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SYSTEMS WITH EXPONENTIAL LIFE AND IFRA COMPONENT LIVES[1]

BY HENRY W. BLOCK AND THOMAS H. SAVITS

University of Pittsburgh

Systems with exponential life formed from independent IFRA components are studied. It is shown that (a) in the case of monotonic systems, the system must be essentially a series system of exponential components and (b) in the case of systems whose life is the sum of component lives, all but one of the components are degenerate at zero while the remaining one is exponential.

1. Introduction. Many kinds of systems can be formed using independent components. It turns out, however, that only in very special cases can the system lifetime be exponential if the components have increasing failure rate average (IFRA). In particular, we show (Theorem 2.1) that if a monotonic system formed with independent IFRA components has exponential life, then it must be essentially a series system with exponential components. Results similar to this have been mentioned in Esary, Marshall and Proschan (1970) and Esary and Marshall (1975), but these require some assumptions on the support of the component lifetime distributions. We need no such assumptions in our proof.

In the case of systems whose life is the sum of its component lives we obtain a similar result. Theorem 2.8 states that if the sum of independent IFRA lifetimes is exponential, then one must be exponential and all others are degenerate at zero. Some related results about parallel systems of two dependent components whose lifetime is exponential are also given in Theorems 2.4–2.6.

In Section three we use these results to give counterexamples which show that the various definitions of multivariate IFRA distributions are not equivalent.

All definitions and notation follow that of Barlow and Proschan (1975).

2. Results.

(2.1). THEOREM. *Let F be the life distribution of a monotonic system of order n formed with independent* IFRA *components having life distributions $F_1(t), \cdots, F_n(t)$ respectively and assume F is exponential. Then there exists a subcollection $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that for all $t \geqslant 0$*

$$\bar{F}(t) = \bar{F}_{i_1}(t) \cdots \bar{F}_{i_k}(t)$$

where each $F_{i_j}(t)$ is exponential and $\bar{F} = 1 - F$.

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Before proving the theorem we introduce a lemma which will be used in the proof of the theorem.

(2.2). LEMMA. *Let $h(\mathbf{p})$ be the reliability function of a monotonic system of n independent components and p_i the probability that the* ith *component is functioning. If there is $0 < \alpha < 1$ such that*

$$p_i^\alpha h^\alpha(1_i, \mathbf{p}) + (1 - p_i^\alpha) h^\alpha(0_i, \mathbf{p}) = h^\alpha(\mathbf{p}),$$

then at least one of the following three conditions must hold:

(1) $p_i = 0$ or $p_i = 1$;

(2) $h(1_i, \mathbf{p}) = h(0_i, \mathbf{p})$;

(3) $h(0_i, \mathbf{p}) = 0$.

PROOF. This follows from the proof of Lemma 2.3, page 84, of Barlow and Proschan (1975) since $f(x) = x^\alpha$ is strictly concave in $x \geq 0$.

PROOF OF THEOREM. Let $\bar{F}(t) = e^{-\lambda t} = h(\bar{F}_1(t), \cdots, \bar{F}_n(t))$ where h is the reliability function of a monotonic system ϕ of order n, and $\bar{F}_i(t)$ are IFRA for $i = 1, \cdots, n$. Now ϕ has the representation $\phi(\mathbf{x}) = \min_{1 \leq j \leq p} \max_{i \in K_j} x_i$, where x_i is the state of the ith relevant component and K_j is a min cut set. So if $T_1, \cdots, T_n$ are the lifetimes, then

$$e^{-\lambda t} = P\left\{\min_{j=1,\cdots,p} \max_{i \in K_j} T_i > t\right\}.$$

We shall now show that there is an i such that $0 < \bar{F}_i(t) < 1$ for all $t > 0$. First, $e^{-\lambda t} > 0$ for all $t > 0$ implies that $P\{\max_{i \in K_j} T_i > t\} > 0$ for each $j = 1, \cdots, p$. Consequently, for each $j = 1, \cdots, p$, there is some $i_j \in K_j$ such that $P\{T_{i_j} > t\} > 0$ for all $t > 0$. Now $1 > e^{-\lambda t}$ for all $t > 0$ implies that there is a j_0 such that

$$1 > P\left\{\max_{i \in K_{j_0}} T_i > t\right\} = 1 - \Pi_{i \in K_{j_0}} P\{T_i \leq t\}$$

and so for all $t > 0$, $P\{T_i > t\} < 1$ for each $i \in K_{j_0}$. Thus for $i_{j_0} \in K_{j_0}$, $0 < \bar{F}_{i_{j_0}}(t) < 1$ for all $t > 0$.

We now use an induction argument on the order of the monotonic system. Clearly the result is true for $n = 1$. Assume it is true for any monotonic system of order less than n and consider a monotonic system of order n. By the pivotal decomposition using the i above

$$e^{-\lambda t} = \bar{F}_i(t)\left[h\left(1_i, \bar{\mathbf{F}}(t)\right) - h\left(0_i, \bar{\mathbf{F}}(t)\right)\right] + h\left(0_i, \bar{\mathbf{F}}(t)\right),$$

where $h((\cdot)_i, \bar{\mathbf{F}}(t)) = h(\bar{F}_1(t), \cdots, (\cdot)_i, \cdots, \bar{F}_n(t))$. Now, by the lemma, since $0 < \bar{F}_i(t) < 1$ for all $t > 0$, then either $h(1_i, \bar{\mathbf{F}}(t)) = h(0_i, \bar{\mathbf{F}}(t))$ or $h(0_i, \bar{\mathbf{F}}(t)) = 0$. Let $\gamma = \inf\{t | h(0_i, \bar{\mathbf{F}}(t)) = 0\}$. Assume $\gamma > 0$. Then for all $0 < t < \gamma$, $h(0_i, \bar{\mathbf{F}}(t)) > 0$ so that $h(1_i, \bar{\mathbf{F}}(t)) = h(0_i, \bar{\mathbf{F}}(t))$ and for $t \geq \gamma$, $h(0_i, \bar{\mathbf{F}}(t)) = 0$. Thus

$$\begin{aligned} e^{-\lambda t} &= h\left(0_i, \bar{\mathbf{F}}(t)\right) = h\left(1_i, \bar{\mathbf{F}}(t)\right) \quad \text{for} \quad 0 < t < \gamma \\ &= \bar{F}_i(t) h\left(1_i, \bar{\mathbf{F}}(t)\right) \quad \text{for} \quad \gamma \leq t. \end{aligned}$$

Now take $t_0 > \gamma$ and $0 < \alpha_0 < 1$ such that $\alpha_0 t_0 < \gamma$. It follows that

$$\exp(-\lambda\alpha_0 t_0) = h\big(1_i, \bar{\mathbf{F}}(\alpha_0 t_0)\big) \geqslant h\big(1_i, \bar{\mathbf{F}}^{\alpha_0}(t_0)\big) \geqslant h^{\alpha_0}\big(1_i, \bar{\mathbf{F}}(t_0)\big)$$
$$\geqslant \bar{F}_i^{\alpha_0}(t_0) h^{\alpha_0}\big(1_i, \bar{\mathbf{F}}(t_0)\big) = \exp(-\lambda\alpha_0 t_0).$$

This implies that $\bar{F}_i(t_0) = 1$, a contradiction. Hence $\gamma = 0$ and so $h(0_i, \bar{\mathbf{F}}(t)) = 0$ for all $t \geqslant 0$. Consequently for all $t \geqslant 0$,

$$e^{-\lambda t} = \bar{F}_i(t) h\big(1_i, \bar{\mathbf{F}}(t)\big).$$

But for any $t \geqslant 0$ and $0 < \alpha < 1$,

$$e^{-\lambda\alpha t} = \bar{F}_i(\alpha t) h\big(1_i, \bar{\mathbf{F}}(\alpha t)\big) \geqslant \bar{F}_i^{\alpha}(t) h^{\alpha}\big(1_i, \bar{\mathbf{F}}(t)\big) = e^{-\lambda\alpha t}$$

and so $\bar{F}_i(\alpha t) = \bar{F}_i^{\alpha}(t)$. This gives that $\bar{F}_i(t) = e^{-\lambda_i t}$, $t \geqslant 0$, for some $\lambda_i > 0$, and so for all $t \geqslant 0$,

$$\bar{F}(t) = e^{-\lambda t} = e^{-\lambda_i t} h\big(1_i, \bar{\mathbf{F}}(t)\big),$$

which implies that $h(1_i, \bar{\mathbf{F}}(t))$ is also exponential. But this is a monotonic system of order $(n - 1)$ and so we can use the induction hypothesis.

(2.2a). Remark. As noticed by Esary and Marshall (1975), it follows immediately from the theorem that a multivariate exponential distribution whose one dimensional marginal distributions are lifetimes of coherent systems of independent IFRA distributions must be the multivariate distribution of Marshall and Olkin (1967).

(2.3). Example. Let X and Y be independent IFRA random variables and assume $\max(X, Y)$ is exponential. Then it follows that one of X or Y is exponential and the other has all its mass at zero.

This example is not peculiar in the sense that the assumption that $\max(X, Y)$ is exponential is prohibitively strong. This is illustrated in the following where X and Y are not assumed to be independent.

(2.4). Theorem. *Let X, Y and* $\max(X, Y)$ *be exponential. Then* $\min(X, Y)$ *is exponential and $P\{X \leqslant Y\} = 1$ or $P\{X \geqslant Y\} = 1$. If X and Y are identically distributed or if the* min *and* max *are identically distributed, then $P\{X = Y\} = 1$.*

Proof. Assume X, Y, $\max(X, Y)$ have means λ_1^{-1}, λ_2^{-1}, $(\lambda'_{12})^{-1}$. Thus $\exp(-\lambda_1 t) = P\{X > t\} \leqslant P\{\max(X, Y) > t\} = \exp(-\lambda'_{12} t)$ for all $t \geqslant 0$ and so $\lambda_1 \geqslant \lambda'_{12}$. Similarly $\lambda_2 \geqslant \lambda'_{12}$. Furthermore, $0 \leqslant P\{\min(X, Y) > t\} = 1 - P\{X \leqslant t\} - P\{Y \leqslant t\} + P\{\max(X, Y) \leqslant t\} = \exp(-\lambda_1 t) + \exp(-\lambda_2 t) - \exp(-\lambda'_{12} t)$, and so $\exp(-\lambda_1 t) + \exp(-\lambda_2 t) \geqslant \exp(-\lambda'_{12} t)$, or $\exp(\lambda'_{12} - \lambda_1)t + \exp(\lambda'_{12} - \lambda_2)t \geqslant 1$. Now if $\lambda'_{12} < \lambda_1$ and $\lambda'_{12} < \lambda_2$, the above inequality fails for large t. Hence $\lambda'_{12} = \lambda_1$ or $\lambda'_{12} = \lambda_2$. In the case $\lambda'_{12} = \lambda_1$, $P\{\min(X, Y) > t\} = \exp(-\lambda_2 t)$. This also implies that $P\{X \geqslant Y\} = 1$. The case $\lambda'_{12} = \lambda_2$ is similar. For the second part of the theorem assume λ_{12}^{-1} is the mean of $\min(X, Y)$. This yields the identity $\exp(-\lambda_{12} t) + \exp(-\lambda'_{12} t) = \exp(-\lambda_1 t) + \exp(-\lambda_2 t)$ for all t. Clearly if $\lambda_1 = \lambda_2$, then $\lambda'_{12} = \lambda_1 = \lambda_2$ and the above gives that $P\{X = Y\} = 1$. The case $\lambda_{12} = \lambda'_{12}$ is similar.

(2.5). COROLLARY. *If* $\max(aX, bY)$ *is exponential for all* $a, b \geqslant 0$, *then* $\min(aX, bY)$ *is exponential for all* $a, b \geqslant 0$.

PROOF. Apply Theorem 2.4.

It turns out that an even stronger result than the one given in the corollary holds.

(2.6). THEOREM. *If* $\max(aX, bY)$ *is exponential for all* $a, b \geqslant 0$, *then there is a* $c > 0$ *such that* $P\{X = cY\} = 1$.

PROOF. From the hypothesis and Corollary 2.5 it follows that X and Y are exponential. Let λ_1^{-1} and λ_2^{-1} be their means respectively. Choose $a, b > 0$ such that $\lambda_1 b = \lambda_2 a$. Then since aX and bY are identically distributed, we know from Theorem 2.4 that $aX = bY$ with probability one.

(2.7). REMARK. Esary and Marshall (1974) have studied the class of distributions whose scaled minimums are exponential. The above theorem shows why this is fruitless for scaled maximums.

The result following is similar to Theorem 2.1 but deals with a system whose lifetime is the sum of its component lifetimes instead of dealing with a monotonic system.

(2.8). THEOREM. *If the convolution of* n IFRA *distributions is exponential, then* $(n - 1)$ *of the distributions are degenerate at zero and the other distribution is exponential.*

PROOF. It is sufficient to consider two independent IFRA random variables X and Y with survival functions $\bar{F}$ and $\bar{G}$ respectively such that for all $t \geqslant 0$, $P\{X + Y > t\} = e^{-\lambda t}$ with $\lambda > 0$. Since $0 < e^{-\lambda t} < 1$ for all $t > 0$, it easily follows that either $0 < \bar{F}(t) < 1$ for all $0 < t < \infty$ or $0 < \bar{G}(t) < 1$ for all $0 < t < \infty$. Assume the latter case holds. Suppose now that X is neither exponential nor degenerate at zero. Then there is some $0 < \alpha < 1$ and, by right-continuity, some interval $I \subset [0, \infty)$ such that $\bar{F}(\alpha x) > \bar{F}^{\alpha}(x)$ for all $x \in I$. Consequently, for sufficiently large t, since $\bar{G}$ is IFRA and $0 < \bar{G}(t) < 1$, we have that

$$e^{-\lambda\alpha t} = \int_0^\infty \bar{F}(\alpha t - y)\, dG(y) > \int_0^\infty \bar{F}^{\alpha}(t - y/\alpha)\, dG(y).$$

But

$$\int_0^\infty \bar{F}^{\alpha}(t - y/\alpha)\, dG(y) \geqslant \left\{\int_0^\infty \bar{F}(t - y)\, dG(y)\right\}^{\alpha} = e^{-\lambda\alpha t}$$

from Lemma 2.1 of Block and Savits (1976). Hence it must be that X is either exponential or degenerate at zero. Now if X is exponential, then $0 < \bar{F}(t) < 1$ for all $0 < t < \infty$ and the above argument can be repeated so that Y is either exponential or degenerate at zero. But since we have assumed $0 < \bar{G}(t) < 1$ for all $t > 0$, it follows that Y must be exponential and so, consequently, $X + Y$ is not exponential. This is a contradiction. Hence X must be degenerate at zero and therefore Y is exponential.

(2.9). REMARK. An immediate corollary is that a multivariate exponential distribution whose univariate marginal distributions are convolutions of independent IFRA distributions must have univariate marginals which are either pairwise independent or pairwise identical.

3. Applications. In this section examples are constructed to show that certain definitions of multivariate IFRA distributions are not equivalent. We compare Condition C of Esary and Marshall (1975), the definition of multivariate IFRA (MIFRA) of Block and Savits (1977) and another definition designated condition Σ. We list the definitions below.

(3.1). DEFINITION. 1. $(T_1, \cdots, T_n)$ is MIFRA if

$$E[h(T_1, \cdots, T_n)] \leqslant E^{1/\alpha}[h^\alpha(T_1/\alpha, \cdots, T_n/\alpha)]$$

for all continuous nonnegative nondecreasing functions h and all $0 < \alpha < 1$.

2. $(T_1, \cdots, T_n)$ satisfies *Condition* C if, for some independent IFRA random variables $X_1, \cdots, X_k$, and some coherent life functions $\tau_1, \cdots, \tau_n$ of order k, $T_i = \tau_i(X_i, \cdots, X_k)$ for $i = 1, \cdots, n$.

3. $(T_1, \cdots, T_n)$ satisfies *Condition* Σ if, for some independent IFRA random variables $X_1, \cdots, X_k$, and some nonempty subsets $S_i \subset \{1, \cdots, k\}$, $i = 1, \cdots, n$,

$$T_i = \Sigma_{j \in S_i} X_j.$$

(3.2). EXAMPLE. (MIFRA $\not\Rightarrow$ C). Consider $F(x, y) = \exp(-(x^2 + y^2)^{\frac{1}{2}})$ which has exponential marginals and is MIFRA. By Remark 2.2a if this distribution satisfied Condition C, it would have the Marshall and Olkin distribution, but it obviously does not.

(3.3). EXAMPLE. (MIFRA $\not\Rightarrow \Sigma$). Let (T_1, T_2) be given by $T_1 = \min(X, Z)$ and $T_2 = \min(Y, Z)$, where X, Y and Z are independent exponential random variables. Then (T_1, T_2) is MIFRA. Now if (T_1, T_2) satisfied Condition Σ, we would have, by Remark 2.9, that either T_1 and T_2 are independent or that $T_1 = T_2$. However, both of these are impossible.

It should be mentioned that both Condition C and Condition Σ imply MIFRA as is shown in Block and Savits (1977).

Added in proof. Theorem 2.1 can be improved by replacing the assumption that the component lives are IFRA with the assumption that they are NBU (new better than used). This fact was observed by the authors and also by Mark Brown and Moshe Shaked.

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Department of Mathematics and Statistics
University of Pittsburgh
Pittsburgh, Pennsylvania 15260