i/(\alpha + 1) - \theta_i^{-1})^2$  in Theorem 1, it suffices to show that

$$(2.20) \quad \frac{1}{(\alpha + 1)^2(\alpha + 2)} \sup_{\theta} \frac{E_{\theta} \left[ \sum_{i=1}^p X_i^2 (\prod_{j=1}^p X_j)^{2/p} \right]}{E_{\theta} \left[ (\prod_{j=1}^p X_j)^{2/p} \sum_{i=1}^p (X_i/(\alpha + 1) - \theta_i^{-1})^2 \right]} < 1.$$

By straightforward calculations, the numerator of (2.20) equals

$$(2.21) \quad \frac{\Gamma(\alpha + 2 + 2/p)}{\Gamma(\alpha + 2/p)} \left( \frac{\Gamma(\alpha + 2/p)}{\Gamma\alpha} \right)^p \left( \prod_j \theta_j^{-2/p} \right) \sum_i \theta_i^{-2}.$$

Also, the denominator of (2.20) equals

$$(2.22) \quad \left\{ \frac{(\alpha + 1 + 2/p)(\alpha + 2/p)}{(\alpha + 1)^2} - \frac{2(\alpha + 2/p)}{\alpha + 1} + 1 \right\} \\ \times \left( \frac{\Gamma(\alpha + 2/p)}{\Gamma\alpha} \right)^p \left( \prod_j \theta_j \right)^{-2/p} \sum_i \theta_i^{-2} \\ = \frac{4/p^2 - 2/p + \alpha + 1}{(\alpha + 1)^2} \left( \frac{\Gamma(\alpha + 2/p)}{\Gamma\alpha} \right)^p \left( \prod_j \theta_j^{-2/p} \right) \left( \sum_i \theta_i^{-2} \right).$$

Hence, in the notation of Theorem 1,

$$(2.23) \quad A(\theta) \equiv \frac{1}{(\alpha + 1)^2(\alpha + 2)} \frac{(\alpha + 1 + 2/p)(\alpha + 2/p)(\alpha + 1)^2}{4/p^2 - 2/p + \alpha + 1}.$$

Interestingly enough, this ratio is exactly 1 for both  $p = 1$  and 2 and thus the inadmissibility result of Theorem 1 does not apply. However, for  $p \geq 3$ , this ratio is less than 1 and hence uniform domination can be achieved by using improved loss estimators  $X_i^2/[(\alpha + 1)^2(\alpha + 2)] + c(\prod_j X_j)^{2/p}$ , where  $c > 0$  is a suitable number. Thus, although the best equivariant estimate of  $\theta_i^{-1}$  is itself inadmissible for  $p \geq 2$ , a natural estimator of its loss can be shown to be inadmissible only for  $p \geq 3$ , and is probably admissible for  $p = 1, 2$ . This somewhat surprising phenomenon in the gamma case thus completely agrees with the recent results of Johnstone (1987) in the normal case.

In the general scale-parameter family, the expression analogous to (2.19) is given as

$$(2.24) \quad (c_i - a_i^2)EX_i^4 + 2a_iE(X_i^3) - E(X_i^2) = 0 \\ \Leftrightarrow c_i = \frac{EX_i^2 + a_i^2EX_i^4 - 2a_iEX_i^3}{EX_i^4},$$

where  $E(\cdot)$  again denotes expectation under  $\theta_i = 1$ . If the coordinate distributions under  $\theta_i = 1$  are assumed to be identical, then  $c_i \equiv c$  (say) and  $a_i \equiv a$  (say)

for every  $i$ . The constant ratio analogous to (2.23) now is given as

$$(2.25) \quad A(\theta) \equiv \left[ cm \left( 2 + \frac{2}{p} \right) / m \left( \frac{2}{p} \right) \right] / \left[ a^2 \frac{m(2 + 2/p)}{m(2/p)} - 2a \frac{m(1 + 2/p)}{m(2/p)} + 1 \right],$$

where  $m(r) = EX_i^r$ .

Because of the result we have just proved for the gamma case, there is no hope of showing that this ratio is  $\neq 1$  for every  $p \geq 2$ . Interestingly, however, in virtually *every other* standard scale-parameter family, this ratio is different from 1 for  $p = 2$ . Our theorem would then imply that the best scale-equivariant estimator of the loss is inadmissible in *all of these* distributions for  $p = 2$  and, *hence*, for every  $p \geq 2$ .

We will now give a simple sufficient condition for the ratio in (2.25) to be different from 1 when  $p = 2$  (in view of Theorem 1, that is all one needs for inadmissibility).

Simple algebra shows that the ratio in (2.25) is different from 1 for  $p = 2$  iff

$$(2.26) \quad \frac{1}{2} \frac{m(4)}{m(3)} + \frac{1}{2} \frac{m(2)}{m(1)} \neq \frac{m(3)}{m(2)}.$$

Now let  $\psi(r) = [m(r + 1)]/[m(r)]$ , where  $r > 0$ . It is clear that if  $\psi(r)$  is strictly convex or strictly concave in the interval  $1 \leq r \leq 3$ , then (2.26) holds. For many standard scale-parameter distributions, showing that  $\psi(r)$  is strictly convex (concave) is actually easier than verifying (2.26) directly. For the sake of completeness, we give a list of the appropriate  $\psi$ 's in a few standard examples:

Distribution	Density under $\theta = 1$	$\psi(r)$
gamma	$\frac{e^{-x}x^{\alpha-1}}{\Gamma\alpha}$	$\alpha + r$
Pareto	$ax^{-a-1}, x > 1, a > 4$	$\frac{a-r}{a-r-1}$
lognormal	$\frac{1}{\sqrt{2\pi}x} e^{-1/2(\log x)^2}$	$e^{r+1/2}$
rectangular	$1(0 < x < 1)$	$\frac{r+1}{r+2}$
$F$	$\frac{1}{\sqrt{2}x} \left( \sqrt{n} B \left( \frac{1}{2}, \frac{n}{2} \right) \right)^{-1} \left( 1 + \frac{x}{n} \right)^{-(n+1)/2}$	$\frac{n(1+2r)}{n-2r-2}$
Weibull	$\beta x^{\beta-1} e^{-x^\beta}$	$\frac{\Gamma(\alpha + (r+1)/\beta)}{\Gamma(\alpha + (r/\beta))}$

In all of these cases, except the Weibull, it is very easy to check that  $\psi(r)$  is indeed either strictly convex or strictly concave by simply looking at  $\psi''(r)$ . In the Weibull case with  $\beta \neq 1$ , denoting  $\alpha + r/\beta$  as  $s$  and  $1/\beta$  as  $t$ , the convexity (concavity) of  $\psi(s) = [\Gamma(s+t)]/[\Gamma(s)]$  follows from the facts that  $[\Gamma(s+t)]/[\Gamma(s)] = EX^t$  where  $X \sim \text{gamma}(s, 1)$ , that the gamma( $s, 1$ ) densities are strict Polya type  $\infty$  and for any  $t \neq 1$ ,  $x^t$  crosses any line at most twice and

hence Karlin's sign change theorem [see Karlin (1957)] implies that  $\psi(s)$  crosses any line at most twice. This argument was sketched by an anonymous referee to whom we are very grateful.

**3. Final remarks, Bayesian motivation, risk-improvements and scope for choice.** As stated in Section 2, in the Wishart eigenvalue problem, Example 7 gives a wide class of improved estimators. This gives us a good scope for choice and past evidence suggests that the function  $\lambda(S)$  can be chosen to incorporate prior information; for example, one can choose that  $\lambda$  which gives the smallest Bayes risk with respect to a given prior.

There are some important problems in which our theorem does *not* apply without nontrivial modifications. These include estimation of characteristic roots of the correlation matrix, estimation of the "proportions of variance explained" in principal components analysis, i.e.,  $\lambda_i/\sum_{i=1}^p \lambda_i$ , where  $\lambda_i$ 's are the eigenvalues of  $\Sigma$ , and estimation of canonical correlations. However, we hope that the techniques of this paper will provide mathematical insight into those problems.

One nice feature of the dominating estimators provided by our theorem is that in almost all applications, the risk improvement is exactly analytically calculable. This not only makes subsequent Bayesian calculations analytically feasible, it also makes simulation unnecessary for the study of risk-improvement. Using these analytical expressions in the Wishart case, we found that by using  $\lambda(S) = \text{tr} S$  and  $c$  as the midpoint of the admissible range, about 22.2% risk-improvement can be attained over the eigenvalues of  $S/k$  when  $p = 2$  and  $k$  is small. The improvement, however, starts decreasing as  $k$  increases.

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