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SPECTRAL FACTORIZATION OF NONSTATIONARY MOVING AVERAGE PROCESSES

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We solve here the general nonstationary multivariate MA *spectral factorization problem*, i.e. the problem of obtaining all the possible MA models (with time-dependent coefficients) corresponding to a given (time-dependent) autocovariance function. Our result (Theorem 8) relies on a symbolic generalization (Theorem 1) of the classical factorization property of the characteristic polynomial associated with stationary autocovariance functions, and is obtained by means of a matrix extension of ordinary continued fractions. We also give necessary and sufficient conditions for an autocovariance function to be an MA autocovariance function and for a process to be an MA one (Theorems 6 and 7).

1. Introduction. The usual approach to nonstationary problems, in time series analysis, is to assume some suitable difference of the stochastic process under study to be (second-order) stationary (in the univariate case; the idea is somewhat more subtle for multivariate processes—a simultaneous differencing of all the component processes being not always required—but essentially amounts to the same).

Such an assumption is, of course, very convenient for practical purposes, since it reduces any problem to the well-known stationary case. It is, however, exceedingly restrictive. The only nonstationary processes satisfying this assumption are indeed, in Box and Jenkins (1976)'s terminology, *homogeneous nonstationary processes*; such processes are second-order explosive, and, if time ranges from $-\infty$ to $+\infty$, they even do not have a well-defined variance, which, in a second-order theory, is quite a limitation.

As a consequence, processes with time-varying autocovariances—even the simplest ones, such as periodical moving averages—are hopelessly excluded from the analysis (periodic autoregressions can be treated by means of stationary multivariate methods, Cf. Troutman (1979); periodic moving averages cannot).

The alternative—in a time-domain approach—would be the use of models with time-dependent coefficients; and, since ARMA models with constant coefficients proved to be very efficient in stationary problems, ARMA models with time-dependent coefficients should also provide a very efficient framework for nonstationary cases. This is, most plausibly, true for AR models: we showed in Hallin and Ingenbleek (1981 and 1983) that—in the nonstationary case as well as in the stationary one—AR(p) models describe the class of processes whose autocovariance functions satisfy difference equations of order p (namely, the

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generalized Yule-Walker equations). As for the MA models, the very strong argument in favor of their use (a process admits an MA(q) representation iff it is q-dependent) is shown to hold also in the nonstationary case (Theorem 7 below).

The main statistical problem, in time series analysis, is the estimation, from a finite length realization, of the coefficients of some adequate model for a given process. Among several others, a well-known paper by Priestley (1965) uses a frequency-domain approach, and a series of works by Mélard (1982) and Kiehm and Mélard (1981) are based on a time-domain likelihood method for this estimation problem; both are restricted to particular cases (*evolutionary* spectra and *linear/exponential* models). However, a common feature of all the likelihood methods is that they do not, strictly speaking, estimate the coefficients of a model: more or less explicitly, they all provide an estimation of the process autocovariance function, the "estimated model" being then *any* element of the set (possibly restricted to a single element) of ARMA models associated with this estimated autocovariance function. Accordingly, before starting with estimation problems, the relationship between (nonstationary, i.e. time-dependent) autocovariance functions and the possible models for them should be carefully studied. The problem of obtaining the set of models corresponding to a given autocovariance function is what we call here—in a time series analysis context—the *theoretical model-building problem*; it is also known, in the engineering literature, as the *covariance factorization* or (time varying) *spectral factorization* problem.

In the autoregressive case, the links between an autocovariance function and the corresponding AR model can be described by means of difference equations (generalized Yule-Walker equations—cf. the above quoted papers).

Problems are much more difficult in the moving average case. In Hallin (1981b), using continued fraction methods, we obtained a complete and explicit solution for the univariate MA(1) problem. For lack of an adequate generalization of continued fractions, we used another approach (Hallin, 1982a) in the MA(2) case, but could not achieve as complete a solution as in the MA(1) case.

The present paper gives a complete, explicit and unified solution for the general p-variate MA(q) model-building problem.

Section 2 presents the basic symbolic factorization property of the autocovariance difference operator which is the nonstationary generalization of the well-known factorization of the autocovariance characteristic polynomial.

In the third section, we introduce the theoretical model-building problem, and show how it brings about a matrix generalization of the concept of (positive definite) continued fractions: M-*fractions*. Section 4 is entirely devoted to the study of the convergence and positive definiteness of M-fractions, which constitute the mathematical setting required for Section 5. Finally, Theorems 6, 7 and 8 of Section 5 present the main results: a characterization of nonstationary MA autocovariance functions, a generalization of a theorem by T. W. Anderson bearing out the usefulness of time-varying MA models, and the explicit solution to the general MA theoretical model-building problem. Section 6 gives an illustration of these results in the univariate MA(1) and MA(2) cases; Section 7 briefly discusses the potential application of the results and their connections with related papers in the engineering and applied mathematics literature.

Throughout this paper, we use the following notation:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}, \quad \mathbb{N} = \{0, 1, 2, \cdots\}$$
$$k\mathbb{Z} = \{0, \pm k, \pm 2k, \cdots\}$$

$\mathbb{C}$ stands for the set of complex numbers. Unprimed vectors refer to column vectors, primes denote transposes. $M = (\psi_1 \cdots \psi_m)$, where the ψ_i's are $m \times 1$ vectors, is the square matrix whose columns are $\psi_1 \cdots \psi_m$; $|M| = |\psi_1 \cdots \psi_m|$ is the corresponding determinant. $O_{k\times l}$ stands for the $k \times l$ null matrix, and $1_{m\times m}$ for the $m\times m$ unit matrix. E denotes mathematical expectation.

2. A factorization property of moving average autocovariance operators.

2a. MA *processes and* MA *models.* Let $\{\varepsilon_t; t \in \mathbb{Z}\}$ denote a real, p-dimensional, second-order white noise:

$$E(\varepsilon_u) = O_{p\times 1}, \quad E(\varepsilon_u \varepsilon'_v) = \delta_{uv} 1_{p\times p}, \quad u, v \in \mathbb{Z}.$$

Let A_{tj} $(j = 0, \cdots, q)$ be real $p \times p$ matrices, and

$$(1) \qquad A_t(L) = A_{t0} + A_{t1}L + \cdots + A_{tq}L^q, \quad t \in \mathbb{Z}$$

(L denotes the lag operator) a linear difference operator of order q and dimension p (hence, A_{t0} and A_{tq} have to be of full rank). Consider the p-variate process $\{z_t; t \in \mathbb{Z}\}$ generated by

$$(2) \qquad z_t = A_t(L)\varepsilon_t = A_{t0}\varepsilon_t + A_{t1}\varepsilon_{t-1} + \cdots + A_{tq}\varepsilon_{t-q}, \quad t \in \mathbb{Z}$$

(in a quadratic mean sense, i.e. the variance of $z_t - A_t(L)\varepsilon_t$ is zero): z_t is a p-variate *moving average process* of order q(more briefly, an MA(q) *process*), and (2) is a *model* for z_t (an MA(q) *model*); $A_t(L)$ will be called the *model difference operator.* z_t is generally nonstationary, although time-dependent models may generate stationary processes (see Hallin, 1981b, for an example).

Notice that the concept of an MA model or operator bears an intrinsic indetermination: let indeed $(O_t; t \in \mathbb{Z})$ be an *arbitrary* sequence of orthogonal $p \times p$ matrices: the model

$$A_{t0}O_t\varepsilon_t + A_{t1}O_{t-1}\varepsilon_{t-1} + \cdots + A_{tq}O_{t-q}\varepsilon_{t-q} = A_t(L)O_t\varepsilon_t$$

cannot be considered distinct from (2), since $\{O_t\varepsilon_t; t \in \mathbb{Z}\}$ is still a second-order white noise. We shall say that $A_t(L)$ and $A_t(L)O_t$ are *equivalent* MA *model-operators*, defining *equivalent* MA *models.*

2b. *The autocovariance difference operator.* Let $\Gamma_{tj} = E(z_t z'_{t-j})$; of course, Γ_{tj} vanishes for $j > q$. The matrices Γ_{tj} are what we call the *process autocovariances* at time $t(j = 0, 1, 2, \cdots)$ (we use the terms *autocovariance, autocovariance matrix* and *autocovariance function*, which we find convenient, instead of discriminating between *auto*covariances and *cross-covariances*); they are the blocks of the covariance matrix Γ of the process. Γ is an infinite symmetric band matrix, with bandwidth not greater than $2p(q+1)-1$; as a block matrix, it has rows of the form

$$(3)\qquad \cdots\ 0\,\Gamma'_{t+q,q}\Gamma'_{t+q-1,q-1}\cdots\Gamma'_{t+1,1}\Gamma_{t0}\Gamma_{t1}\cdots\Gamma_{t,q-1}\Gamma_{tq}0\ \cdots\quad t\in\mathbb{Z}$$

and hence a "*block-bandwidth*" of $(2q+1)$ blocks.

A difference operator of order $2q$ (and dimension p) can be associated with Γ: the *autocovariance difference operator* $\Gamma_t(L)$

$$(4)\qquad \begin{aligned}\Gamma_t(L) &= \Gamma'_{t+q,q}L^0 + \Gamma'_{t+q-1,q-1}L + \cdots\\ &\quad + \Gamma'_{t+1,1}L^{q-1} + \Gamma_{t0}L^q + \Gamma_{t1}L^{q+1} + \cdots + \Gamma_{tq}L^{2q},\quad t\in\mathbb{Z}.\end{aligned}$$

In the case of stationary processes z_t, generated by time-independent models, the autocovariance and model difference operators take the form of matrix polynomials

$$\Gamma(L) = \Gamma'_q + \Gamma'_{q-1}L + \Gamma'_{q-2}L^2 + \cdots + \Gamma'_1L^{q-1} + \Gamma_0L^q + \cdots + \Gamma_qL^{2q}$$

and

$$A(L) = A_0 + A_1L + \cdots + A_qL^q$$

satisfying the well-known factorization property

$$(5)\qquad \Gamma(x) = A(x)A'(1/x)x^q,\quad x\in\mathbb{C}$$

($A'(\cdot)$ denotes the polynomial obtained by transposing the matrix coefficients in $A(\cdot)$) or, in a more classical form,

$$\Gamma(1/x)x^q = A(1/x)A'(x),\quad x\in\mathbb{C}.$$

(5) cannot be expected to hold straightforwardly in the nonstationary case; however, a Yule-Walker approach to the problem (cf. Hallin, 1981c), suggests a *symbolic* generalization of this important factorization property. In order to introduce this generalization, we first need to recall some concepts about difference operators.

2c. *The adjoint difference operator.* Let $a_t(L) = a_{t0} + a_{t1}L + \cdots + a_{tq}L^q$ be a *scalar* difference operator of order q. The one-sided *Green's function* $G(t, s)$ associated with $a_t(L)$(cf. Miller, 1968) can be defined as the value, for $t\in\mathbb{Z}$, of the (unique) solution of the homogeneous equation $a_t(L)x_t = 0$ with "initial" values $x_s = a_{s0}^{-1}$, $x_{s-1} = \cdots = x_{s-q+1} = 0$. The adjoint operator $a_t^*(L)$ is then defined as

$$(6)\qquad a_t^*(L) = a_{t0}^* + a_{t1}^*L + \cdots + a_{tq}^*L^q,\ a_{tj}^* = a_{t+q-j,q-j},\quad t\in\mathbb{Z}$$

with the characteristic property that the Green's functions $G^*(t, s)$ associated with $a_t^*(L)$ satisfy

$$(7)\qquad G(t, s) = -G^*(s-q, t),\quad t, s\in\mathbb{Z}.$$

This definition of an adjoint difference operator can be extended to the case of multivariate difference operators: define the adjoint operator $A_t^*(L)$ of $A_t(L) = A_{t0} + A_{t1}L + \cdots + A_{tq}L^q$ as

$$A_t^*(L) = A_{t0}^* + A_{t1}^*L + \cdots + A_{tq}^*L^q, \quad A_{tj}^* = A'_{t+q-j,q-j}, \quad t \in \mathbb{Z}. \tag{8}$$

Such an extension should be legitimatized by some property analogous with (7). Denote here by $G(t, s)$ the matrix solution of $A_t(L)X_t = 0$(X_t a $p \times p$ matrix; $t \in \mathbb{Z}$) taking "initial" values $X_s = A_{s0}^{-1}$, $X_{s-1} = \cdots = X_{s-q+1} = 0$, and designate $G(t, s)$ as the *one-sided Green's matrices* associated with the operator $A_t(L)$; accordingly, let $G^*(t, s)$ stand for the *one-sided Green's matrices* associated with $A_t^*(L)$. (A linear difference operator of order q defines *vector difference equations* and *matrix difference equations*, denoted here by $A_t(L)\psi_t = \cdot$ and $A_t(L)X_t = \cdot$, respectively; of course, the columns of a *matrix solution* are themselves *vector solutions.*) The following result can then be established (Hallin, 1982b):

$$G'(t, s) = -G^*(s - q, t), \quad t, s \in \mathbb{Z}. \tag{7'}$$

2d. *Factorization of the autocovariance operator.* We can now state the following result, characterizing the links between MA autocovariance operators and MA model operators:

THEOREM 1. *$\Gamma_t(L)$ is the autocovariance difference operator associated with some* MA(q) *model* (*of the form* (2)) *iff it can be factored into*

$$\Gamma_t(L) = A_t(L) \circ A_t^*(L), \quad t \in \mathbb{Z} \tag{9}$$

($\circ$ *denotes the symbolic product of difference operators*; *recall that the symbolic product of two difference operators is obtained by applying usual noncommutative multiplication rules, the lag operator L operating on any time index appearing on its right.* EXAMPLE: $c_tL^2 \circ d_{t+1}L = c_td_{t-1}L^3$).

PROOF. The proof is immediate by expanding the symbolic product $(\Sigma_j A_{tj}L^j) \circ(\Sigma_j A_{tj}^*L^j)$. If $\Gamma_t(L) = \Gamma(L)$, (9) takes the form, for models with constant coefficients, of an ordinary product of polynomials $\Gamma(L) = A(L) \cdot A^*(L)$; moreover, $A^*(L) = A'(1/L)L^q$: (9) thus reduces to the classical result (5). □

3. A first look at the model-building problem.

3a. *Factorizing the autocovariance operator.* How can the symbolic factorization (9), which expresses the link between an autocovariance function and the possible MA(q) models for it, be used for model-building purposes? Roughly speaking, how can we solve equation (9) for $A_t(L)$ in terms of $\Gamma_t(L)$?

The most immediate consequence of (9) is that any vector solution of $A_t^*(L)\psi_t = 0$ is also a solution of $\Gamma_t(L)\psi_t = 0$. Hence, if $A_t(L)$ provides an MA(q) model for a process with associated autocovariance operator $\Gamma_t(L)$, the pq-dimensional solution space of $A_t^*(L)\psi_t = 0$ is a subspace of $\Gamma_t(L)\psi_t = 0$'s $2pq$-dimensional one. We may thus expect to solve the theoretical model-building problem by choosing appropriately pq linearly independent vector solutions of $\Gamma_t(L)\psi_t = 0$, just as the appropriate selection of q out of the $2q$ roots of the characteristic autocovariance polynomial led, in the univariate MA(q) stationary case, to an adequate MA(q) model.

Such an approach was previously applied to the univariate MA(1) and MA(2) cases in Hallin (1981 and 1982a). Continued fractions results and methods (more precisely, properties of *positive definite continued fractions*—cf. Wall, 1948) allowed for a complete and explicit solution in the MA(1) case only; our purpose is to introduce here a quite natural matrix extension of the concept of continued fraction, which will provide a complete and explicit solution for the general p-variate MA(q) problem. Let us briefly show how such an extension naturally arises from (9).

Consider again the MA(q) model (2); putting

$$\mathbf{z}_t = \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-q+1} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}, \quad \mathbf{A}_{t0} = \begin{bmatrix} A_{t0} & A_{t1} & \cdots & A_{t,q-1} \\ 0 & A_{t-1,0} & \cdots & A_{t-1,q-2} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & A_{t-q+1,0} \end{bmatrix},$$

$$\mathbf{A}_{t1} = \begin{bmatrix} A_{tq} & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ A_{t-q+2,2} & \cdots & A_{t-q+2,q} & 0 \\ A_{t-q+1,1} & \cdots & A_{t-q+1,q-1} & A_{t-q+1,q} \end{bmatrix}, \quad t \in \mathbb{Z},$$

we have, of course,

$$\mathbf{z}_t = \mathbf{A}_{t0}\boldsymbol{\varepsilon}_t + \mathbf{A}_{t1}\boldsymbol{\varepsilon}_{t-q}, \quad t \in \mathbb{Z}; \tag{2'}$$

$\{\mathbf{z}_t;\ t \in q\mathbb{Z}\}$ is thus an m-variate MA(1) process (with $m = pq$), so that we may concentrate most of our attention on the first-order multivariate case

$$z_t = A_{t0}\varepsilon_t + A_{t1}\varepsilon_{t-1}, \quad t \in \mathbb{Z}. \tag{10}$$

Let $\Sigma_t = E(z_t z_t')$ and $\Gamma_t = E(z_t z_{t-1}')$ stand for Γ_{t0} and Γ_{t1}: if z_t is generated by (10), Σ_t and Γ_t are nonsingular, and

$$\Sigma_t = A_{t0}A_{t0}' + A_{t1}A_{t1}', \quad \Gamma_t = A_{t1}A_{t-1,0}' \quad t \in \mathbb{Z}.$$

We shall refer to the sequence $(\Sigma_t, \Gamma_t;\ t \in \mathbb{Z})$ as z_t's (time-dependent) *autocovariance function.*

Now, let (Σ_t, Γ_t) be a given MA(1) autocovariance function, and suppose we want to solve the model-building problem by means of the symbolic factorization (9). Consider m linearly independent solutions, $\psi_t^1 \cdots \psi_t^m$, of the homogeneous vector equation $\Gamma_t(L)\psi_t = 0$:

$$\Gamma_{t+1}'\psi_t + \Sigma_t\psi_{t-1} + \Gamma_t\psi_{t-2} = 0, \quad t \in \mathbb{Z}. \tag{11}$$

The vector space spanned by $\psi_t^1 \cdots \psi_t^m$ is the solution space of any difference equation $B_t^*(L)\psi_t = (B'_{t+1,1} + B'_{t0}L)\psi_t = 0$ such that

$$B_{t0}'^{-1}B'_{t+1,1} = -(\psi_{t-1}^1 \cdots \psi_{t-1}^m)(\psi_t^1 \cdots \psi_t^m)^{-1}, \quad t \in \mathbb{Z}, \tag{12}$$

the operators $B_t^*(L)$ being thus symbolic divisors of $\Gamma_t(L)$.

Suppose that one of these operators, $A_t^*(L)$, say, is such that $\Gamma_t(L) = A_t(L)\circ A_t^*(L)$, hence providing an appropriate model for (Σ_t, Γ_t): (12) can be written as

$$A_{t0}A'_{t0} = -\Gamma'_{t+1}(\psi_t^1 \cdots \psi_t^m)(\psi_{t-1}^1 \cdots \psi_{t-1}^m)^{-1}, \quad t \in \mathbb{Z} \tag{12'}$$

or, equivalently,

$$A_{t1}A'_{t1} = -\Gamma_t(\psi_{t-2}^1 \cdots \psi_{t-2}^m)(\psi_{t-1}^1 \cdots \psi_{t-1}^m)^{-1}, \quad t \in \mathbb{Z}. \tag{13}$$

Denote by Φ_t the right-hand side of (13): Φ_t is positive definite (p.d.).

Conversely, suppose that an m-tuple $(\psi_t^1 \cdots \psi_t^m)$ is such that

$$-\Gamma_t(\psi_{t-2}^1 \cdots \psi_{t-2}^m)(\psi_{t-1}^1 \cdots \psi_{t-1}^m)^{-1} = \Phi_t$$

is p.d. for any $t \in \mathbb{Z}$: we can choose a sequence of nonsingular matrices $(A_{t1};\ t \in \mathbb{Z})$ such that $A_{t1}A'_{t1} = \Phi_t$, $t \in \mathbb{Z}$. Define A_{t0} as $\Gamma'_{t+1}A_{t+1,1}'^{-1}$: the corresponding model operator $A_t(L)$ provides an appropriate solution to the model-building problem, and it is easy to see that the set of models obtained from $(\psi_t^1 \cdots \psi_t^m)$ by trying all the possible factorizations of Φ_t is precisely the equivalence class of $A_t(L)$ (cf.2a.).

Summing up, we established the following result:

THEOREM 2. *An m-tuple* $\{\psi_t^1 \cdots \psi_t^m\}$ *of linearly independent (i.e. such that* $|\psi_t^1 \cdots \psi_t^m| \neq 0$, $t \in \mathbb{Z}$*) solutions of* $\Gamma_t(L)\psi_t = 0$ *defines a solution to the model-building problem iff* $\Phi_t = -\Gamma_t(\psi_{t-2}^1 \cdots \psi_{t-2}^m)(\psi_{t-1}^1 \cdots \psi_{t-1}^m)^{-1}$ *is positive definite for any* $t \in \mathbb{Z}$*. This solution is unique up to the orthogonal transformation equivalence defined in Section 1a.*

3b. *Matrix continued fractions.* What appears from Theorem 2 is the important role played by the sequence of matrices Φ_t, and most of the next section will be devoted to a study of their positive definiteness properties. Starting from the definition, we have, substituting for ψ_{t-1}^i in terms of ψ_{t-2}^i and ψ_{t-3}^i,

$$\begin{aligned}\Phi_t &= -\Gamma_t[(\psi_{t-1}^1 \cdots \psi_{t-1}^m)(\psi_{t-2}^1 \cdots \psi_{t-2}^m)^{-1}]^{-1}\\ &= -\Gamma_t[-\Gamma_t'^{-1}(\Sigma_{t-1}(\psi_{t-2}^1 \cdots \psi_{t-2}^m) + \Gamma_{t-1}(\psi_{t-3}^1 \cdots \psi_{t-3}^m))(\psi_{t-2}^1 \cdots \psi_{t-2}^m)^{-1}]^{-1},\end{aligned}$$

hence the *prospective recursion*

$$\Phi_t = \Gamma_t(\Sigma_{t-1}\ \Phi_{t-1})^{-1}\Gamma'_t, \quad t \in \mathbb{Z}; \tag{14}$$

similarly, substituting for ψ_{t-2}^i in tems of ψ_{t-1}^i and ψ_t^i,

$$\Phi_t = (\Gamma'_{t+1}(\psi_t^1 \cdots \psi_t^m) + \Sigma_t(\psi_{t-1}^1 \cdots \psi_{t-1}^m))(\psi_{t-1}^1 \cdots \psi_{t-1}^m)^{-1}$$

yields the *retrospective recursion*

$$\Phi_t = \Sigma_t - \Gamma'_{t+1}\Phi_{t+1}^{-1}\Gamma_{t+1}, \quad t \in \mathbb{Z}. \tag{15}$$

Iterating (14) and (15) brings infinite expressions of the form

$$\Gamma_t(\Sigma_{t-1} - \Gamma_{t-1}(\Sigma_{t-2} - \Gamma_{t-2}(\cdots)^{-1}\Gamma'_{t-2})^{-1}\Gamma'_{t-1})^{-1}\Gamma'_t \tag{16}$$

and

$$\Sigma_t - \Gamma'_{t+1}(\Sigma_{t+1} - \Gamma'_{t+2}(\Sigma_{t+2} - \Gamma'_{t+3}(\cdots)^{-1}\Gamma_{t+3})^{-1}\Gamma_{t+2})^{-1}\Gamma_{t+1}; \tag{17}$$

formally, and all convergence considerations apart, (16) and (17) are matrix generalizations of continued fractions such as

$$\frac{\gamma_t^2\,|}{|\,\sigma_{t-1}^2} - \frac{\gamma_{t-1}^2|}{|\,\sigma_{t-2}^2} - \frac{\gamma_{t-2}^2|}{|\,\sigma_{t-3}^2} - \cdots \quad \text{and} \quad \sigma_t^2 - \left(\frac{\gamma_{t+1}^2|}{|\,\sigma_{t+1}^2} - \frac{\gamma_{t+2}^2|}{|\,\sigma_{t+2}^2} - \frac{\gamma_{t+3}^2|}{|\,\sigma_{t+3}^2} - \cdots\right),$$

which appear in the univariate MA(1) spectral factorization problem (Hallin, 1981c). Two main questions thus arise: Do such expressions converge? Are their limits positive definite?

4. A brief theory of matrix continued fractions (M-fractions)

4a. *Definitions.* (The value, when it exists, of our M-*fractions* is itself a matrix; the elements of this matrix can be seen as *generalized continued fractions* in the sense of Magnus (1977); the leading elements of our M-*fractions* are also *generalized continued fractions* in the sense of Rutishauser (1958); the connection with other definitions of generalized continued fractions is less evident (cf. De Bruin, 1978, Van der Cruyssen, 1979). In this section, we adopt the same presentation as Wall (1948, Chapter 1).

Formally, a *matrix continued fraction* (shortly, an M-*fraction*) is an infinite expression of the form

$$B_0 + A_1(B_1 + A_2(B_2 + A_3(\cdots)^{-1})^{-1})^{-1}, \tag{18}$$

where the $m \times m$ matrices A_p and $B_p (p \in \mathbb{N})$, called *elements*, are real matrices: A_p is called the pth *partial numerator*, and B_p the pth *partial denominator.* The finite expression

$$F^{(n)} = B_0 + A_1(B_1 + A_2(\cdots(B_{n-1} + A_nB_n^{-1})^{-1}\cdots)^{-1})^{-1} \tag{19}$$

is called (whether it has a well-defined value or not) the nth *approximant.* The 0-th approximant is taken as B_0.

As in the univariate case, the nth approximant can also be seen as the value, in $W = 0$ (the $m \times m$ null matrix), of $B_0 + t_1 \circ t_2 \circ \cdots \circ t_n(W)$, with t_p a linear transformation

$$t_p(W) = A_p(B_p + W)^{-1}, \quad p = 1, 2, \cdots.$$

By mathematical induction, it is easy to show that

$$t_1 \circ t_2 \circ \cdots \circ t_n(W)$$
$$= A_1(G'(n-1, 1)W + G'(n, 1))(G'(n-1, 0)W + G'(n, 0))^{-1},$$

where $G(t, s)$ is the Green's matrix associated with the second-order equation

$$X_t - B'_t X_{t-1} - A'_t X_{t-2} = 0, \quad t \geq 1. \tag{20}$$

Hence, $F^{(n)}$ can be written as $B_0 + A_1 G'(n, 1)(G'(n, 0))^{-1}$.

Up to this point, we made only formal descriptions, disregarding existence, nonsingularity and convergence problems; we are now able to make the following definition.

DEFINITION. The M-fraction (18) is said to *converge* if at most a finite number of its associated Green's matrices $G(n, 0)$ are singular, and if the limit of its sequence of approximants

$$F = \lim_{n=\infty} F^{(n)} = B_0 + A_1 \lim_{n=\infty} G'(n, 1)(G'(n, 0))^{-1} \tag{21}$$

exists (componentwise) and is finite; the *value* of a convergent M-fraction is defined to be the limit (21).

4b. *Positive definite* M-*fractions*. Of particular interest, in our model-building context, are the M-fractions with elements of the form

$$A_p = -C'^{-1}_{p-2} C_{p-1}, \quad B_p = C'^{-1}_{p-1} S_p, \quad (p = 1, 2, \cdots),$$

with $C_{-1} = 1$ and C_p a full-rank $m \times m$ real matrix such that the infinite symmetric block-tridiagonal matrix

$$C = \begin{pmatrix} S_0 & C_0 & 0 & 0 & \cdots \\ C'_0 & S_1 & C_1 & 0 & \cdots \\ 0 & C'_1 & S_2 & C_2 & \cdots \\ 0 & 0 & C'_2 & S_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{22}$$

is positive definite (p.d.—Recall that, if we denote by $M_{(n)}$ the finite segment of dimension n of a (semi-) infinite matrix M (i.e. the $n \times n$ square block in the upper left corner of M), M is positive definite (*positive semidefinite*) iff all its finite principal minors are p.d. (*positive semidefinite*) iff $M_{(1)}, M_{(2)}, M_{(3)}, \cdots$ are p.d. (*positive semidefinite*)).

Such an M-fraction will be called a *positive definite* M-*fraction*. It can be written more conveniently

$$S_0 - C_0(S_1 - C_1(S_2 - C_2(\cdots)^{-1} C'_2)^{-1} C'_1)^{-1} C'_1)^{-1} C'_0 \tag{23}$$

in the sense that its approximants $F^{(n)}$ take the form

$$F^{(n)} = S_0 - C_0(S_1 - C_1(S_2 - \cdots C_{n-1} S_n^{-1} C'_{n-1} \cdots)^{-1} C'_1)^{-1} C'_0.$$

The coefficients of the difference equation (20) characterizing the approximants of (18) are expressed in terms of the elements A_p and B_p of (18); the corresponding equation for (23) is

$$X_t - S_t C_{t-1}^{-1} X_{t-1} + C'_{t-1} C_{t-2}^{-1} X_{t-2} = 0, \quad t \geq 0,$$

with Green's matrices $G(t, s)$. It is easy to see that the Green's matrices $H(t, s)$ of

$$C_t X_t + S_t X_{t-1} + C'_{t-1} X_{t-2} = 0, \quad t \geq 0$$

are such that

$$H(t, 0) = (-1)^t C_t^{-1} G(t, 0) \quad \text{and} \quad H(t, 1) = (-1)^{t-1} C_t^{-1} G(t, 1);$$

hence $G'(t, 1)(G'(t, 0))^{-1} = -H'(t, 1)(H'(t, 0))^{-1}$, and the approximants $F^{(n)}$ of (23) take the form $F^{(n)} = S_0 + C_0 H'(t, 1)(H'(t, 0))^{-1}$.

Clearly, expressions (16) and (17) involve p.d. M-fractions associated with the autocovariance difference equation $\Gamma_t(L) X_t = 0$.

4c. *Some properties of (finite) positive definite matrices.* Before studying the convergence of (23), we first state without proof a few lemmas on (finite) p.d. matrices. All the matrices in this section are $m \times m$.

LEMMA 1. *Let A, B, C be such that A–B and B–C are* p.d.: *then A–C is also* p.d.

LEMMA 2.

(i) *Let $M_0, M_1, M_2, \cdots$ denote a sequence of* p.d. *matrices such that $M_n - M_{n+1}$ is itself* p.d. *for $n \in \mathbb{N}$. Then $M = \lim_{n=\infty} M_n$ exists, and is positive semidefinite.*

(ii) *However, M is strictly* p.d. iff *there exists some* p.d. *matrix D such that $M_n - D$ is* p.d. *for all $n \in \mathbb{N}$, or, equivalently,* iff *the determinants $|M_n|$ are uniformly bounded from below by some strictly positive number d: $|M_n| \geq d > 0$, $n \in \mathbb{N}$.*

(iii) *In any case, $M_n - M$ is* p.d., *$n \in \mathbb{N}$.*

LEMMA 3. *Let A and B be* p.d. *such that A–B is itself* p.d.: *then $B^{-1} - A^{-1}$ (as well, of course, as B^{-1} and A^{-1}) is still* p.d.

4d. *Convergence of positive definite* M*-fractions.* We are now able to establish the main theorem on the convergence of p.d. M-fractions.

THEOREM 3. *Consider the positive definite* M*-fraction* (23):

(i) *all its approximants $F^{(n)}$ are* p.d.;
(ii) $F^{(n-1)} - F^{(n)}$ *is* p.d., $n \in \mathbb{N}$;
(iii) *it converges, and its value is a positive semidefinite matrix F;*
(iv) *moreover, if we consider the* p.d. M*-fraction*

$$S_1 - C_1(S_2 - C_2 (\cdots)^{-1} C'_2)^{-1} C'_1, \tag{24}$$

it converges to a (strictly) p.d. *value $F_1 = C'_0(S_0 - F)^{-1} C_0$; of course its approximants also satisfy* (i) *and* (ii).

The successive points of the theorem give an indication for the proof. As for point (i), it follows from the following lemma (as in Section 4b, denote by $M_{(n)}$ the finite segment of dimension n of a (finite or infinite) matrix M):

LEMMA 4. $(F^{(n)})^{-1} = (C_{(n+1)}^{-1})_{(m)}$, $m < n \in \mathbb{N}$.

PROOF OF LEMMA 4. In order to prove that $F^{(n)}$ admits an inverse $(F^{(n)})^{-1}$ which is the $m \times m$ block in the left upper corner of $(C_{(n+1)})^{-1}$, we shall establish that the solution of the system (the unknowns $X_1, X_2, \cdots, X_{n+1}$ being $m \times m$ blocks)

$$C_{(n+1)} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e.

$$\begin{cases} S_0 X_1 + C_0 X_2 & = 1 \\ C'_0 X_1 + S_1 X_2 + C_1 X_3 & = 0 \\ \qquad C'_1 X_2 + S_2 X_3 + C_2 X_4 & = 0 \\ \qquad\qquad \ddots & \vdots \\ \qquad\qquad C'_{n-1} X_n + S_n X_{n+1} & = 0 \end{cases}$$

is such that $F^{(n)} = X_1^{-1}$. By successive substitution, we obtain

$$\begin{aligned} X_{n+1} &= -S_n^{-1} C'_{n-1} X_n, \\ X_n &= -(S_{n-1} - C_{n-1} S_n^{-1} C'_{n-1})^{-1}, \\ \vdots \quad & \quad \vdots \\ X_2 &= -(S_1 - C_1(S_2 - \cdots C_{n-1} S_n^{-1} C'_{n-1} \cdots)^{-1} C'_1)^{-1} C'_0, \end{aligned}$$

and, replacing in the first equation,

$$X_1 = (S_0 - C_0(S_1 - \cdots C_{n-1} S_n^{-1} C'_{n-1} \cdots)^{-1} C'_0)^{-1},$$

which is precisely $(F^{(n)})^{-1}$. Now X_1 being nothing else than $(C_{(n+1)}^{-1})_{(m)}$ and C being p.d., so is $C_{(n+1)}$, hence also X_1^{-1}. □

PROOF OF THEOREM 3. (i) From Lemma 4, $F^{(n)} = X_1^{-1}$, which is p.d.

(ii) We assumed, in the definition of a p.d. M-fraction, the blocks C_i to be nonsingular; hence

$$\begin{aligned} F^{(n-1)} - F^{(n)} = {} & C_0(S_1 - \cdots C_{n-2}(S_{n-1} - C_{n-1} S_n^{-1} C'_{n-1})^{-1} C'_{n-2} \cdots)^{-1} C'_0 \\ & - C_0(S_1 - \cdots C_{n-2} S_{n-1}^{-1} C'_{n-2} \cdots)^{-1} C'_0 \end{aligned}$$

is p.d. iff

$$(S_1 - \cdots C_{n-1} S_n^{-1} C'_{n-1} \cdots)^{-1} - (S_1 - \cdots C_{n-2} S_{n-1}^{-1} C'_{n-2} \cdots)^{-1}$$

is p.d., which, according to Lemma 3, holds iff

$$C_1(S_2 - \cdots C_{n-1}S_n^{-1}C'_{n-1} \cdots)^{-1}C'_1 - C_1(S_2 - \cdots C_{n-2}S_{n-1}^{-1}C'_{n-2})^{-1}C'_1$$

is itself p.d. Using the same arguments repeatedly, $F^{(n-1)} - F^{(n)}$ is p.d. iff $C_{n-1}S_n^{-1}C'_{n-1}$ is p.d., which completes this part of the proof.

(iii) From (i) and (ii), we know that the sequence of approximants $F^{(n)}$ satisfies the assumptions of Lemma 2(i); hence the M-fraction converges to a positive semidefinite value F.

(iv) Finally, denote by $F_1^{(n)}$ the approximants of (24); they are p.d. and satisfy the assumptions of Lemma 2(i)—however, they also fulfill the additional requirement that $F_1^{(n)} - C'_0S_0^{-1}C_0$ is p.d. for any n. Indeed, $F_1^{(n)} = C'_0(S_0 - F^{(n+1)})^{-1}C_0$; hence $F_1^{(n)} - C'_0S_0^{-1}C_0 = C'_0((S_0 - F^{(n+1)})^{-1} - S_0^{-1})C_0$, which, according to Lemma 3 again (C_0 is nonsingular), is p.d., $F^{(n+1)}$ being itself p.d. (cf. (i)). It thus follows from Lemma 2(ii) that F_1 is (strictly) p.d. □

4e. *Positive definite* M*-fractions and infinite positive definite band matrices.* In this section, we consider infinite band matrices of the form

$$(25)\qquad C = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & C'_2 & S_1 & C_1 & & \\ & & C'_1 & S_0 & C_0 & \\ & & & C'_0 & S_{-1} & C_{-1} \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where the C_i's are nonsingular $m \times m$ blocks. We denote by C_t^+ and C_t^-, respectively, the submatrices

$$(26)\qquad C_t^+ = \begin{bmatrix} S_t & C'_{t+1} & & & \\ C_{t+1} & S_{t+1} & C'_{t+2} & & \\ & C_{t+2} & S_{t+2} & C'_{t+3} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad C_t^- = \begin{bmatrix} S_t & C_t & & & \\ C'_t & S_{t-1} & C_{t-1} & & \\ & C'_{t-1} & S_{t-2} & C_{t-2} & \\ & & \ddots & \ddots & \ddots \end{bmatrix};$$

as in the preceding sections, $(C_t^+)_{(n)}$ and $(C_t^-)_{(n)}$ stand for the finite segments of C_t^+ and C_t^-. Let also $(F_t^+)^{(n)}$ and $(F_t^-)^{(n)}$ denote the approximants of the M-fractions

$$(27)\qquad S_t - C'_{t+1}(S_{t+1} - C'_{t+2}(\cdots)^{-1}C_{t+2})^{-1}C_{t+1}$$

and

$$(28)\qquad S_t - C_t(S_{t-1} - C_{t-1}(\cdots)^{-1}C'_{t-1})^{-1}C'_t$$

associated with C_t^+ and C_t^-; whenever they are convergent, their values will be denoted by F_t^+ and F_t^-.

The following theorem gives a necessary and sufficient condition for C to be a p.d. matrix.

THEOREM 4. *C is* p.d. iff, *for some $t \in \mathbb{Z}$, the following conditions hold*:

(i) *C_t^+ and C_t^- are* p.d. (*hence*, (27) *and* (28) *are* p.d. M-*fractions*)
(ii) *$F_t^+ + F_t^- - S_t$ is positive semidefinite.*

If (i) *and* (ii) *hold for some $t \in \mathbb{Z}$, they hold for any $t \in \mathbb{Z}$.*

PROOF. If C is p.d., (i) trivially holds; let us show that $(F_t^+)^{(k)} + (F_t^-)^{(l)} - S_t$ is p.d. $\forall\, t \in \mathbb{Z}$, $\forall\, k, l \in \mathbb{N}$. We have

$$\begin{aligned}(F_t^+)^{(k)} + (F_t^-)^{(l)} - S_t &\\ &= S_t - C'_{t+1}(S_{t+1} - \cdots C'_{t+k}S_{t+k}^{-1}C_{t+k} \cdots)^{-1}C_{t+1} \\ &\quad - C_t(S_{t-1} - \cdots C_{t-l+1}S_{t-l}^{-1}C'_{t-l+1} \cdots)^{-1}C'_t \\ &= (F_t^+)^{(k)} - C_t((F_{t-1}^-)^{(l-1)})^{-1}C'_t,\end{aligned}$$

which (according to Lemma 3) is p.d. iff

$$(F_{t-1}^-)^{(l-1)} - C'_t((F_t^+)^{(k)})^{-1}C_t = (F_{t-1}^-)^{(l-1)} + (F_{t-1}^+)^{(k+1)} - S_{t-1}$$

is p.d. Repeating the same argument, we obtain that $(F_t^+)^{(k)} + (F_t^-)^{(l)} - S_t$ is p.d. iff $(F_{t-l}^-)^{(0)} + (F_{t-l}^+)^{(k+l)} - S_{t-l}$ is p.d.; now $(F_{t-l}^-)^{(0)} = S_{t-l}$, and $(F_{t-l}^+)^{(k+l)}$, C_{t-l}^+ being p.d., is p.d. (Theorem 3(i)). Let k and $l \to \infty$: $F_t^+ + F_t^- - S_t$, as the limit of a sequence of p.d. matrices, is positive semidefinite, $\forall\, t \in \mathbb{Z}$.

Conversely, suppose that C_0^+ and C_0^- are p.d., and that $F_0^+ + F_0^- - S_0$ is positive semidefinite; in order to prove that C is p.d., it is sufficient to establish that C_1^+, C_1^-, and $F_1^+ + F_1^- - S_1$ are also p.d. and positive semidefinite. The positive definiteness of C_1^+ is obvious; as for $F_1^+ + F_1^- - S_1$, it is positive semidefinite iff

$$\begin{aligned}C_1^{-1}(F_1^+ + F_1^- - S_1)C_1'^{-1} &\\ &= (C'_1(S_1 - C'_2(S_2 - \cdots)^{-1}C_2)^{-1}C_1)^{-1} - (S_0 - C_0(S_{-1} - \cdots)^{-1}C'_0)^{-1}\end{aligned}$$

is positive semidefinite, which, in turn, applying Lemma 3 again, is positive semidefinite iff

$$S_0 - C_0(S_{-1} - \cdots)^{-1}C'_0 - C'_1(S_1 - C'_2(S_2 - \cdots)^{-1}C_2)^{-1}C_1 = F_0^- - (S_0 - F_0^+)$$

is positive semidefinite. Finally, C_1^- is p.d. iff $|(C_1^-)_{(n)}| > 0$ $\forall\, n \in \mathbb{N}$; however classical formulas for determinants of (finite) partitioned matrices yield

$$\begin{aligned}|(C_1^-)_{(n+1)}| &= |(C_0^-)_{(n)}| \cdot \left| S_1 - (C_1 0 \cdots 0)[(C_0^-)_{(n)}]^{-1} \begin{pmatrix} C'_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \\ &= |(C_0^-)_{(n)}| \cdot |S_1 - C_1[(C_0^-)_{(n)}^{-1}]_{(m)}C'_1|.\end{aligned}$$

Now, from Lemma 4, we know that $((C_0^-)_{(n)}^{-1})_{(m)} = ((F_0^-)^{(n-1)})^{-1}$. C_1^- is thus p.d. iff $| S_1 - C_1((F_0^-)^{(n-1)})^{-1}C_1' | > 0$; but this latter determinant is nothing else than $| (F_1^-)^{(n)} |$, whose positive definiteness follows from Theorem 3(i). □

The next theorem establishes the link between p.d. matrices and the p.d. solutions of recursions such as (14) and (15).

THEOREM 5. *Let $(M_t, t \in \mathbb{Z})$ be a sequence of $m \times m$ matrices such that $M_t = C_t(S_{t-1} - M_{t-1})^{-1}C_t'$ (or, equivalently, $M_t = S_t - C_{t+1}'M_{t+1}^{-1} C_{t+1}$) where C, given in* (25), *is* p.d.

M_t is p.d., *$t \in \mathbb{Z}$,* iff *$(F_{t_0}^+ - M_{t_0})$ and $(M_{t_0} + F_{t_0}^- - S_{t_0})$ are simultaneously positive semidefinite for some $t_0 \in \mathbb{Z}$. $(F_t^+ - M_t)$ and $(M_t + F_t^- - S_t)$ are then positive semidefinite for any $t \in \mathbb{Z}$.*

Notice that, if C is p.d., there always exist sequences such as $(M_t, t \in \mathbb{Z})$, since we have then (Theorem 4(ii)) that $F_t^+ + F_t^- - S_t$ is positive semidefinite: particular cases are

$$(F_t^+, t \in \mathbb{Z}); \quad ((S_t - F_t^-), t \in \mathbb{Z});$$
$$(\lambda F_t^+ + (1 - \lambda)(S_t - F_t^-), t \in \mathbb{Z}) \quad (0 < \lambda < 1).$$

Theorem 5 can thus be interpreted as an extension of Theorem 3(i). We also have the following corollary:

COROLLARY. *Let $(M_t, t \in \mathbb{Z})$ and $(N_t, t \in \mathbb{Z})$ satisfy the assumptions of Theorem* 5: *if $M_{t_0} - N_{t_0}$ is* p.d. (*positive semidefinite*) *for some $t_0 \in \mathbb{Z}$, $M_t - N_t$ is* p.d. (*positive semidefinite*) *for any $t \in \mathbb{Z}$.*

PROOF OF THEOREM 5. By definition, $C_t(S_{t-1} - M_{t-1})^{-1}C_t' = M_t = S_t - C_{t+1}'M_{t+1}^{-1}C_{t+1}$; C_t and C_{t+1} being nonsingular, M_t is p.d., $t \in \mathbb{Z}$ iff $(S_{t-1} - M_{t-1})$ and $(C_{t+1}'^{-1} S_t C_{t+1}^{-1} - M_{t+1}^{-1})$ are p.d., $t \in \mathbb{Z}$. But $S_{t-1} = (F_{t-1}^+)^{(0)}$, and $(C_{t+1}'^{-1} S_t C_{t+1}^{-1} - M_{t+1}^{-1})$ is p.d. iff (Lemma 3) $M_{t+1} - C_{t+1} S_t^{-1} C_{t+1}' = M_{t+1} + (F_{t+1}^-)^{(1)} - S_{t+1}$ is p.d. Hence, M_t is p.d., $t \in \mathbb{Z}$ iff $((F_t^+)^{(0)} - M_t)$ and $(M_t + (F_t^-)^{(1)} - S_t)$ are p.d., $t \in \mathbb{Z}$. Reiterating this reasoning, we obtain that M_t is p.d. for $t \in \mathbb{Z}$ iff, simultaneously,

$$(F_t^+)^{(k)} - M_t \quad \text{p.d.}, \quad k \in \mathbb{N}, \quad t \in \mathbb{Z} \tag{29}$$

and

$$M_t + (F_t^-)^{(k)} - S_t \quad \text{p.d.}, \quad k \in \mathbb{N}, \quad t \in \mathbb{Z}. \tag{30}$$

When $k \to \infty$, the expressions appearing in (29) and (30) converge, according to Lemma 2(i), to positive semidefinite matrices; (29) and (30) therefore imply

$$F_t^+ - M_t \text{ and } M_t + F_t^- - S_t \text{ positive semidefinite}, \quad t \in \mathbb{Z}. \tag{31}$$

Conversely, (31), together with Lemma 1 and part (iii) of Lemma 2, imply (29) and (30). So far, we have proved that M_t is p.d., $t \in \mathbb{Z}$ iff (31) holds. It remains to verify that (31) holds for $t \in \mathbb{Z}$ iff it holds for some $t_0 \in \mathbb{Z}$, which immediately follows by applying to $F^+_{t_0} - M_{t_0}$ and $M_{t_0} + F^-_{t_0} - S_{t_0}$ the arguments leading to (29) and (30).□

PROOF OF THE COROLLARY. The corollary again follows by applying to $M_{t_0} - N_{t_0}$ the arguments leading to (29) and (30).□

5. The model-building problem.

5a. *A characterization of* MA *autocovariance functions.* In this section, we adopt for covariance matrices Γ (of the form (33)) the same notational conventions as we used for C in the preceding section: Γ^-_t, Γ^+_t, $(\Gamma_t)_{(n)}$ etc. Whenever Γ^+_t and/or Γ^-_t are p.d., (16) and (17) converge: put then

$$\Phi^-_t = \Gamma_t(\Sigma_{t-1} - \Gamma_{t-1}(\Sigma_{t-2} - \cdots)^{-1}\Gamma'_{t-1})^{-1}\Gamma'_t \tag{32}$$
$$\Phi^+_t = \Sigma_t - \Gamma'_{t+1}(\Sigma_{t+1} - \cdots)^{-1}\Gamma_{t+1}. \qquad t \in \mathbb{Z}.$$

We can state the following characteristic property of MA autocovariances.

THEOREM 6. *$(\Sigma_t, \Gamma_t; t \in \mathbb{Z})$ is an m-variate* MA(1) *autocovariance function* iff

(*i*) $|\Sigma_t| \neq 0 \neq |\Gamma_t|$, $t \in \mathbb{Z}$

and

(*ii*) *the corresponding covariance matrix, of the form*

$$\Gamma = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & \Gamma'_{t+2} & \Sigma_{t+1} & \Gamma_{t+1} & & \\ & & \Gamma'_{t+1} & \Sigma_t & \Gamma_t & \\ & & & \Gamma'_t & \Sigma_{t-1} & \Gamma_{t-1} \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{33}$$

is p.d.

Condition (ii) *can be replaced by*:

(ii)′ $\Gamma^+_{t_0}$ *and* $\Gamma^-_{t_0}$ *are* p.d. *and* $\Phi^+_{t_0} - \Phi^-_{t_0}$ *is positive semidefinite for some* $t_0 \in \mathbb{Z}$.

COROLLARY. *The set of all m-variate* MA(1) *autocorrelation functions is a convex set,* i.e. $(\lambda\Sigma'_t + (1-\lambda)\Sigma''_t, \lambda\Gamma'_t + (1-\lambda)\Gamma''_t; t \in \mathbb{Z})$ *is an* MA(1) *autocovariance function* $(0 \le \lambda \le 1)$ *if* $(\Sigma'_t, \Gamma'_t; t\ \mathbb{Z})$ *and* $(\Sigma''_t, \Gamma''_t; t \in \mathbb{Z})$ *are* (*cf. Anderson* (1976) *for a more complete but stationary result*).

PROOF. The equivalence between (ii) and (ii)′ immediately follows from Theorem 4 and the definitions of Φ^+_t and Φ^-_t. Also, under conditions (i) and (ii), Φ^+_t and Φ^-_t are (strictly) p.d.: Φ^-_t satisfies the assumptions of Theorem 3(iv), and $\Phi^-_t = \Gamma_t(F^-_{t-1})^{-1}\Gamma'_t$, with F^-_{t-1} the value of the p.d. M-fraction associated with Γ^-_{t-1}, and therefore satisfying the same assumptions.

Now, suppose that (i) and (ii) hold; as we noticed at the end of Theorem 5, there always exists at least one sequence $(M_t;\ t \in \mathbb{Z})$ of p.d. matrices satisfying recursions (14) and (15). Consider an m-tuple $(\psi_t^1 \cdots \psi_t^m)$ of linearly independent solutions of (11) such that

$$\Phi_{t_0} = -\Gamma_{t_0}(\psi_{t_0-2}^1 \quad \cdots \quad \psi_{t_0-2}^m)(\psi_{t_0-1}^1 \cdots \psi_{t_0-1}^m)^{-1} \text{ be equal to } M_{t_0}$$

(take, for example, $(\psi_{t_0-1}^1 \cdots \psi_{t_0-1}^m) = 1$ and $(\psi_{t_0-2}^1 \cdots \psi_{t_0-2}^m) = -\Gamma_{t_0}^{-1} M_{t_0}$ as "initial" values). Since Φ_t and M_t are solutions of the same recursions, we have $\Phi_t = M_t$ for any $t \in \mathbb{Z}$, and Φ_t p.d. Theorem 2 thus applies, which completes the "if" part of the proof. The "only if" part is immediate, as well as the proof of the Corollary. □

In his 1971 book, T. W. Anderson (pages 224–225) implicitly gives an important theorem, which can be stated as "*a second-order q-dependent* (*univariate*) *stationary process always admits* (*in a quadratic mean sense*) *an MA representation*". This result has been extended for seasonal moving averages by O. D. Anderson (1978), and is certainly the most convincing argument for using MA models. We give it here in a general p-variate (by a p-variate process, we mean a nondegenerate one, whose covariance matrix Σ_t is never singular) nonstationary context.

THEOREM 7. *A p-variate process* $\{z_t;\ t \in \mathbb{Z}\}$ *with mean zero is an* MA(q) *process* iff *it is second-order q-dependent*, i.e. iff *its autocovariances* $\Gamma_{uv} = E(z_u z'_v)$ *exist and are such that* $|\Gamma_{u,u+q}| \neq 0$ *and* $\Gamma_{u,u+q+n+1} = 0$ *for* $u \in \mathbb{Z}$, $n \in \mathbb{N}$.

PROOF. Let $\{z_t;\ t \in \mathbb{Z}\}$ be q-dependent. Its block-band covariance matrix Γ ("block-bandwidth" of $(2q+1)$ blocks) can also be seen to be a block-tridiagonal matrix of the form (33), with block-triangular blocks Γ_t (after renumbering) of dimension $m = pq$

$$\Gamma_t = \begin{bmatrix} \Gamma_{u,u-q} & & & & \\ \Gamma_{u-1,u-q} & \Gamma_{u-1,u-q-1} & & & 0 \\ \vdots & & \ddots & & \\ \Gamma_{u-q+1,u-q} & \Gamma_{u-q+1,u-q-1} & \Gamma_{u-q-1,u-q-2} & \cdots & \Gamma_{u-q-1,u-2q-1} \end{bmatrix};$$

these $m \times m$ blocks Γ_t are nonsingular, since the $p \times p$ "subblocks" $\Gamma_{u,u-q}$ are nonsingular, and Γ is p.d. Hence Theorem 6 provides an equivalence class of pq-variate MA(1) models for the $pq \times 1$ vector $\mathbf{z}_t = (z'_u z'_{u-1} \cdots z'_{u-q+1})'$. Within this class there exists at least one operator $\mathbf{A}_{t0} + \mathbf{A}_{t1} L$ with nonsingular and lower triangular $\mathbf{A}_{t1}$; the corresponding $\mathbf{A}_{t0}$ is then $\Gamma'_{t+1}(\mathbf{A}_{t+1,1}^{-1})'$ and is consequently upper block-triangular. $\mathbf{A}_{t0}$ and $\mathbf{A}_{t1}$ are thus of the upper and lower block-triangular forms required in (2′). □

5b. *The main result.* The solution of the model-building problem now immediately follows from Theorems 2 and 6.

THEOREM 8. *Let* $(\Sigma_t, \Gamma_t;\ t \in \mathbb{Z})$ *be an m-variate* MA(1) *autocovariance function,* Φ_t^- *and* Φ_t^+ *the* p.d. *matrices defined in* (32).

(i) *If* $\Phi_{t_0}^- = \Phi_{t_0}^+$ *for some* $t_0 \in \mathbb{Z}$, *then* $\Phi_t^- = \Phi_t^+$ *for any* $t \in \mathbb{Z}$, *and there exists one (and only one) class of equivalent models*

$$(34)\qquad \begin{cases} z_t = \Gamma'_{t+1}(A_{t+1,1}^{-1})'O_t\varepsilon_t + A_{t1}O_{t-1}\varepsilon_{t-1}, & t \in \mathbb{Z} \\ \text{with} \quad A_{t1} \quad \text{such that} \quad A_{t1}A'_{t1} = \Phi_t, & t \in \mathbb{Z} \\ (O_t;\ t \in \mathbb{Z}) \quad \text{an arbitrary sequence of orthogonal matrices} \end{cases}$$

admitting (Σ_t, Γ_t) *as its autocovariance function.*

(ii) *If* $\Phi_{t_0}^+ - \Phi_{t_0}^-$ *is (positive semidefinite) of rank* l $(0 < l \le m)$, *then* $\Phi_t^+ - \Phi_t^-$ *is (positive semidefinite) of rank* l *for any* $t \in \mathbb{Z}$, *and there exists a* $(l(l+1)/2)$-*dimensional set of classes of equivalent models admitting* (Σ_t, Γ_t) *as their autocovariance function. Any* p.d. *matrix* Φ_{t_0} *such that* $\Phi_{t_0}^+ - \Phi_{t_0}$ *and* $\Phi_{t_0} - \Phi_{t_0}^-$ *are simultaneously positive semidefinite defines such a class, according to* (34), Φ_t *being obtained from the "initial" value* Φ_{t_0} *by means of the recursions* (14) *and* (15).

PROOF. The proof is contained in Theorems 2 and 6. $l(l+1)/2$ is the dimension of the convex cone of the real positive semidefinite $m \times m$ matrices of rank l; notice that $\{\Phi_{t_0} \mid \Phi_{t_0}$ p.d., $\Phi_{t_0}^+ - \Phi_{t_0}$ and $\Phi_{t_0} - \Phi_{t_0}^-$ positive semidefinite$\}$ is a bounded closed subset of the $l(l+1)/2$-dimensional real space.

6. The univariate MA(1) and MA(2) cases. The univariate MA(1) case has been treated in (Hallin, 1981), in an autocorrelation function setting. The MA(1) autocovariance operator reduces to $\Gamma_t(L) = \gamma_{t+1,1} + \gamma_{t0}L + \gamma_{t1}L^2$, with $\gamma_{ti} = E(z_t z_{t-i})$, and the M-fractions Φ_t^- and Φ_t^+ are then the classical continued fractions

$$\varphi_t^- = \frac{\gamma_{t1}^2 \mid}{\mid \gamma_{t-1,0}} - \frac{\gamma_{t-1,1}^2 \mid}{\mid \gamma_{t-2,0}} - \frac{\gamma_{t-2,1}^2 \mid}{\mid \gamma_{t-3,0}} - \cdots;$$

$$\varphi_t^+ = \gamma_{t0} - \frac{\gamma_{t+1,1}^2 \mid}{\mid \gamma_{t+1,0}} - \frac{\gamma_{t+2,1}^2 \mid}{\mid \gamma_{t+2,0}} - \frac{\gamma_{t+3,1}^2 \mid}{\mid \gamma_{t+3,0}} - \cdots.$$

According to Theorem 6, we have $\varphi_t^+ \ge \varphi_t^- > 0$, $t \in \mathbb{Z}$. For simplicity sake, let $t_0 = 0$; applying Theorem 8, each point $\varphi_0 \in [\varphi_0^-, \varphi_0^+]$ defines a set of equivalent solutions to the model-building problem, $(a_{t0} + a_{t1}L)$, with

$$a_{t1}^2 = -\gamma_{t1}\psi_{t-2}/\psi_{t-1}, \qquad \text{sign}(a_{t1}a_{t-1,0}) = \text{sign}(\gamma_{t1})$$
$$a_{t0}^2 = -\gamma_{t+1,1}\psi_t/\psi_{t-1},$$

ψ_t being the solution of the homogeneous equation of order two $\Gamma_t(L)\psi_t = 0$ taking on "initial" values $\psi_{-1} = 1$ and $\psi_{-2} = -\varphi_0/\gamma_{01}$.

The univariate MA(2) case is detailed in Hallin (1982a). Letting again $\gamma_{ti} =$

$E(z_t z_{t-i})$, we have

$$\Phi_t^+ = \Sigma_t - \lim_{k=\infty} \Gamma'_{t+2}$$
$$\cdot (\Sigma_{t+2} - \Gamma'_{t+4}(\Sigma_{t+4} - \cdots \Gamma'_{t+2k}\Sigma_{t+2k}^{-1}\Gamma_{t+2k} \cdots)^{-1}\Gamma_{t+4})^{-1}\Gamma_{t+2}$$
$$\Phi_t^- = \lim_{k=\infty} \Gamma_t$$
$$\cdot (\Sigma_{t-2} - \Gamma_{t-2}(\Sigma_{t-4} - \Gamma_{t-4}(\cdots \Gamma_{t-2k}\Sigma_{t-2(k+1)}\Gamma'_{t-2k} \cdots)^{-1}\Gamma'_{t-4})^{-1}\Gamma'_{t-2})^{-1}\Gamma'_t$$

with

$$\Sigma_t = \begin{pmatrix} \gamma_{t0} & \gamma_{t1} \\ \gamma_{t1} & \gamma_{t-1,0} \end{pmatrix}, \quad \Gamma_t = \begin{pmatrix} \gamma_{t2} & 0 \\ \gamma_{t-1,1} & \gamma_{t-1,2} \end{pmatrix}.$$

Still assuming $t_0 = 0$, the operator $A_t(L) = a_{t0} + a_{t1}L + a_{t2}L^2$ provides an adequate MA(2) model for a process z_t with autocovariances γ_{ti} iff

$$\begin{cases} a_{t+2,2} = \gamma_{t+2,2}/(\gamma_{t0} - a_{t1}^2 - a_{t2}^2)^{1/2} \\ a_{t+1,1} = (\gamma_{t+1,1} - a_{t+1,2}a_{t1})/(\gamma_{t0} - a_{t1}^2 - a_{t2}^2)^{1/2}, \end{cases} \quad t \in \mathbb{Z}$$

or, equivalently,

$$\begin{cases} a_{t-1,1} = (\gamma_{t1} - a_{t1}\gamma_{t2}/a_{t+1,2})/a_{t2} \\ a_{t-1,2}^2 = \gamma_{t-1,0} - a_{t-1,1}^2 - (\gamma_{t2}/a_{t+1,2})^2 \end{cases} \quad t \in \mathbb{Z}$$

$$\operatorname{sign}(a_{t-1,2}) = \operatorname{sign}(\gamma_{t-1,2})$$

with "initial" values belonging to the nonempty set

$$\left\{ a_{01}, a_{02}, a_{12} \,\middle|\, \Phi_0 = \begin{pmatrix} a_{12}^2 & a_{01}a_{12} \\ a_{01}a_{12} & a_{01}^2 + a_{02}^2 \end{pmatrix} \text{p.d.}, \right.$$
$$\left. \Phi_0^+ - \Phi_0 \text{ and } \Phi_0 - \Phi_0^- \text{ positive semidefinite} \right\}.$$

An alternative expression of these models is also given, in Hallin (1982a), in terms of solutions of the homogeneous (scalar) difference equation of order four

$$\gamma_{t+2,2}\psi_t + \gamma_{t+1,1}\psi_{t-1} + \gamma_{t0}\psi_{t-2} + \gamma_{t1}\psi_{t-3} + \gamma_{t2}\psi_{t-4} = 0, \quad t \in \mathbb{Z}.$$

7. Concluding remarks. The three theorems in Section 5 (Theorems 6, 7 and 8) provide a characterization of nonstationary moving average processes in terms of their (time-varying) autocovariances and an explicit solution to the corresponding model-building (or spectral factorization) problem.

These theoretical results have immediate applications in *system theory*, where the spectral factorization problem has been extensively studied (cf. below) and in related areas of the mathematical theory of control and optimization. Our main concern, however, and leading motivation in deriving those results, is the *statistical analysis of nonstationary time series.*

Though we cannot exhibit at present a completed statistical application, we are convinced that Theorem 8 provides the appropriate theoretical basis for the analysis of moving average processes with simple time-varying covariance structures. Simple time-varying covariances does not mean necessarily slowly evolving ones. The structure of the likelihood function for MA processes with periodically varying autocovariances, for example, is not very different from the corresponding stationary one; it should therefore be quite possible to adapt the currently existing estimation methods to this new situation. It follows indeed from Theorem 7 that this class of periodic MA processes is precisely the class of (second-order) q-dependent periodic processes, which of course covers a very broad and attractive range of phenomena, and should be most useful in the analysis of seasonal time series.

The estimation problem should also be tractable for processes generated by MA models whose coefficients themselves obey some linear homogeneous equation, and the case of models whose coefficients are known analytical functions of time has already been considered, from a practical point of view, by Mélard (1982) and Kiehm and Mélard (1981).

Invertibility properties. It is well known that, for predictability as well as for causal interpretation purposes, a moving average should be *invertible* in some sense: "full" (linear) invertibility or, at least, the weaker asymptotic invertibility property of Granger and Andersen (1978) is thus required.

The models obtained in Theorem 8 for a given autocovariance function should therefore be classified according to their invertibility properties. In the univariate MA(1) case, we established (Hallin, 1981b) that, for a given time-varying autocovariance function (and in the notation of Section 6),

(i) the only invertible model—if any—is obtained by choosing $\varphi_0 = \varphi_0^-$: denote by a_{tj}^- its coefficients.
(ii) the only Granger-Andersen noninvertible model—if any—is obtained by choosing $\varphi_0 = \varphi_0^+$: denote by a_{tj}^+ its coefficients.
(iii) for any model corresponding to an intermediate value $\varphi_0^- < \varphi_0 < \varphi_0^+$ (denote by a_{tj} its coefficients), the following hold:

$$\lim_{t=-\infty}(a_{tj} - a_{tj}^+) = \lim_{t=+\infty}(a_{tj} - a_{tj}^-) = 0.$$

We are presently investigating these invertibility properties in the general multivariate MA(q) case, in connection with Theorem 8. Up to now, we have been able to show that (i) still holds (the proof relies on asymptotic dominance properties of certain subspaces of solutions of homogeneous difference equations); but (ii) and (iii) obviously need to be modified.

Related results. As mentioned above, the spectral factorization problem has a long history in the engineering literature on time-varying systems (see B. D. O. Anderson and P. J. Moylan (1974) or Halyo and McAlpine (1974) for more extensive references). Anderson and Moylan (1974), more particularly, seem to address the same problem that we do. Moreover, they are working in a more general framework: continuous time, general ARMA models, singular as well as nonsingular autocovariances. However, they do not give their solution in an explicit form, and they restrict their attention to *finite* time intervals $t \in [0, t_1]$; this simplifies the problem of choosing an initial value Φ_0 for Φ_t — indeed, if the process starts in $t = 0$, $\Phi_0 = 0$ is always an adequate choice, which automatically guarantees positive definiteness of Φ_t and positive semidefiniteness of $\Phi_t^+ - \Phi_t$. Furthermore, because they do not have a necessary and sufficient condition equivalent to our Theorem 6 for an autocovariance function to be an ARMA one, they have to make an *extendability* assumption which cannot be expressed in terms of the autocovariances.

Our results, together with the invertibility properties of the model associated with Φ_t, make possible a causal interpretation for any variation in a given covariance function.

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