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# NONPARAMETRIC ESTIMATION OF LOCATION PARAMETER AFTER A PRELIMINARY TEST ON REGRESSION[1]

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For a simple regression model, the problem of estimating the intercept after a preliminary test on the regression coefficient is considered here. Some nonparametric procedures for this problem are formulated and their various asymptotic properties studied. Comparison with the conventional estimation procedures (for both the situations where the regression coefficient is treated as a nuisance parameter or not) has also been made.

**1. Introduction.** Let $Y_1, \cdots, Y_n$ be independent random variables (rv) with absolutely continuous distribution functions (df)

$$F_i(x) = P\{Y_i \leqq x\} = F(x - \theta - \beta t_i), \quad -\infty < x < \infty, \; i = 1, \cdots, n, \tag{1.1}$$

where $\mathbf{t}_n = (t_1, \cdots, t_n)$ is a vector of known constants (not all equal) and $\theta$, $\beta$ are unknown parameters; the form of $F$ may or may not by specified. We are primarily concerned with the estimation of $\theta$. If $\beta = 0$, then the $Y_i$ are identically distributed with location $\theta$, and a host of (parametric as well as nonparametric) estimators of $\theta$ is available in the literature; we designate such an estimator by $\hat{\theta}_n$. On the other hand, if $\beta$ is unknown, the estimator of $\theta$ depends on the estimator of $\beta$ and generally results in a larger mean squared error (m.s.e.); such an estimator is denoted by $\tilde{\theta}_n$. When the true $\beta$ is not specified, but is suspected to be close to 0, often a preliminary test of significance concerning $\beta$ is made: if $H_0$: $\beta = 0$ is tenable, the estimator $\hat{\theta}_n$ is used, while $\tilde{\theta}_n$ is used when $H_0$ is not tenable. Such an estimator after a preliminary test on regression is denoted by $\theta_n^*$. Usually $\theta_n^*$ is not strictly unbiased, though it has generally a smaller m.s.e. than $\tilde{\theta}_n$.

The effects of such a preliminary test of significance (viz., bias and m.s.e.) upon estimation have been studied in various special cases by Bancroft (1944), Han and Bancroft (1968) and Mosteller (1948), among others. Ahsanullah and Saleh (1972) considered the model (1.1) and studied these effects for the classical least squares estimators. The object of the current investigation is to employ robust, nonparametric estimators of $\theta$ and $\beta$ and to study the effects of preliminary tests on $\beta$ on the estimation of $\theta$.

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Along with the preliminary notions, the proposed estimators are introduced in Section 2. Section 3 deals with the asymptotic distribution theory of the estimators. Expressions for the "asymptotic bias" and the "asymptotic mean squared error" of the estimators are studied in Section 4. Asymptotic relative efficiency (a.r.e.) results are presented in the last section.

**2. The proposed estimators.** Let $\mathscr{F}$ be the class of all absolutely continuous symmetric (about 0) df's with (almost everywhere) absolutely continuous probability density functions (pdf) having finite Fisher information:

$$I(f) = \int_{-\infty}^{\infty} \{f'(x)/f(x)\}^2 \, dF(x) \quad (< \infty), \tag{2.1}$$

where $f'(x) = (d/dx)f(x) = (d^2/dx^2)F(x)$. Also, let

$$\bar{t}_n = n^{-1} \sum_{i=1}^{n} t_i \quad \text{and} \quad Q_n = \sum_{i=1}^{n} (t_i - \bar{t}_n)^2 . \tag{2.2}$$

We assume that

$$(1.1) \text{ holds with } F \in \mathscr{F}, \tag{2.3}$$

$$\lim_{n\to\infty} n^{-1}Q_n = Q^* \quad (0 < Q^* < \infty) \quad \text{and} \quad \lim_{n\to\infty} \bar{t}_n = \bar{t} \quad (|\bar{t}| < \infty) \text{ both exist,} \tag{2.4}$$

$$\text{the } t_i \text{ are all bounded } (\Rightarrow \max_{1\le i\le n} (t_i - \bar{t}_n)^2/Q_n \to 0 \text{ as } n \to \infty) . \tag{2.5}$$

Let $\phi = \{\phi(u), 0 < u < 1\}$ be a nondecreasing, skew-symmetric (i.e., $\phi(u) + \phi(1-u) = 0$, $\forall\, 0 < u < 1$) and square integrable score function, $\phi^* = \{\phi^*(u) = \phi((1+u)/2), 0 < u < 1\}$, and for every $n$ ($\geq 1$), let

$$a_n(i) = E\phi(U_{ni}) \quad \text{or} \quad \phi\left(\frac{i}{n+1}\right), \tag{2.6}$$
$$a_n^*(i) = E\phi^*(U_{ni}) \quad \text{or} \quad \phi^*\left(\frac{i}{n+1}\right), \qquad i = 1, \cdots, n,$$

where $U_{n1} < \cdots < U_{nn}$ are the ordered rv's of a sample of size $n$ from the rectangular (0, 1) df. Finally, let $\mathbf{Y}_n = (Y_1, \cdots, Y_n)$ and for every real $(a, b)$, define $\mathbf{Y}_n(a, b) = \mathbf{Y}_n - a\mathbf{1}_n - b\mathbf{t}_n$ where $\mathbf{1}_n = (1, \cdots, 1)$ and $\mathbf{t}_n$ is defined after (1.1). Consider then the statistics

$$T_n(a, b) = T(\mathbf{Y}_n(a, b)) = n^{-1} \sum_{i=1}^{n} \operatorname{sgn}(Y_i - a - bt_i)a_n^*(R_{ni}^+(a, b)) , \tag{2.7}$$

$$L_n(a, b) = L(\mathbf{Y}_n(a, b)) = n^{-1} \sum_{i=1}^{n} (t_i - \bar{t}_n)a_n(R_{ni}(a, b)) , \tag{2.8}$$

where $R_{ni}(a, b)$ (or $R_{ni}^+(a, b)$) is the rank of $Y_i - a - bt_i$ (or $|Y_i - a - bt_i|$) among $Y_1 - a - bt_1, \cdots, Y_n - a - bt_n$ (or $|Y_1 - a - bt_1|, \cdots, |Y_n - a - bt_n|$), for $i = 1, \cdots, n$. Note that $R_{ni}(a, b) = R_{ni}(0, b)$ for every real $a$, and hence, $L_n(a, b)$ does not depend on $a$; we write it as $L_n(b)$. Also, we write $R_{ni}(0, 0) = R_{ni}$ for $i = 1, \cdots, n$.

Note that for every given $\mathbf{Y}_n$ and $b$, $T_n(a, b)$ is $\searrow$ in $a$: $-\infty < a < \infty$, and for every given $\mathbf{Y}_n$, $L_n(b)$ is $\searrow$ in $b$: $-\infty < b < \infty$ (see Theorem 6.1 of Sen (1969) in this context). Also, if in the model (1.1), we let $\theta = \beta = 0$, then

$T_n(0, 0)$ and $L_n(0)$ both (marginally) have distributions symmetric about 0. As such, as in Adichie (1967), we consider the following estimators. Let

$$\hat{\theta}_n^{(1)} = \sup\{a : T_n(a, 0) > 0\}, \qquad \hat{\theta}_n^{(2)} = \inf\{a : T_n(a, 0) < 0\}; \tag{2.9}$$

$$\hat{\theta}_n = \tfrac{1}{2}(\hat{\theta}_n^{(1)} + \hat{\theta}_n^{(2)}); \tag{2.10}$$

$$\tilde{\beta}_n^{(1)} = \sup\{b : L_n(b) > 0\}, \qquad \tilde{\beta}_n^{(2)} = \inf\{b : L_n(b) < 0\}; \tag{2.11}$$

$$\tilde{\beta}_n = \tfrac{1}{2}(\tilde{\beta}_n^{(1)} + \tilde{\beta}_n^{(2)}); \tag{2.12}$$

$$\tilde{\theta}_n^{(1)} = \sup\{a : T_n(a, \tilde{\beta}_n) > 0\}, \qquad \tilde{\theta}_n^{(2)} = \inf\{a : T_n(a, \tilde{\beta}_n) < 0\}; \tag{2.13}$$

$$\tilde{\theta}_n = \tfrac{1}{2}(\tilde{\theta}_n^{(1)} + \tilde{\theta}_n^{(2)}). \tag{2.14}$$

Then, $\hat{\theta}_n$ is a translation-invariant, robust and consistent estimator of $\theta$ when $\beta = 0$, while $\tilde{\theta}_n$ is a similar estimator when $\beta$ is unspecified.

For the preliminary test on regression, we use the nonparametric test based on $L_n = L_n(0)$. Thus, for the one-sided test (viz., $H_0: \beta = 0$ vs. $H_1: \beta > 0$), our test consists in

$$\text{accepting or rejecting } H_0 \text{ according as } L_n \text{ is } < \text{ or } \geqq L_{n,\alpha}, \quad \text{where } P\{L_n \geqq L_{n,\alpha} \mid H_0\} \leqq \alpha, \quad 0 < \alpha < 1, \tag{2.15}$$

and $\alpha$ is the level of significance of the test. If we let

$$A_n^2 = (n-1)^{-1} \sum_{i=1}^n \{a_n(i) - n^{-1} \sum_{j=1}^n a_n(j)\}^2, \qquad n \geqq 2, \tag{2.16}$$

and if $\tau_\alpha$ is the upper $100\alpha\%$ point of the standard normal df, then

$$n Q_n^{-\frac{1}{2}} A_n^{-1} L_{n,\alpha} \to \tau_\alpha \qquad \text{as} \quad n \to \infty. \tag{2.17}$$

For small $n$, $L_{n,\alpha}$ can be computed by direct enumeration of the null distribution of $L_n$, generated by the $n!$ equally likely permutations of the ranks $R_{n1}, \cdots, R_{nn}$ (over the set $(1, \cdots, n)$ of natural integers).

Our proposed estimator of $\theta$ is then as follows:

$$\begin{aligned} \theta_n^* &= \hat{\theta}_n, && \text{if} \quad L_n < L_{n,\alpha} \\ &= \tilde{\theta}_n, && \text{if} \quad L_n \geqq L_{n,\alpha}. \end{aligned} \tag{2.18}$$

For a two-sided test, we replace in (2.15), (2.17) and (2.18), $L_n$ by $|L_n|$ and $\tau_\alpha$ by $\tau_{\alpha/2}$.

As is usually the case with estimators based on preliminary tests, $\theta_n^*$ is not (generally) an unbiased estimator of $\theta$. Our contention is to study the nature of the bias and m.s.e. of $\theta_n^*$. In passing, we may remark that, in general, $\hat{\theta}_n$, $\tilde{\beta}_n$ and $\tilde{\theta}_n$ (and hence, $\theta_n^*$) are to be obtained by trial and error solutions. For some specific scores (viz., Wilcoxon's), in some specific cases (viz., $\hat{\theta}_n$), an exact expression may be available. Also, if instead of the linear rank statistic $L_n$, one uses the estimator of $\beta$ (and the test for $\beta = 0$) based on Kendall's tau [viz., Sen (1968)], then for the Wilcoxon estimator in (2.10) and (2.14), we have some exact expressions. The linearized rank and signed-rank estimators of Kraft and van Eeden (1972) can be used and they are computationally simpler. However, they are asymptotically equivalent to the estimators in (2.10), (2.12) and (2.14) only if $\phi(u) = \psi(u) = -f'(F^{-1}(u))/f(F^{-1}(u))$, $0 < u < 1$. For $\phi(u) \neq \psi(u)$, the Kraft–van Eeden estimators can still be used, but these will have asymptotic distributions different from those of $\hat{\theta}_n$, $\tilde{\beta}_n$ and $\tilde{\theta}_n$, and we do not intend to pursue the case here.

**3. Asymptotic distribution of the estimator $\theta_n^*$.** Let us denote by

$$A_\phi^2 = \int_0^1 \phi^2(u)\,du - \left(\int_0^1 \phi(u)\,du\right)^2, \tag{3.1}$$

$$\psi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)), \quad 0 < u < 1; \tag{3.2}$$

$$A_\psi^2 = I(f) = \int_0^1 \psi^2(u)\,du,$$

$$\gamma(\psi, \phi) = \left(\int_0^1 \psi(u)\phi(u)\,du\right). \tag{3.3}$$

Then, by the basic theorems of Chapter V of Hájek and Šidák (1967), it follows that under (2.4), (2.5) and (2.6) and $\theta = \beta = 0$ (where $T_n = T_n(0, 0)$),

$$\mathscr{L}(n^{\frac{1}{2}}(T_n, L_n)) \to \mathscr{N}_2(\mathbf{0}, A_\phi^2 \operatorname{diag}(1, Q^*)), \quad \text{as} \quad n \to \infty \tag{3.4}$$

(so that under $\theta = \beta = 0$, the two statistics are asymptotically independent, too). Secondly, from Theorem 3.1 of Jurečková (1969), we have the following result where $K$ $(0 < K < \infty)$ is a positive constant.

Under (2.1) through (2.6) and for $\beta = 0$, as $n \to \infty$,

$$\sup\{n^{\frac{1}{2}}|L_n(n^{-\frac{1}{2}}b) - L_n(0) + n^{-\frac{1}{2}}bQ^*\gamma(\psi, \phi)| : |b| \leqq K\} \to_p 0. \tag{3.5}$$

Finally, note that under $\theta = \beta = 0$, $Y_i$ and $-Y_i$ both have the same df. Also, we note that by (2.5), there exists a $d$ $(0 < d < \infty)$ such that $d|t_i| \leqq 1$ for all $i \geqq 1$, so that $(1 + dt_i) \geqq 0$ for every $i \geqq 1$. Moreover, (2.4) and (2.5) insure that for $\mathbf{x}_i = (1, t_i)$, $i \geqq 1$,

$$n^{-1} \sum_{i=1}^n \mathbf{x}_i'\mathbf{x}_i \to \begin{pmatrix} 1 & \bar{t} \\ \bar{t} & Q^* + \bar{t}^2 \end{pmatrix} \quad \text{as} \quad n \to \infty,$$

and also $\max_{1 \leq i \leq n} \{t_i^2 / \sum_{i=1}^n t_i^2\} \to 0$ as $n \to \infty$. Since the first coordinate of $\mathbf{x}_i$ is equal to 1 for all $i \geqq 1$, it follows that $(|1| - |1|)(|1 + dt_i| - |1 + dt_{i'}|) = 0$ for all $i, i' = 1, \cdots, n$ and $n \geqq 1$. As such, by our (2.1) through (2.6) and Theorem 7.2 of Kraft and van Eeden (1972), we arrive at the following.

Under (2.1) through (2.6) and for $\theta = \beta = 0$, as $n \to \infty$,

$$\begin{aligned} \sup\{n^{\frac{1}{2}}|T_n(n^{-\frac{1}{2}}(a, b)) - T_n(0, 0) + n^{-\frac{1}{2}}(a + b\bar{t}_n)\gamma(\psi, \phi)| &: \\ |a| \leqq K, |b| \leqq K\} \to 0, &\quad \text{in probability,} \end{aligned} \tag{3.6}$$

where $K(0 < K < \infty)$ is a positive constant.

From (2.9) through (2.14), (3.4), (3.5) and (3.6), it follows by some standard computations that under (2.1) through (2.6),

$$\begin{aligned} &\mathscr{L}(n^{\frac{1}{2}}(\tilde{\theta}_n - \theta, \tilde{\beta}_n - \beta)) \\ &\quad \to \mathscr{N}_2\left(\mathbf{0}; \{A_\phi^2/\gamma^2(\psi, \phi)\} \begin{pmatrix} 1 + \bar{t}^2/Q^*, & -\bar{t}/Q^* \\ -\bar{t}/Q^*, & 1/Q^* \end{pmatrix}\right), \end{aligned} \tag{3.7}$$

and when $\beta = 0$,

$$\mathscr{L}(n^{\frac{1}{2}}(\hat{\theta}_n - \theta)) \to \mathscr{N}_1(0; A_\phi^2/\gamma^2(\psi, \phi)) . \tag{3.8}$$

Note that the (one-sided) test based on $L_n$ in (2.15) is consistent against $\beta > 0$, so that asymptotically, $P\{\theta_n^* = \tilde{\theta}_n \mid \beta > 0\} \to 1$. Hence, by (3.7), for every $\beta > 0$, as $n \to \infty$,

$$\mathscr{L}(n^{\frac{1}{2}}(\theta_n^* - \theta)) \to \mathscr{N}_1(0; A_\phi^2(1 + \bar{t}^2/Q^*)/\gamma^2(\psi, \phi)) . \tag{3.9}$$

Similarly, $P\{L_n < L_{n,\alpha} \mid \beta < 0\} \to 1$ as $n \to \infty$, so that $P\{\theta_n^* = \hat{\theta}_n \mid \beta < 0\} \to 1$ as $n \to \infty$. On the other hand, for $\beta \neq 0$ and $\bar{t} \neq 0$, $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ does not have any asymptotic distribution, so that for every real $x$ $(-\infty < x < \infty)$,

$$P\{n^{\frac{1}{2}}(\theta_n^* - \theta) \leqq x \mid \beta < 0\} \to 0 \quad \text{or} \quad 1 \quad \text{according as} \quad \bar{t} \quad \text{is} \quad < \quad \text{or} \quad > 0 . \tag{3.10}$$

For the two-sided preliminary test, (3.9) holds for any $\beta \neq 0$. In either case, for $\beta = 0$ or close to 0, the asymptotic distribution will be different. For this purpose, we conceive of a sequence of alternative hypotheses $\{K_n\}$ where

$$K_n : \beta = \beta_{(n)} = n^{-\frac{1}{2}}\lambda , \qquad \lambda \text{ real.} \tag{3.11}$$

Then, we have the following theorem.

THEOREM 3.1. *Under* (2.1) *through* (2.6) *and* $\{K_n\}$ *in* (3.11), *as* $n \to \infty$,

$$\mathscr{L}(n^{\frac{1}{2}}(\tilde{\theta}_n - \theta, L_n)) \to \mathscr{N}_2(0, \lambda Q^*\gamma(\psi, \phi); \Sigma_1) , \tag{3.12}$$

$$\mathscr{L}(n^{\frac{1}{2}}(\hat{\theta}_n - \theta, L_n)) \to \mathscr{N}_2(\lambda(\bar{t}, Q^*\gamma(\psi, \phi)); \Sigma_2) , \tag{3.13}$$

*where*

$$\Sigma_1 = A_\phi^2 \begin{pmatrix} (1 + \bar{t}^2/Q^*)\gamma^2(\psi, \phi) , & -\bar{t}/\gamma(\psi, \phi) \\ -\bar{t}/\gamma(\psi, \phi) , & Q^* \end{pmatrix} , \tag{3.14}$$

$$\Sigma_2 = A_\phi^2 \begin{pmatrix} 1/\gamma^2(\psi, \phi) , & 0 \\ 0 , & Q^* \end{pmatrix} .$$

OUTLINE OF THE PROOF. Note that both $\tilde{\theta}_n$ and $\hat{\theta}_n$ are translation-invariant estimators, and hence, for proving (3.12) and (3.13), we may, without any loss of generality, assume that $\theta = 0$. Also, by (2.11)—(2.12) and (3.4)—(3.5), $n^{\frac{1}{2}}|\tilde{\beta}_n - \beta| = O_p(1)$, while, under (3.11), $n^{\frac{1}{2}}\beta = \lambda = O(1)$. Thus, under (3.11), $n^{\frac{1}{2}}|\tilde{\beta}_n| = O_p(1)$. Observe that under $\theta = \beta = 0$, by (3.5), $n^{\frac{1}{2}}L_n(0) = n^{\frac{1}{2}}\tilde{\beta}_n Q^*\gamma(\psi, \phi) + o_p(1)$, while, by (3.6), $n^{\frac{1}{2}}T_n(0, 0) = n^{\frac{1}{2}}(\tilde{\theta}_n + \bar{t}_n\tilde{\beta}_n)\gamma(\psi, \phi) + o_p(1)$, so that $n^{\frac{1}{2}}\tilde{\theta}_n\gamma(\psi, \phi) = n^{\frac{1}{2}}T_n(0, 0) - (\bar{t}_n/Q^*)n^{\frac{1}{2}}L_n(0) + o_p(1)$. Hence, utilizing the contiguity of the probability measures under $\{K_n^* : \theta = 0, \beta = n^{-\frac{1}{2}}\lambda\}$ to those under $H_0^* : \theta = \beta = 0$, we obtain from the above that under $\{K_n^*\}$, as $n \to \infty$,

$$n^{\frac{1}{2}}\tilde{\theta}_n\gamma(\psi, \phi) = n^{\frac{1}{2}}T_n(0, 0) - (\bar{t}_n/Q^*)n^{\frac{1}{2}}L_n(0) + o_p(1) . \tag{3.15}$$

Finally, $n^{\frac{1}{2}}(T_n(0, 0), L_n(0))$, under $K_n^*$, has the same joint distribution as $n^{\frac{1}{2}}(T_n(0, -n^{-\frac{1}{2}}\lambda), L_n(-n^{-\frac{1}{2}}\lambda))$ under $H_0^*$; by (3.4) through (3.6), the latter is asymptotically normal with mean vector $\lambda\gamma(\psi, \phi)(\bar{t}, Q^*)$ and dispersion matrix $A_\phi^2 \operatorname{diag}(1, Q^*)$.

Thus, under $\{K_n^*\}$, as $n \to \infty$,

$$\mathscr{L}(n^{\frac{1}{2}}(T_n(0,0), L_n(0))) \to \mathscr{N}_2(\{\lambda\gamma(\psi,\phi)\}(\bar{t}, Q^*); A_\phi^2 \operatorname{diag}(1, Q^*)) . \tag{3.16}$$

The proof of (3.12) follows from (3.15) and (3.16). Also, noting that by (2.9), (2.10) and (3.6), under $\{K_n^*\}$, $n^{\frac{1}{2}}\hat{\theta}_n\gamma(\psi,\phi) = n^{\frac{1}{2}}T_n(0,0) + o_p(1)$, the proof of (3.13) again follows from (3.16). □

Let $P_{K_n}$ denote the probability under $K_n$ in (3.11). Then, by (2.18), we obtain that for every real $x$ $(-\infty < x < \infty)$,

$$\begin{aligned} P_{K_n}\{n^{\frac{1}{2}}(\theta_n^* - \theta) \leqq x\} &= P_{K_n}\{n^{\frac{1}{2}}(\theta_n^* - \theta) \leqq x, L_n < L_{n,\alpha}\} \\ &\quad + P_{K_n}\{n^{\frac{1}{2}}(\theta_n^* - \theta) \leqq x, L_n \geqq L_{n,\alpha}\} \\ &= P_{K_n}\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \leqq x, L_n < L_{n,\alpha}\} \\ &\quad + P_{K_n}\{n^{\frac{1}{2}}(\tilde{\theta}_n - \theta) \leqq x, L_n \geqq L_{n,\alpha}\} . \end{aligned} \tag{3.17}$$

Note that $A_n^2$, defined by (2.16), converges to $A_\phi^2$, as $n \to \infty$. Hence, if we denote by $G(x)$ (and $g(x)$) the df (and the pdf) of a standard normal distribution, from (2.5), (2.17) and (3.17) and Theorem 3.1, we obtain by some routine computations that under $\{K_n\}$ in (3.11) and (2.1) through (2.6), as $n \to \infty$, for every real $x$,

$$\begin{aligned} &P_{K_n}\{n^{\frac{1}{2}}(\theta_n^* - \theta)\gamma(\psi,\phi)/A_\phi \leqq x\} \\ &\quad = P_{K_n}\{n^{\frac{1}{2}}(\theta_n^* - \theta) \leqq xA_\phi/\gamma(\psi,\phi)\} \\ &\quad \to G(x - \lambda\nu_1)G(\tau_\alpha - \lambda\nu_2) + \textstyle\int_{\tau_\alpha - \lambda\nu_2}^{\infty} G(x + w\nu_1/\nu_2)\, dG(w) \\ &\quad = G_1^*(x) , \quad \text{say,} \end{aligned} \tag{3.18}$$

where

$$\nu_1 = \bar{t}\gamma(\psi,\phi)/A_\phi , \qquad \nu_2 = (Q^*)^{\frac{1}{2}}\gamma(\psi,\phi)/A_\phi \qquad \text{and} \qquad \nu_1/\nu_2 = \bar{t}/(Q^*)^{\frac{1}{2}} . \tag{3.19}$$

In a similar manner, it can be shown that for the two-sided preliminary test, the asymptotic distribution is given by

$$\begin{aligned} G_2^*(x) &= G(x - \lambda\nu_1)\{G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2)\} \\ &\quad + \{\textstyle\int_{-\infty}^{-\tau_{\alpha/2}-\lambda\nu_2} + \int_{\tau_{\alpha/2}-\lambda\nu_2}^{\infty} G(x + w\nu_1/\nu_2)\, dG(w)\} . \end{aligned} \tag{3.20}$$

Note that both $G_1^*$ and $G_2^*$ depend on $\alpha$, $\lambda$, $\bar{t}$, $Q^*$, $\phi$ and $\psi$. Thus, we arrive at the following

THEOREM 3.2. *For the one and two-sided preliminary tests on* $\beta$, *under* (3.11), *the asymptotic distributions of* $n^{\frac{1}{2}}(\theta_n^* - \theta)\gamma(\psi,\phi)/A_\phi$ *are* $G_1^*$ *and* $G_2^*$, *respectively, defined by* (3.18) *and* (3.20).

We conclude this section with the note that for the density functions $g_1^*$ and $g_2^*$ corresponding to the df's $G_1^*$ and $G_2^*$, we have

$$g_1^*(x) = g(x - \lambda\nu_1)G(\tau_\alpha - \lambda\nu_2) + \textstyle\int_{\tau_\alpha-\lambda\nu_2}^{\infty} g(x + w\nu_1/\nu_2)\, dG(w) , \tag{3.21}$$

$$\begin{aligned} g_2^*(x) &= g(x - \lambda\nu_1)\{G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2)\} \\ &\quad + \{\textstyle\int_{-\infty}^{-\tau_{\alpha/2}-\lambda\nu_2} + \int_{\tau_{\alpha/2}-\lambda\nu_2}^{\infty} g(x + w\nu_1/\nu_2)\, dG(w)\} , \\ &\qquad -\infty < x < \infty . \end{aligned} \tag{3.22}$$

**4. Asymptotic bias and mean squared error of the estimator.** Theorem 3.2 gives us the asymptotic distribution of $n^{\frac{1}{2}}(\theta_n^* - \theta)\gamma(\psi, \phi)/A_\phi$ for both the situations. We now define for the two cases

$$\text{(4.1)} \qquad \textit{Asymptotic bias of } n^{\frac{1}{2}}(\theta_n^* - \theta) = \xi_j = \{A_\phi/\gamma(\psi, \phi)\}\{\textstyle\int_{-\infty}^{\infty} x g_j^*(x)\,dx\} \qquad \text{for } j = 1, 2,$$

and

$$\text{(4.2)} \qquad \textit{Asymptotic mean squared error } \text{(a.m.s.e.) } \textit{of } n^{\frac{1}{2}}(\theta_n^* - \theta) = \zeta_j = \{A_\phi^2/\gamma^2(\psi, \phi)\}\{\textstyle\int_{-\infty}^{\infty} x^2 g_j^*(x)\,dx\} \qquad \text{for } j = 1, 2.$$

We may remark here that the actual bias and m.s.e. of $n^{\frac{1}{2}}(\theta_n^* - \theta)$ may not be asymptotically equal to the expressions in (4.1) and (4.2); such an asymptotic equivalence demands conditions more restrictive than the ones insuring the convergence in law in (3.18) and (3.20). Nevertheless, (4.1) and (4.2) are important tools for studying the asymptotic properties of the estimator $\theta_n^*$.

Note that for a standard normal density $g\ (= G')$ and $a < b$,

$$\text{(4.3)} \qquad \textstyle\int_a^b x g(x)\,dx = g(a) - g(b),$$

$$\text{(4.4)} \qquad \textstyle\int_a^b x^2 g(x)\,dx = \{ag(a) - bg(b)\} + \{G(b) - G(a)\},$$

$$\text{(4.5)} \qquad \textstyle\int_{-\infty}^{\infty} (x + h)^2 g(x)\,dx = 1 + h^2, \qquad \text{for all real } h.$$

As such, from (3.21), (3.22), (4.1) and (4.3), we obtain that

$$\text{(4.6)} \qquad \begin{aligned} \xi_1 &= \{A_\phi/\gamma(\psi, \phi)\}\{\lambda\nu_1 G(\tau_\alpha - \lambda\nu_2) + g(\tau_\alpha - \lambda\nu_2)(-\nu_1/\nu_2)\} \\ &= \lambda\bar{t}G(\tau_\alpha - \lambda\nu_2) - g(\tau_\alpha - \lambda\nu_2)\bar{t}A_\phi/\gamma(\psi, \phi)(Q^*)^{\frac{1}{2}} \\ &= \bar{t}\{\lambda G(\tau_\alpha - \lambda\nu_2) - g(\tau_\alpha - \lambda\nu_2)/\nu_2\}; \end{aligned}$$

$$\text{(4.7)} \qquad \begin{aligned} \xi_2 = \bar{t}\{\lambda[&G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2)] \\ &- \nu_2^{-1}[g(\tau_{\alpha/2} - \lambda\nu_2) - g(-\tau_{\alpha/2} - \lambda\nu_2)]\}. \end{aligned}$$

Also, from (3.21), (3.22), (4.2), (4.4) and (4.5), we obtain that

$$\text{(4.8)} \qquad \begin{aligned} \zeta_1 &= \{A_\phi/\gamma(\psi, \phi)\}^2[\{1 + (\lambda\nu_1)^2\}G(\tau_\alpha - \lambda\nu_2) + \textstyle\int_{\tau_\alpha - \lambda\nu_2}^{\infty} \{1 + (\nu_1/\nu_2)^2 w^2\, dG(w)\}] \\ &= \{A_\phi/\gamma(\psi, \phi)\}^2[\{1 + (\lambda\nu_1)^2\}G(\tau_\alpha - \lambda\nu_2) + \{1 - G(\tau_\alpha - \lambda\nu_2)\} \\ &\qquad + (\nu_1/\nu_2)^2\{1 - G(\tau_\alpha - \lambda\nu_2) + (\tau_\alpha - \lambda\nu_2)g(\tau_\alpha - \lambda\nu_2)\}] \\ &= \{A_\phi/\gamma(\psi, \phi)\}^2(1 + \bar{t}^2/Q^*) \\ &\qquad + \bar{t}^2\{G(\tau_\alpha - \lambda\nu_2)(\lambda^2 - \nu_2^{-2}) + \nu_2^{-2}(\tau_\alpha - \lambda\nu_2)g(\tau_\alpha - \lambda\nu_2)\}; \end{aligned}$$

$$\text{(4.9)} \qquad \begin{aligned} \zeta_2 &= \{A_\phi/\gamma(\psi, \phi)\}^2(1 + \bar{t}^2/Q^*) \\ &\qquad + \bar{t}^2\{(G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2))(\lambda^2 - \nu_2^{-2}) \\ &\qquad + \nu_2^{-2}[(\tau_{\alpha/2} - \lambda\nu_2)g(\tau_{\alpha/2} - \lambda\nu_2) + (\tau_{\alpha/2} + \lambda\nu_2)g(\tau_{\alpha/2} + \lambda\nu_2)]\}. \end{aligned}$$

Note that the asymptotic distribution in (3.8) holds when $\beta = 0$ and we need to study the situation when $\{K_n\}$ in (3.11) holds. Towards this, we define $\{K_n^*\}$ and $H_0^*$ as in before (3.15) and note that for every (fixed) real $a$ $(-\infty < a < \infty)$,

$$\text{(4.10)} \qquad P_{K_n^*}\{n^{\frac{1}{2}}T_n(n^{-\frac{1}{2}}a, 0) < 0\} = P_{H_0^*}\{n^{\frac{1}{2}}T_n(n^{-\frac{1}{2}}(a, -\lambda)) < 0\}.$$

Hence, by (2.9), (2.10), (3.6) and (4.10), we obtain by a few standard steps that for every (fixed) real $a$,

$$P\{n^{\frac{1}{2}}(\hat{\theta}_n - \theta)\gamma(\psi, \phi)/A_\phi \leqq a \mid K_n\} = P\{\hat{\theta}_n \leqq n^{-\frac{1}{2}}aA_\phi/\gamma(\psi, \phi) \mid K_n^*\} \rightarrow G(a - \lambda\bar{t}\gamma(\psi, \phi)/A_\phi) . \tag{4.11}$$

Thus, under $\{K_n\}$ in (3.11), $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ has asymptotically a normal distribution with mean $\lambda\bar{t}$ and variance $A_\phi^2/\gamma^2(\psi, \phi)$, and hence, the asymptotic bias of $n^{\frac{1}{2}}(\hat{\theta}_n - \theta)$ is equal to

$$\xi_0 = \{A_\phi/\gamma(\psi, \phi)\} \textstyle\int_{-\infty}^{\infty} x \, dG(x - \lambda\bar{t}\gamma(\psi, \phi)/A_\phi) = \lambda\bar{t} \tag{4.12}$$

and its a.m.s.e. is equal to

$$\zeta_0 = A_\phi^2/\gamma^2(\psi, \phi) + \lambda^2\bar{t}^2 . \tag{4.13}$$

On the other hand, by (3.12), the asymptotic bias and a.m.s.e. of $n^{\frac{1}{2}}(\tilde{\theta}_n - \theta)$ are (respectively) given by

$$\xi_3 = 0 \quad \text{and} \quad \zeta_3 = \{A_\phi^2/\gamma^2(\psi, \phi)\}(1 + \bar{t}^2/Q^*) . \tag{4.14}$$

We define the *relative asymptotic bias* as $\mu_j = \xi_j/\zeta_j^{\frac{1}{2}}$, for $j = 1, 2$. The expressions for $\mu_1$ and $\mu_2$ can be obtained directly from (4.6) through (4.9). For the null hypothesis case (i.e., for $\lambda = 0$), $\mu_2 = 0$ and $\mu_1$ reduces to

$$-(\bar{t}/(Q^*)^{\frac{1}{2}})g(\tau_\alpha)/\{1 + (\bar{t}^2/Q^*)(\alpha + \tau_\alpha g(\tau_\alpha))\}^{\frac{1}{2}} . \tag{4.15}$$

Note that (4.15) depends on $\bar{t}$, $Q^*$ as well as $\alpha$. For the special case of the two-sample location problem (with equal sample sizes), we obseve that (4.15) reduces to

$$-g(\tau_\alpha)/\{1 + \alpha + \tau_\alpha g(\tau_\alpha)\}^{\frac{1}{2}} . \tag{4.16}$$

For $\alpha = 0.01$, 0.05 and 0.10, the values of (4.16) are $-0.02$, $-0.08$ and $-0.152$, respectively.

Some interesting features of (4.6) through (4.16) are the following. First, for any $\lambda$ (not necessarily equal to 0),

$$\bar{t} = 0 \Rightarrow \xi_0 = \xi_1 = \xi_2 = \xi_3 = 0 \quad \text{and} \quad \zeta_0 = \zeta_1 = \zeta_2 = \zeta_3 = A_\phi^2/\gamma^2(\psi, \phi) . \tag{4.17}$$

Thus, for $\bar{t} = 0$, all the three estimators in (2.10), (2.14) and (2.18) have asymptotic bias equal to 0 and a.m.s.e. equal to $A_\phi^2/\gamma^2(\psi, \phi)$. Hence, in this case, the preliminary test of significance (on $\beta$) does no entail any asymptotic difference in the properties of the three estimators. In fact, $\bar{t} = 0 \Rightarrow \nu_1 = 0$, so that both $g_1^*(x)$ and $g_2^*(x)$ in (3.21)—(3.22) reduce to the standard normal pdf $g(x)$, insuring the asymptotic normality of $n^{\frac{1}{2}}(\theta_n^* - \theta)$. Secondly, if $\bar{t} \neq 0$ but $\lambda = 0$ (i.e., $H_0$: $\beta = 0$ holds), then $\xi_0 = \xi_2 = \xi_3 = 0$, while $\xi_1 = -(\bar{t}/\nu_2)g(\tau_\alpha)$ and is negative or positive according as $\bar{t}$ is positive or negative. Thus, for the null hypothesis case, the two-sided preliminary test has an advantage over the one-sided test so far as the asymptotic bias is concerned. As regards the a.m.s.e., we have the following.

THEOREM 4.1. *Under the null hypothesis* (i.e., $\lambda = 0$), *when* $\bar{t} \neq 0$,

$$0 < \zeta_0 < \zeta_1 < \zeta_2 < \zeta_3 < \infty , \qquad \textit{for every} \quad \alpha \in (0, 1) . \tag{4.18}$$

PROOF. Note that for $\lambda = 0$,

$$\begin{aligned} &\zeta_0 = A_\phi^2/\gamma^2(\psi, \phi) , \qquad \zeta_3 = \zeta_0(1 + \bar{t}^2/Q^*) , \\ &\zeta_1 = \zeta_0(1 + (\bar{t}^2/Q^*)\{\alpha + \tau_\alpha g(\tau_\alpha)\}) , \\ &\zeta_2 = \zeta_0(1 + (\bar{t}^2/Q^*)\{\alpha + 2\tau_{\alpha/2} g(\tau_{\alpha/2})\}) . \end{aligned} \tag{4.19}$$

Hence,

$$\begin{aligned} &\zeta_1 - \zeta_0 = (\zeta_0 \bar{t}^2/Q^*)(\alpha + \tau_\alpha g(\tau_\alpha)) ; \\ &\zeta_2 - \zeta_1 = (\zeta_0 \bar{t}^2/Q^*)(2\tau_{\alpha/2} g(\tau_{\alpha/2}) - \tau_\alpha g(\tau_\alpha)) ; \\ &\zeta_3 - \zeta_2 = (\zeta_0 \bar{t}^2/Q^*)(1 - \alpha - 2\tau_{\alpha/2} g(\tau_{\alpha/2})) . \end{aligned} \tag{4.20}$$

Note that for every real $x$,

$$\begin{aligned} 1 - G(x) + xg(x) &= \textstyle\int_x^\infty g(y)\, dy + xg(x) \\ &= [yg(y)]_x^\infty - \textstyle\int_x^\infty yg'(y)\, dy + xg(x) \\ &= \textstyle\int_x^\infty y^2 g(y)\, dy , \end{aligned} \tag{4.21}$$

so that

$$0 < \alpha + \tau_\alpha g(\tau_\alpha) = \textstyle\int_{\tau_\alpha}^\infty y^2 g(y)\, dy < 1 , \quad \forall\, 0 < \alpha < 1 . \tag{4.22}$$

Also,

$$\begin{aligned} &1 - \alpha - 2\tau_{\alpha/2} g(\tau_{\alpha/2}) \\ &\qquad = 1 - 2(\alpha/2 + \tau_{\alpha/2} g(\tau_{\alpha/2})) = 1 - 2 \textstyle\int_{\tau_{\alpha/2}}^\infty y^2 g(y)\, dy \\ &\qquad = \textstyle\int_{-\tau_{\alpha/2}}^{\tau_{\alpha/2}} y^2 g(y)\, dy > 0 , \qquad \text{for all} \quad 0 < \alpha < 1 . \end{aligned} \tag{4.23}$$

Finally, for $\frac{1}{2} \leqq \alpha < 1$, $\tau_\alpha$ is $\leqq 0$ while $\tau_{\alpha/2}$ is $> 0$, and hence, $2\tau_{\alpha/2} g(\tau_{\alpha/2}) - \tau_\alpha g(\tau_\alpha)$ is $> 0$. On the other hand, for $0 < \alpha < \frac{1}{2}$, $\tau_{\alpha/2} > \tau_\alpha > 0$ and

$$\begin{aligned} 2\tau_{\alpha/2} g(\tau_{\alpha/2}) - \tau_\alpha g(\tau_\alpha) &= 2(\alpha/2 + \tau_{\alpha/2} g(\tau_{\alpha/2})) - (\alpha + \tau_\alpha g(\tau_\alpha)) \\ &= 2 \textstyle\int_{\tau_{\alpha/2}}^\infty y^2 g(y)\, dy - \int_{\tau_\alpha}^\infty y^2 g(y)\, dy \\ &= \textstyle\int_{\tau_{\alpha/2}}^\infty y^2 g(y)\, dy - \int_{\tau_\alpha}^{\tau_{\alpha/2}} y^2 g(y)\, dy \\ &> \tau_{\alpha/2}^2(\alpha/2) - \tau_{\alpha/2}^2(\alpha - \alpha/2) = 0 , \qquad 0 < \alpha < \tfrac{1}{2} . \end{aligned} \tag{4.24}$$

Hence, (4.18) follows from (4.20) through (4.24). □

From Theorem 4.1, we conclude that from the point of view of a.m.s.e., $\theta_n^*$ is better than $\tilde{\theta}_n$ and $\hat{\theta}_n$ is better than $\theta_n^*$; the one-sided preliminary test (on $\beta$) is better than the two-sided one. In Section 5, we shall consider parallel results for the case when the null hypothesis is not true. Continuing the study under the null hypothesis case, we note that by (4.19) and (4.20),

$$\zeta_1/\zeta_0 = 1 + (\bar{t}^2/Q^*)(\textstyle\int_{\tau_\alpha}^\infty y^2 g(y)\, dy) . \tag{4.25}$$

For small $\alpha$, the integral on the right-hand side (r.h.s.) of (4.25) is small, indicating that the relative increase in the a.m.s.e. is also small. For the particular case of the two-sample location problem where the $t_i$ are either 0 or 1, we obtain that for the equal sample sizes case, $\bar{t} = \frac{1}{2}$ and $Q^* = \frac{1}{4}$, so that (4.25) reduces to

$$\zeta_1/\zeta_0 = 1 + \int_{\tau_\alpha}^{\infty} y^2 g(y)\, dy . \tag{4.26}$$

For $\alpha = 0.01$, 0.05 and 0.10, the values for the r.h.s. of (4.26) are 1.073, 1.221 and 1.328, respectively. Finally, for $\lambda = 0$ and $\bar{t} \neq 0$, we have noticed that $|\xi_1| = (|\bar{t}|/\nu_2)g(\tau_\alpha)$ where by (3.2), (3.3) and (3.19)

$$\begin{aligned}\bar{t}^2/\nu_2 &= (\bar{t}^2/Q^*)(A_\phi^2/\gamma^2(\psi, \phi)) \\ &= (\bar{t}^2/Q^*)A_\psi^2\{\gamma^2(\psi, \phi)/A_\phi^2 A_\psi^2\}^{-1} .\end{aligned} \tag{4.27}$$

Now, the first factor on the r.h.s. of (4.27) depends only on the design of the set of independent variables $t_1, \cdots, t_n$. If the choice of the design is left to us, we can minimize $(\bar{t}^2/Q^*)$ by setting $\bar{t}_n \to 0$ as $n \to \infty$. On the other hand, if the $t_i$ are given, we do not have much control in this respect, and hence, the prospect of minimizing (4.27) rests on the minimization of the last factor on the r.h.s. of (4.27). Towards this, note that

$$\gamma^2(\psi, \phi)/A_\psi^2 A_\phi^2 \leqq 1 , \qquad \text{for all} \quad \phi , \tag{4.28}$$

where the strict equality sign holds when $\psi = \phi$. Hence, an optimal choice of the source function relates to $\phi = \psi$.

**5. Asymptotic comparison of the estimators when $\lambda \neq 0$.** Here, we shall be mainly concerned with the asymptotic comparison of the bias and mean squared errors of the estimators $\hat{\theta}_n$, $\theta_n^*$ and $\tilde{\theta}_n$ when $\bar{t} \neq 0$ and $H_0$: $\beta = 0$ may not hold.

First, let us consider the asymptotic bias for these estimators. For real $(t, u)$, let us define $G$ and $g$ as in Section 3 and let

$$h_t(u) = uG(t-u) - g(t-u) = G(t-u)\{u - g(t-u)/G(t-u)\} , \tag{5.1}$$

$$\bar{h}_t(u) = u\{G(t-u) - G(-t-u)\} - \{g(t-u) - g(-t-u)\} , \quad t \geqq 0 . \tag{5.2}$$

Then, $h_t'(u) = (d/du)h_t(u) = G(t-u) - tg(t-u) = G(t-u)\{1 - tg(t-u)/G(t-u)\}$. Note that for $t = 0$, $h_0(0) = -g(0) < 0$, $h_0'(u) = G(-u) \geqq 0$, $\forall$ real $u$ and $h_0(\infty) = 0$, so that $h_0(u)$ is a monotonically nondecreasing and negative function of $u \in (-\infty, \infty)$. Let us next consider the case of $0 < t \leqq (\pi/2)^{\frac{1}{2}}$. Then, $h_t(0) = -g(t) < 0$, for $0 \leqq u \leqq t$, $h_t'(u) = G(t-u)\{1 - tg(t-u)/G(t-u)\} \geqq G(0)\{1 - tg(0)/G(0)\} = \frac{1}{2}\{1 - t(2/\pi)^{\frac{1}{2}}\} \geqq 0$ and $h_t(t) = tG(0) - g(0)$ is $\leqq$ or $> 0$ according as $0 < t \leqq (2/\pi)^{\frac{1}{2}}$ or not. Further, $g(x)/G(x)$ is $\downarrow$ in $x$ $(-\infty < x < \infty)$ and $|g(x)/G(x) + x| \to 0$ as $x \to -\infty$. Thus, for every $t \in (0, (\pi/2)^{\frac{1}{2}})$, there exists an $u_0 = u_0(t)$ $(\geqq t)$, such that $h_t'(u)$ is $<$ or $\geqq 0$ according as $u$ is $> u_0$ or $t \leqq u \leqq u_0$, while we have observed that $h_t'(u) \geqq 0$ for $u \leqq t$; for large $u$, $h_t(u)$ behaves as $G(t-u)\{t + o(1)\}$ and is positive. Hence, $h_t(u)$ monotonically increases from $-g(t)$ $(< 0)$ to $h_t(u_0)$ $(0 < h_t(u_0) < u_0)$ as $u$ increases from 0 to $u_0$ and then it monotonically converges to 0 as $u \to \infty$. Finally, if $t > (\pi/2)^{\frac{1}{2}}$, $h_t(0) = -g(t)$ $(< 0)$, $h_t'(0) = G(t) - tg(t) > 0$, $h_t(t) = tG(0) - g(0) > 0$ and $h_t'(t) = \frac{1}{2}\{1 - 2tg(0)\} < 0$. Since, for $0 \leqq u \leqq t$, $tg(t-u)/G(t-u)$ is $\uparrow$ in $u$, there exists an $u_0 = u_0(t)$ $(0 < u_0 < t)$, such that $h_t'(u)$

is $>$ or $\leqq 0$ according as $u$ is $<$ or $\geqq u_0$, whereas as before, $h_t'(u) < 0$ for every $u \geqq t$. Hence, here $h_t(u)$ is $\uparrow$ in $u \in (-\infty, u_0)$ and $\downarrow$ in $u \in (u_0, \infty)$ where $h_t(0) < 0 < h_t(u_0) < u_0 < t$ and $h_t(\infty) = 0$. In a similar manner, it follows that $\bar{h}_0(u) = 0$ for every real $u$, while for $t > 0$, there exists an $u_0 = u_0(t)$ $(0 < u_0 < \infty)$, such that $\bar{h}_t(u)$ is monotonically increasing in $u \in (0, u_0)$ and decreasing in $u \in (u_0, \infty)$. Besides, $\bar{h}_t(u)$ is symmetric in $u$, $\bar{h}_t(0) = \bar{h}_t(\infty) = 0$ and $0 < \bar{h}_t(u_0) < u_0$.

Let us consider the expressions for the asymptotic bias of the estimators in (4.6), (4.7) and (4.12). Define $\lambda_0$ by $\lambda_0 \nu_2 = u_0$ where $u_0$ is defined as in above. Then, from (5.1), (5.2), (4.6), (4.7), (4.12) and the above discussion, we arrive at the following.

LEMMA 5.1. *For* $0 < \alpha < \frac{1}{2}$ *and* $\bar{t} > 0$, *there exist two numbers* $(\lambda_0, \lambda_1): 0 < \lambda_1 \leqq \lambda_0 < \infty$, *such that* (i) $\xi_1 = \xi_1(\lambda)$ *is* $\leqq$ *or* $>0$ *according as* $\lambda$ *is* $\leqq$ *or* $> \lambda_1$ *and* (ii) $\xi_1(\lambda)$ *is* $\uparrow$ *in* $\lambda \in (-\infty, \lambda_0)$ *and* $\downarrow$ *in* $\lambda \in (\lambda_0, \infty)$ *with* $\xi_1(0) < 0 < \xi_1(\lambda_0) < \lambda_0 \bar{t}$ *and* $\xi_1(\infty) = 0$. *For* $\alpha = \frac{1}{2}$, $\xi_1(\lambda)$ *is* $\uparrow$ *in* $\lambda \in (-\infty, \infty)$ *with* $\xi_1(\lambda) < 0$ $\forall$ *real* $\lambda$ *and* $\xi_1(\infty) = 0$. *For* $\alpha = 0$, $\xi_2 = \xi_2(\lambda) = 0$ for all $\lambda \in (-\infty, \infty)$, *while for* $\bar{t} > 0$ *and* $0 < \alpha < 1$, $\xi_2(\lambda)$ *is a symmetric and nonnegative function of* $\lambda$, $\xi_2(0) = \xi_2(\infty) = 0$ *and* $\xi_2(\lambda)$ *is* $\uparrow$ *in* $\lambda \in (0, \lambda_0)$ *and is* $\downarrow$ *in* $\lambda \in (\lambda_0, \infty)$ *where* $0 < \xi_2(\lambda_0) < \lambda_0 \bar{t}$. *Finally,* $\xi_0 = \xi_0(\lambda) = \bar{t}\lambda$ *and is* $\uparrow$ *in* $\lambda$ *when* $\bar{t} > 0$. *For* $\bar{t} < 0$, *all the results hold with the* $\xi_j(\lambda)$ *replaced by* $-\xi_j(\lambda)$, $j = 0, 1, 2$.

Actually, it can be shown along the same line as in above that both $(d/d\lambda)\{\xi_0(\lambda) - \xi_1(\lambda)\}$ and $(d/d\lambda)\{\xi_0(\lambda) - \xi_2(\lambda)\}$ are nonnegative for all real $\lambda$, so that by Lemma 5.1, $\xi_0(\lambda) - \xi_1(\lambda)$ and $\xi_0(\lambda) - \xi_2(\lambda)$ both monotonically go to $\infty$ as $\lambda \to \infty$. Since, for the one-sided preliminary test in (2.15), we are primarily interested in the set of alternatives $\lambda > 0$, it appears that as regards the asymptotic bias, excepting for $\lambda$ close to 0, $\theta_n^*$ performs better than $\hat{\theta}_n$. For the two-sided test, $\hat{\theta}_n$ has an asymptotic bias never less than that of $\theta_n^*$.

Let us next compare the a.m.s.e.'s $\zeta_0$, $\zeta_1$ and $\zeta_2$. By (4.8), (4.9) and (4.13), we have

$$\begin{aligned}\zeta_1 - \zeta_0 = \bar{t}^2\{(A_\phi^2/Q^*\gamma^2(\psi, \phi))[1 - G(\tau_\alpha - \lambda\nu_2) + (\tau_\alpha - \lambda\nu_2)g(\tau_\alpha - \lambda\nu_2)] \\ - \lambda^2[1 - G(\tau_\alpha - \lambda\nu_2)]\}\,; \end{aligned} \tag{5.3}$$

$$\begin{aligned}\zeta_2 - \zeta_0 = \bar{t}^2\{(A_\phi^2/Q^*\gamma^2(\psi, \phi))[1 - G(\tau_{\alpha/2} - \lambda\nu_2) + G(-\tau_{\alpha/2} - \lambda\nu_2) \\ + (\tau_{\alpha/2} - \lambda\nu_2)g(\tau_{\alpha/2} - \lambda\nu_2) + (\tau_{\alpha/2} + \lambda\nu_2)g(\tau_{\alpha/2} + \lambda\nu_2)] \\ - \lambda^2[1 - G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2)]\}\,. \end{aligned} \tag{5.4}$$

Though in some neighborhood of $\lambda = 0$ (depending on $Q^*$, $\bar{t}$, $\alpha$ and $\gamma(\psi, \phi)$), $\zeta_1 - \zeta_0$ is positive, it goes to 0 as $\lambda \to -\infty$ and there exists a $\lambda_0 > 0$ such that $\zeta_1 < \zeta_0$ for $\lambda > \lambda_0$. A similar case holds for (5.4): it is symmetric in $\lambda$, is positive in some neighborhood of $\lambda = 0$ and is negative for $|\lambda| > \lambda_0$. Thus, for the general case, when $\bar{t}$ and $\lambda$ are not necessarily equal to 0, $\theta_n^*$ may have a smaller a.m.s.e. than that of $\hat{\theta}_n$.

Let us next compare the a.m.s.e. of $\theta_n^*$ and $\tilde{\theta}_n$. By (4.8), (4.9) and (4.14), we have

$$\zeta_1 - \zeta_3 = \bar{t}^2\{G(\tau_\alpha - \lambda\nu_2)(\lambda^2 - \nu_2^{-2}) + \nu_2^{-2}(\tau_\alpha - \lambda\nu_2)g(\tau_\alpha - \lambda\nu_2)\}\,; \tag{5.5}$$

$$\begin{aligned}\zeta_2 - \zeta_3 = \bar{t}^2\{&(G(\tau_{\alpha/2} - \lambda\nu_2) - G(-\tau_{\alpha/2} - \lambda\nu_2))(\lambda^2 - \nu_2^{-2}) \\ &+ \nu_2^{-2}[(\tau_{\alpha/2} - \lambda\nu_2)g(\tau_{\alpha/2} - \lambda\nu_2) + (\tau_{\alpha/2} + \lambda\nu_2)g(\tau_{\alpha/2} + \lambda\nu_2)]\}\,.\end{aligned} \tag{5.6}$$

Note that for $\bar{t} = 0$, both the quantities in (5.5) and (5.6) are equal to 0. Also, for $\bar{t} \neq 0$ but $\lambda = 0$, we have observed in (4.18) that $\zeta_1 < \zeta_2 < \zeta_3$. So that, in such a case, $\theta_n^*$ has a smaller a.m.s.e. than that of $\tilde{\theta}_n$, for both the cases of one and two-sided preliminary tests of significance (on $\beta$). This explains the asymptotic superiority of $\theta_n^*$ to that of $\tilde{\theta}_n$. For the particular case of the two-sample location model (with equal sample sizes), we obtain from (5.5) that for $\lambda = 0$,

$$2 \geqq \zeta_3/\zeta_1 = 2/\{1 + \alpha + \tau_\alpha g(\tau_\alpha)\} \geqq \tfrac{4}{3}\,, \qquad 0 < \alpha \leqq \tfrac{1}{2}\,, \tag{5.7}$$

where the lower bound is attained for $\alpha = \frac{1}{2}$ and for small $\alpha$, it is close to its upper bound 2; by (4.21), $\zeta_3/\zeta_1$ is $> 1$ for every $\alpha \in (0, 1)$.

The picture can be somewhat different when $\lambda \neq 0$; the presence of the asymptotic bias of $\theta_n^*$ may shoot up its a.m.s.e. and reduce its a.r.e. with respect to $\tilde{\theta}_n$. Note that for the case of the one-sided preliminary test (on $\beta$), the a.r.e. of $\{\theta_n^*\}$ with respect to $\{\tilde{\theta}_n\}$ (as judged by their a.m.s.e. in (4.8) and (4.14)) is given by

$$e_1(\theta^*, \tilde{\theta}) = \zeta_3/\zeta_1 = \frac{1 + \bar{t}^2/Q^*}{[1 + (\bar{t}^2/Q^*)q_1(\tau_\alpha, \lambda\nu_2)]} \tag{5.8}$$

where for $-\infty < x, y < \infty$,

$$\begin{aligned} q_1(x, y) &= y^2G(x - y) + 1 - G(x - y) + (x - y)g(x - y) \\ &= y^2G(x - y) + \textstyle\int_{(x-y)}^{\infty} u^2g(u)\,du\,, \quad \text{by (4.21)}. \end{aligned} \tag{5.9}$$

Note that $q_1(x, y)$ is nonnegative for all real $x$, $y$, and hence, (5.8) can never exceed $(1 + \bar{t}^2/Q^*)$. Also, $q_1(x, 0) = 1 - G(x) + xg(x) \in (0, 1)$ for all $-\infty < x < \infty$; $q_1(x, x) = \frac{1}{2}(1 + x^2)$ ($> 1$ if $|x| > 1$), $q_1(x, y) \to +\infty$ as $y \to -\infty$ (for a fixed $x$) and $q_1(x, y) \to 1$ as $y \to +\infty$ (for a fixed $x$). In fact, as $y$ increases from 0, $q_1(x, y)$ also increases first, attains a maximum at some $y$ and then it gradually converges to 1 as $y \to \infty$. Similarly, as $y$ decreases from 0, $q_1(x, y)$ first decreases and then it shoots up to $+\infty$ as $y \to -\infty$. This implies that there exists an interval $J = J(\alpha, \nu_2)$ containing 0 as an inner point, such that

$$q_1(\tau_\alpha, \lambda\nu_2) \leqq 1 \qquad \text{for every} \quad \lambda \in J \tag{5.10}$$

and the opposite inequality holds for $\lambda \notin J$. Consequently, by (5.8), (5.9) and (5.10), we conclude that

$$e_1(\theta^*, \tilde{\theta}) \geqq 1 \qquad \text{for every} \quad \lambda \in J\,, \tag{5.11}$$

and the opposite inequality holds outside the interval $J$. Actually, for negative $\lambda$, as $\lambda \to -\infty$, the a.r.e. converges to 0. Of course, for the one-sided test on $\beta$ in (2.15), we are primarily concerned with alternatives on the positive part of the real line, and hence, a highly negative value of $\lambda$ cases to be of much real interest.

In a similar manner, it follows from (4.9) and (4.14) that for the two-sided preliminary test (on $\beta$), the a.r.e. of $\{\theta_n^*\}$ with respect to $\{\tilde{\theta}_n\}$ is given by

$$e_2(\theta^*, \tilde{\theta}) = \zeta_3/\zeta_2 = \frac{1 + \bar{t}^2/Q^*}{[1 + (\bar{t}^2/Q^*)q_2(\tau_{\alpha/2}, \lambda\nu_2)]}, \tag{5.12}$$

where $\tau_{\alpha/2} > 0$ for every $\alpha \in (0, 1)$ and for $x \in [0, \infty)$ and $y \in (-\infty, \infty)$,

$$\begin{aligned} q_2(x, y) &= y^2\{G(x-y) - G(-x-y)\} + 1 - G(x-y) \\ &\quad + G(-x-y) + (x-y)g(x-y) + (x+y)g(x+y) \\ &= y^2\{G(x-y) - G(-x-y)\} + \textstyle\int_{-\infty}^{-x-y} u^2 g(u)\,du \\ &\quad + \textstyle\int_{x-y}^{\infty} u^2 g(u)\,du \quad (> 0)\,. \end{aligned} \tag{5.13}$$

Note that for all $0 \leqq x < \infty$, (i) $q_2(x, 0) \in (0, 1)$, (ii) $q_2(x, x) = x^2(\frac{1}{2} - G(-2x)) + \int_{-\infty}^{-2x} u^2 g(u)\,du + \frac{1}{2} > \frac{1}{2}(1 + x^2) + (4x^2 - 1)G(-2x)$ ($> 1$ if $x > 1$), (iii) $q_2(x, y) = q_2(x, -y)$ for all real $y$, and (iv) $q_2(x, y) \to 1$ as $y \to +\infty$. In fact, as $y$ increases from 0, $q_2(x, y)$ also increases, attains a maximum at some $y$ ($> 0$) and then gradually converges to 1 as $y \to \infty$. Thus, there exists an interval $J = J(\alpha, \nu_2)$, symmetric about 0 (an inner point), such that

$$q_2(\tau_{\alpha/2}, \lambda\nu_2) \leqq 1 \qquad \text{for every} \quad \lambda \in J\,, \tag{5.14}$$

while the opposite inequality holds outside $J$. Hence, we have the same type of picture for the a.r.e. as in the case of the one-sided preliminary test, excepting that (5.12) does not converge to 0 as $\lambda \to -\infty$.

For the two-sample location model (equal sample sizes), (5.8) reduces to

$$e_1(\theta^*, \tilde{\theta}) = 2/\{1 + q_1(\tau_\alpha, \lambda\nu_2)\}\,. \tag{5.15}$$

The following table relates to the a.r.e. in (5.15) for some typical $\alpha$ and $\lambda\nu_2$.

It appears from Table 1 that the smaller is the value of $\alpha$, the greater is the

TABLE 1

*Table for the* a.r.e. *in* (5.15) *for some specific* $\alpha$ *and* $\lambda\nu_2$

| $\lambda\nu_2$ | $e_1(\theta^*, \tilde{\theta})$ | | |
|---|---|---|---|
| | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.10$ |
| −1.00 | 0.997 | 0.984 | 0.967 |
| −0.50 | 1.571 | 1.483 | 1.404 |
| −0.20 | 1.839 | 1.658 | 1.534 |
| 0.00 | 1.864 | 1.638 | 1.500 |
| 0.20 | 1.745 | 1.520 | 1.413 |
| 0.50 | 1.414 | 1.264 | 1.217 |
| 1.00 | 0.901 | 0.906 | 0.949 |
| 1.50 | 0.620 | 0.727 | 0.823 |
| 2.00 | 0.500 | 0.679 | 0.805 |
| 3.00 | 0.529 | 0.806 | 0.915 |

variation in the a.r.e.; in any case, for $\lambda\nu_2$ close to 0, the a.r.e. exceeds one. By actual computations we have verified that for $0.01 \leqq \alpha \leqq 0.10$, the a.r.e. is greater than one for every $\lambda\nu_2$: $-0.96 \leqq \lambda\nu_2 \leqq 0.88$. A more or less similar case holds for the two-sided preliminary test of significance, though there the a.r.e. is a symmetric function of $\lambda\nu_2$. We may also remark that if the sample sizes are not equal, the a.r.e. will be higher or lower than the tabulated values according as the ratio of the first and second sample sizes is greater or smaller than one.

For two nonparametric estimators (after preliminary tests on $\beta$) $\theta_{n,1}^*$ and $\theta_{n,2}^*$ based respectively on the score functions $\phi_1$ and $\phi_2$, satisfying the regularity conditions of Section 2; we obtain from (4.8) that the a.r.e. of $\{\theta_{n,1}^*\}$ relative to $\{\theta_{n,2}^*\}$ (under $\{K_n\}$ in (3.11)) is

$$e_{12}^{(1)}(\lambda) = \{A_{\phi_2}^2 \gamma^2(\psi, \phi_1)/A_{\phi_1}^2 \gamma^2(\psi, \phi_2)\}h(\bar{t}, Q^*, \alpha, \lambda)\,, \tag{5.16}$$

where

$$h(\bar{t}, Q^*, \alpha, \lambda) = \frac{[1 + (\bar{t}^2/Q^*)q_1(\tau_\alpha, \lambda\nu_{22})]}{[1 + (\bar{t}^2/Q^*)q_1(\tau_\alpha, \lambda\nu_{21})]} \tag{5.17}$$

and $\nu_{2j}$ is defined by (3.19) for $\phi = \phi_j$, $j = 1, 2$. When $\lambda = 0$, $h = 1$, so that under the null hypothesis $H_0: \beta = 0$, (5.16) is equal to $\{A_{\phi_2}^2 \gamma^2(\psi, \phi_1)/A_{\phi_1}^2 \gamma^2(\psi, \phi_2)\} =$ the Pitman a.r.e. in the conventional location problem. Hence, in this respect, $\phi = \psi$ is the optimal score function. A similar case holds with the two-sided preliminary test of significance. On the other hand, for $\lambda \neq 0$, (5.16) depends on $\lambda$ as well as on $\bar{t}$, $Q^*$, $\nu_{21}$, $\nu_{22}$ and $\alpha$; the Pitman-optimality may not hold therefore for all $\lambda$.

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