The Annals of Statistics
1977, Vol. 5, No. 6, 1249

FURTHER COMMENTS ON SOME COMMENTS ON A PAPER BY BRADLEY EFRON

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The intrinsic geometry of a space of distributions is represented by means of an embedding in Hilbert space, which has been previously studied.

In comments on the paper by Efron (1975), Dawid (1975) investigated the intrinsic geometry of a space of distributions by somewhat abstract means. In fact, a concrete representation of this structure was given, effectively, by Rao (1945), and is implicit in Jeffreys (1961, page 179 ff.). Let $\mathscr{Q}$ be the set of finite measures on $(\mathscr{X}, \mathscr{A})$ equivalent to a σ-finite measure μ, and $\mathscr{P}$ the subset of probability measures. The map $Q \mapsto 2(dQ/d\mu)^{\frac{1}{2}}$ embeds $\mathscr{Q}$ into the Hilbert space $\mathscr{L}_2 = \mathscr{L}_2(\mathscr{X}, \mathscr{A}, \mu)$. The induced metric on $\mathscr{Q}$ is twice the Hellinger distance and has $\rho(Q_\theta, Q_{\theta+d\theta}) = (i_\theta)^{\frac{1}{2}}\, d\theta$, where $i_\theta = \int [\partial\{\log q_\theta(x)\}/\partial\theta]^2 q_\theta(x)\, d\mu$ $(q_\theta = dQ_\theta/d\mu)$. In particular, $\mathscr{P}$ maps onto the set $\{f \in \mathscr{L}_2 : ||f|| = 2 \text{ and } f > 0 \text{ p.p.}\}$, a portion of a spherical surface of radius 2, and the induced metric for $\mathscr{P}$ is just the information metric. Thus the results of Dawid (1975) on the geometry of $\mathscr{P}$ under the information metric follow trivially.

This spherical representation immediately shows that the geodesic distance, in $\mathscr{P}$, between two distributions P and P' is $\rho^* = 2\cos^{-1} \int (pp')^{\frac{1}{2}}\, d\mu$ $(p = dP/d\mu$, $p' = dP'/d\mu)$, effectively the Bhattacharya (1943) distance. The geodesic curve is an arc of a "circle" through P and P', say $\{P_\theta : 0 \leqq \theta \leqq \rho^*\}$, where $(dP_\theta/d\mu)^{\frac{1}{2}} = p^{\frac{1}{2}}\cos(\frac{1}{2}\theta) + f\sin(\frac{1}{2}\theta)$. Here $f = \{(p')^{\frac{1}{2}} - p^{\frac{1}{2}}\cos(\frac{1}{2}\rho^*)\}/\sin(\frac{1}{2}\rho^*)$, so that $(p^{\frac{1}{2}}, f)$ form an orthonormal basis for the two-dimensional space in $\mathscr{L}_2$ spanned by $p^{\frac{1}{2}}$, $(p')^{\frac{1}{2}}$. We find $p_\theta = dP_\theta/d\mu = A\{1 + \cos(t - \theta)\}$ where $A = \frac{1}{2}\{p + p' - 2(pp')^{\frac{1}{2}}\cos(\frac{1}{2}\rho^*)\}/\sin^2(\frac{1}{2}\rho^*)$ and $\tan(\frac{1}{2}t) = \{(p'/p)^{\frac{1}{2}} - \cos(\frac{1}{2}\rho^*)\}/\sin(\frac{1}{2}\rho^*)$.

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Received October 1976.

AMS 1970 *subject classifications.* 62B10, 53C40.

Key words and phrases. Geometry of distribution space, Hellinger distance, Bhattacharya distance, Fisher information, geodesic.