The Annals of Statistics
1977, Vol. 5, No. 5, 932–945

ON THEORY AND APPLICATIONS OF BIB DESIGNS WITH REPEATED BLOCKS[1]

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Consider BIB designs with parameters v, b, r, k and λ. Define the *support* of a BIB design to be the set of its distinct blocks and let the cardinality of the support be b^*. If $b^* < b$ then the design is said to be a BIB design with repeated blocks. Some potential applications of such designs to experimental design and controlled sampling are given. Some necessary and sufficient conditions for the existence of these designs and some algorithms for their constructions are provided. Bounds on b^* have been obtained. A necessary and sufficient condition under which a set of blocks can be the support of a BIB design are found. A table of BIB designs with $22 \leqq b^* \leqq 56$ for $v = 8$ and $k = 3$ is included.

1. Introduction and summary. From the point of view of application, there is no reason to exclude the possibility that a BIB design would contain repeated blocks. Indeed, the statistical optimality of BIB designs is unaffected by the presence of repeated blocks.

Following the standard notation we consider BIB designs with parameters v, b, r, k and λ. We define the *support* of a BIB design to be the set of its distinct blocks and we denote the cardinality of the support by b^*. The question of whether, for a given v, b and k, there exists a BIB design with repeated blocks has interested researchers in the area of experimental design. As van Lint (1973) has pointed out, many of the BIB designs constructed by Hanani (1961) have repeated blocks. Parker (1963) and Seiden (1963) proved that there is no BIB design with repeated blocks with parameters, $v = 2x + 2$, $b = 4x + 2$, $k = x + 1$. The case x even was settled by Parker and x odd by Seiden. Stanton and Sprott (1964) showed, among other results, that if s blocks of a BIB design are identical, then $b \geqq sv - (s - 2)$. Mann (1969) sharpened this result and showed that $b \geqq sv$. Note that the result of Parker and Seiden follows immediately from either of the above inequalities. Ho and Mendelsohn (1974) gave a generalization of the Mann inequality for t-design. More recently van Lint and Ryser (1972) and van Lint (1973, 1974) systematically studied the problem of the construction of BIB designs with repeated blocks. Their basic interest was in constructing a BIB design with repeated blocks with parameters v, b, r, k, λ such that b, r, and λ are relatively prime.

Wynn (1975) constructed a BIB design with $v = 8$, $b = 56$, $k = 3$ and $b^* = 24$

Received September 1976; revised January 1977.

[1] Research supported in part by AFOSR No. 76-3050 and NSF No. MCS76-07217.

AMS 1970 *subject classifications*. Primary 62K10, 62K05, 62D05; Secondary 05B05.

Key words and phrases. BIB designs with repeated blocks, support of a BIB design, controlled sampling.

and discussed an application of such designs in sampling. From the point of view of applications it is desirable to have techniques for producing BIB designs with various support sizes for a given v and k. We have studied the problem of BIB designs with repeated blocks from the later point of view. In Section 3 we outline some potential applications of such designs. In Section 4 we prove some necessary and sufficient conditions for the existence of these designs and provide algorithms for their constructions. In particular, we show that the combinatorial problem of searching for BIB designs with repeated blocks is equivalent to the algebraic problem of finding solutions to a set of homogeneous linear equations. Using this equivalence we have produced a table of designs based on $v = 8$ and $k = 3$ with $22 \leqq b^* \leqq 56$. In Section 5 we give bounds for the size of the support of BIB designs. In Section 6 we reformulate some of our results in the terminology of linear programming. In Section 7 we present some conditions under which a set of blocks can be the support of a BIB design.

2. Definitions and notation. Let $V = \{1, 2, \cdots, v\}$ and let $v\Sigma k$ be the set of all distinct subsets of size k based on V. Denote the cardinality of $v\Sigma k$ by vCk. For convenience we shall order the elements of $v\Sigma k$ lexicographically. A *balanced incomplete block design*, d, with parameters v, b, r, k and λ, written BIB (v, b, r, k, λ), is a collection of b elements of $v\Sigma k$, referred to as blocks, with the properties that:

(i) each element of V occurs in exactly r blocks,

and

(ii) each pair of distinct elements of V appears together in exactly λ blocks.

We emphasize that this definition does not require that the blocks of a BIB design be *distinct* elements of $v\Sigma k$. As discussed in Section 3, it may at times be to the experimenter's advantage to implement a BIB (v, b, r, k, λ) with less than b distinct blocks. In this paper we investigate BIB designs from the point of view of reducing the number of distinct blocks. To formalize this concept, we introduce the following definition.

DEFINITION 2.1. The *support* of a BIB design, d, is the collection of distinct blocks in d, denoted by d^*. The number of elements in d^* is denoted by b^* and called the *support size* of d.

We will denote a BIB (v, b, r, k, λ) with support size b^* by BIB $(v, b, r, k, \lambda \mid b^*)$. Any incomplete block design may be specified by the number of times that each element of $v\Sigma k$ is repeated in that design. We write f_i for the frequency of the ith element of $v\Sigma k$ in the design. Thus, we identify an incomplete block design, d, with $v\Sigma k$ and the frequency vector $F_d \equiv (f_{d1}, f_{d2}, \cdots, f_{dvCk})$. It is clear that $b = f_{d1} + f_{d2} + \cdots + f_{dvCk}$ and that b^* is the number of nonzero entries in the vector F_d. The BIB design, d, is said to be a *uniform* BIB *design* if the nonzero components of F_d are all identical. A BIB design with $b = b^* = vCk$ is denoted by $d(v, k)$ and referred to as the *trivial* BIB *design* based on v

and k. A BIB design with $b < vCk$ is said to be a *reduced* BIB *design*. In some context we will not demand that a BIB design based on v and k contains a particular number of blocks. In those cases we will simply say that d is a BIB (v, k) or a BIB $(v, k \mid b^*)$ if we wish to specify the support size.

3. Applications of BIB designs with repeated blocks.

A. *In finite population sampling.* A *sampling design* with k observations on a population V of size v is a probability measure on $v\Sigma k$. The support of a sampling design is the set of elements $v\Sigma k$ which have positive measure. A sampling design is said to be uniform if the probability measure is uniform on the support of the design. Denote by Π_i and Π_{ij} the sum of the probabilities associated with those elements of the support of the sampling design which contain $\{i\}$ and $\{i, j\}$, respectively. A simple random sample, SRS (v, k), is a uniform sampling design whose support is $v\Sigma k$. For the SRS (v, k), $\Pi_i = k/v$ and $\Pi_{ij} = k(k-1)/v(v-1)$.

The usual linear unbiased estimator of the mean of a characteristic of the population based on the SRS (v, k) is the sample mean, whose variance is $\sigma^2/k(1 - k/v)$, where σ^2 is the variance of the population. A question of interest is whether it is possible to construct other sampling designs whose sample means are unbiased estimates of the population mean and also have variances $\sigma^2/k(1 - k/v)$. In particular, we might require that these other designs (1) be nonuniform; or (2) have supports whose cardinalities are smaller than vCk.

Such sampling designs have practical applications. For example, it may be that some samples (i.e., some elements of $v\Sigma k$) may be more expensive or difficult to collect due to geographic dispersion or to some other factors. As another example, we may wish to guarantee that the selected sample contains at least a certain number of elements of some particular subsets of the population. In either example we would like to have low or possibly zero probability of selecting the less preferred samples, and to have higher probability of selecting the more preferred samples. For actual examples of applications of such sampling designs, see, for example, Goodman and Kish (1950) and Avadhani and Sukhatme (1973).

Chakrabarti (1963) discussed the use of reduced BIB designs to produce sampling designs. Wynn (1975) noticed that to find a sampling design with reduced support one can use nonreduced BIB designs with repeated blocks. If d is a BIB $(v, b, r, k, \lambda \mid b^*)$, then we can associate with d a sampling design by assigning to the lth element of $v\Sigma k$ probability measure f_l/b. It is easy to check that Π_i and Π_{ij} for this design will be the same as for the SRS (v, k), and that the sample mean will be an unbiased estimator of the population mean and that the variance will be the same as in SRS (v, k).

B. *In design of experiments.* It is well known that BIB designs are optimal for a number of criteria under the usual homoscedastic linear additive model for observations. This optimality holds whether or not the BIB designs contain repeated blocks. But in some cases it may well be to the experimenter's advantage to implement a BIB design with repeated blocks. For a variety of reasons the experimenter may not wish to run certain treatment combinations. One situation in which certain combinations of three or more treatments must be avoided is when it is physically impossible to run those combinations in the same block. In some other cases certain combinations of three or more treatments may produce observations which no longer conform to the homoscedastic linear additive model. If we are only interested in estimating treatment effects, we should use a BIB design in which no block contains any combinations which give rise to observations which violate the model.

Suppose the experimenter wishes to implement a BIB design, but he also wishes to avoid certain treatment combinations. He can search among the nonisomorphic uniform BIB designs to see if there are any which do not contain those combinations. If necessary, he can relabel his treatments. There are cases where one cannot avoid a set of combinations using uniform BIB designs, but can do so by using designs with repeated blocks. In fact, one must always search among the BIB designs with repeated blocks when there does not exist a reduced BIB design for the specified v and k.

4. Construction of BIB designs with reduced support. In this section we will provide algorithms for the construction of BIB designs based on v and k with support size less than vCk.

Label the elements of $v\Sigma 2$ from 1 to $vC2$. Let $p_{ij} = 1$ if the ith element of $v\Sigma 2$ is contained in the jth element of $v\Sigma k$, and let $p_{ij} = 0$ otherwise. By the *pair inclusion vector* associated with the jth element of $v\Sigma k$, we mean $P_j \equiv (p_{1j}, p_{2j}, \cdots, p_{vC2,j})'$. Let

$$P = [P_1, P_2, \cdots, P_{vCk}] .$$

LEMMA 4.1. *The frequency vector F determines a* BIB *design if and only if*

$$PF = \lambda \mathbf{1}$$

where λ is a positive integer.

The proof follows from the fact that $\sum_j f_j p_{ij}$ is the number of times the ith pair appears in the design.

The problem of constructing BIB designs based on v, k and λ is precisely the problem of finding all nonnegative integer solutions, F, to the equation $PF = \lambda\mathbf{1}$. We use this fact in many of the results which follow. This last result and several others in this section and in Section 5 could be phrased in the language of mathematical programming. We will briefly outline in Section 6 how this might be done.

Given a set of frequency vectors of BIB designs based on v and k, it is natural to inquire how these vectors might lead us to new BIB designs. We record here for reference some facts of this nature. In what follows, we say of the vectors $F_i' = (f_1^{(i)}, f_2^{(i)}, \cdots, f_{vCk}^{(i)})$, $i = 1, 2$, that $F_1 < F_2$ if and only if $f_j^{(1)} \leqq f_j^{(2)}$ for all j and $f_i^{(1)} < f_i^{(2)}$ for some i. Let $\mathscr{F}$ be the set of frequency vectors of all BIB designs based on v and k.

PROPOSITION 4.1. *Suppose F, F_1 and F_2 are elements of $\mathscr{F}$.*

(i) *If c is a positive integer, then cF is in $\mathscr{F}$.*
(ii) *If g is a common divisor of the entries of F, then $g^{-1}F$ is in $\mathscr{F}$.*
(iii) *$F_1 + F_2$ is in $\mathscr{F}$.*
(iv) *If $F_1 < F_2$, then $F_2 - F_1$ is in F.*

PROOF. (i) and (iii) are immediate from Lemma 4.1. (ii) and (iv) also follow from this lemma after observing that $P(g^{-1}F)$ and $P(F_2 - F_1)$ are both vectors of nonnegative integers.

Note that (ii) and (iv) can be used to construct from BIB designs new designs with smaller numbers of blocks, and that (iv) can be used to reduce support size. It follows from (ii) that if there is no BIB design with $b < vCk$, then there is no uniform design with $b^* < vCk$.

If d is a BIB (v, k), it is natural to ask whether the support, d^*, of this design properly contains the support of another design, d_1. Theorem 4.1 below shows that the combinatorial problem of searching for such a design is equivalent to the algebraic problem of finding solutions to a set of homogeneous linear equations. If d is a BIB $(v, k \mid b^*)$, then we denote by P_{d^*} the matrix whose columns are the pair inclusion vectors associated with the blocks of d^*.

THEOREM 4.1. *Given d, a BIB $(v, b, r, k, \lambda \mid b^*)$, there exists d_1, a BIB (v, k) such that d^* properly contains d_1^* if and only if there exists a nonzero vector h such that $P_{d^*}h = 0$.*

PROOF. Let F_{d^*} be the $b^* \times 1$ vector whose components consist of the nonzero entries of F_d. Thus

$$P_{d^*}F_{d^*} = PF_d = \lambda\mathbf{1} .$$

If P_{d^*} is not of full column rank, then there exists a nonzero vector h, each of whose entries are rational, such that $P_{d^*}h = 0$. Let

$$m = \min\{-f_{d^*i}/h_i : h_i < 0\}$$

where f_{d^*i} is the ith component of F_{d^*}. Let $g_i = f_{d^*i} + mh_i$, $i = 1, 2, \cdots, b^*$. Let t be the smallest positive integer such that $\bar{f}_{d^*i} \equiv tg_i$ is an integer for all $i = 1, \cdots, b^*$. Define

$$F_{d_1^*} \equiv (\bar{f}_{d^*1}, \cdots, \bar{f}_{d^*b^*}) = t(F_{d^*} + mh) .$$

We claim that $F_{d_1^*}$ is the frequency vector of a new BIB design, d_1, where $\bar{f}_{d^*i}$ is the frequency in d_1 of the ith block of the old design, d. To prove this claim, first notice that $\bar{f}_{d^*i} \geqq 0$, $i = 1, \cdots, b^*$. Further

$$P_{d^*}F_{d_1^*} = tP_{d^*}F_{d^*} + tmP_{d^*}h = t\lambda\mathbf{1} .$$

Since P_{d^*} and $F_{d_1^*}$ have only integer entries, $t\lambda$ is an integer; thus, by Lemma

4.1, $F_{d_1^*}$ is the frequency vector of a BIB design. Also, there is at least one i such that $m = -f_{d^*i}/h_i$, and for this i, $\bar{f}_{d_i^*} = 0$. Therefore, d^* properly contains d_1^*.

Suppose now there exists d_1, a BIB $(v, b_1, r_1, k, \lambda_1 \mid b_1^*)$ such that d^* properly contains d_1^*. Let $\bar{f}_{d^*i}$ be the number of times that the ith block in d^* occurs in d_1. Let

$$h_i = f_{d^*i} - \lambda_1/(\lambda_2 f_{d^*i}) , \qquad h = (h_1, \cdots, h_{b^*}) .$$

Then $P_{d^*}h = 0$.

In the proof of the above theorem, we have actually provided an algorithm for the construction of BIB designs. The designs listed in Table 1 were obtained by a computer program implementing this algorithm.

In the following three propositions we present some techniques for producing BIB designs whose support is contained within a given design. These propositions may be thought of as special cases of Theorem 4.1.

PROPOSITION 4.2. *Suppose d_i is a* BIB $(v', b_i, r_i, k, \lambda_i \mid b_i^*)$, $i = 1, 2$, *based on the same set of v' treatments, and suppose $d_1^* \cap d_2^* = \varnothing$. If $v > v'$, then there exists a* BIB $(v, b, r, k, \lambda \mid b^*)$ *with*

$$d = e(vCk) \qquad \textit{and} \qquad b^* = vCk - b_1^* ,$$

where $e = \lambda_2/gcd(\lambda_1, \lambda_2)$.

PROOF (by construction). Take e copies of the trivial design, $d(v, k)$. Add an additional $e_1 \equiv \lambda_1/gcd(\lambda_1, \lambda_2)$ copies of d_2 and remove all e copies of d_1. We have removed $e\lambda_1$ copies of each pair occurring in d_1 and increased the number of times each pair in d_2 occurs by $e_1\lambda_2$. But d_1 and d_2 contain precisely the same pairs, and

$$e_1\lambda_2 = \lambda_1\lambda_2/gcd(\lambda_1, \lambda_2) = e\lambda_1 .$$

That the values of b and b^* are as claimed is clear from the construction.

EXAMPLE 4.1. Let $v = 8$ and $k = 3$ and $v' = 7$. Let

d_1:	124	561	and	d_2:	356	723
	235	672			467	134
	346	713			571	245 .
	457				612	

In this case $\lambda_1 = \lambda_2 = 1$ and $e = e_1 = 1$. Thus by replacing d_1 in $d(8, 3)$ by an additional copy of d_2, we obtain a BIB $(8, 56, 21, 3, 6 \mid 49)$.

COROLLARY 4.1. *Suppose d_3 is a* BIB $(v', b_3, r_3, k, \lambda_3 \mid b_3)$, $v' < v$. *Then there exists a* BIB $(v, b, r, k, \lambda \mid b^*)$ *with $b^* = vCk - b_3$.*

PROOF. Let d_4 be all of the blocks of $d(v', k)$ which are not contained in d_3. Then apply Proposition 4.2 to d_3 and d_4.

EXAMPLE 4.1 (continued). Let d_4 be the set of all blocks based on $v' = 7$ and $k = 3$ which are not contained in $d_3 = d_1$. We can then apply Proposition 4.2 to d_3 and d_4.

A BIB is still obtained if, in Proposition 4.2, $d_1^* \cap d_2^* = \emptyset$; but the support size will not be reduced to the same extent, as the following proposition shows.

PROPOSITION 4.3. *Suppose d_i is a* BIB $(v', b_i, r_i, k, \lambda_i \mid b_i^*)$, $i = 5, 6$, *based on the same set of $v' < v$ treatments. If $|d_5^* \cap d_6^*| = t$, then there exists a* BIB $(v, b, r, k, \lambda \mid b^*)$ *with $b^* = vCk - (b_5^* - t)$ and $b = e(vCk)$ where $e = \lambda_6/gcd(\lambda_5, \lambda_6)$.*

The proof is analogous to that of Proposition 4.2.

EXAMPLE 4.1 (continued). Let $d_5 = d_1$ and let

$$d_6 = \begin{matrix} 124 & 435 & 562 \\ 467 & 715 & \\ 273 & 316 & \end{matrix} .$$

Here $|d_5^* \cap d_6^*| = 1$ and thus by adding one additional copy of d_6 to $d(8, 3)$ and deleting one copy of d_5 we obtain a BIB $(8, 56, 21, 3, 6 \mid 50)$. Note that the block $\{1, 2, 4\}$ appears only once in the new design.

One may generalize the method of Proposition 4.3 by allowing the BIB designs, d_5 and d_6, to be based on a smaller block size, $k' < k$, as long as $v - v' \geqq k - k'$. Let V' be the set of v' treatments upon which d_5 and d_6 are based, and let $V'' = V - V'$. Fix some subset, A, of $k - k'$ treatments from V''. Then augment each block of d_5 and d_6 by A. Let d_5' and d_6' be these two sets of augmented blocks. Then by a construction exactly like that in the proof of Proposition 4.2, we obtain a BIB $(v, b, r, k, \lambda \mid b^*)$ with $b^* = vCk - (b^* - t)$, where $t = |d_5^* \cap d_6^*|$. We could repeat this process for each distinct subset A of $k - k'$ treatments from V''. Thus we have:

PROPOSITION 4.4. *Suppose d_i is a* BIB $(v', b_i, r_i, k', \lambda_i \mid b_i^*)$, $i = 5, 6$, *based on the same set of $v' < v$ treatments, with $|d_5^* \cap d_6^*| = t$. If $v - v' \geqq k - k'$, then there exists a* BIB $(v, b, r, k, \lambda \mid b_n^*)$, *where $b_n^* = vCk - n(b_5^* - t)$, $n = 1, 2, \cdots, {}_{v-v'}C_{k-k'}$, and $b = e(vCk)$, $e = \lambda_6/gcd(\lambda_5, \lambda_6)$.*

EXAMPLE 4.1 (continued). Let $v = 8$, $k = 4$. We can augment d_5 and d_6 to produce

$$d_5' : \begin{matrix} 8124 & 8561 \\ 8235 & 8672 \\ 8346 & 8713 \\ 8457 & \end{matrix} \qquad d_6' : \begin{matrix} 8124 & 8715 \\ 8467 & 8316 \\ 8273 & 8562 \\ 8435 & \end{matrix} .$$

Note that d_5' and d_6' are not BIB designs. Now by adding one additional copy of d_6' to $d(8, 4)$ and removing d_5', we obtain a BIB $(8, 70, 35, 4, 15 \mid 64)$.

In the above propositions, one set of blocks based on a BIB design was replaced by another. But, in fact, it is not necessary that these sets be based on BIB designs.

EXAMPLE 4.1 (continued).

$$d_7: \begin{matrix} 156 & 235 \\ 134 & 246 \end{matrix} \qquad d_8: \begin{matrix} 135 & 256 \\ 146 & 234 \end{matrix}.$$

Add one copy of d_7 to $d(8, 3)$ and delete one copy of d_8 to produce a BIB $(8, 56, 21, 3, 6 \mid 52)$.

5. Bounds on the support size of a BIB design. The literature of BIB designs contains lower bounds on the total number of blocks in a BIB (v, b, r, k, λ). One such result is the well-known Fisher's inequality: $b \geqq v$. In this section some bounds on the number of distinct blocks in a BIB design are given.

Let $b^*_{\min}$ be the smallest support size for BIB designs based on a given v and k. We bound $b^*_{\min}$ in the following proposition. Here for each real number x, let $\{x\}$ denote the smallest integer greater than or equal to x.

PROPOSITION 5.1. *For all* BIB *designs based on v and k,*

$$\left\{\frac{v}{k}\left\{\frac{v-1}{k-1}\right\}\right\} \leqq b^*_{\min} \leqq vC2 .$$

Before proving this proposition, we introduce the following. For $t \leqq k$ let W_{tk} be the $vCt \times vCk$ matrix whose rows are indexed by the elements of $v\Sigma t$ and whose columns are indexed by the elements of $v\Sigma k$. The entry at a given row and column is 1 if the row index is a subset of the column index and 0 otherwise. For example, the matrix P introduced at the beginning of Section 4 is W_{2k}.

LEMMA 5.1. *If $t \leqq k$ and $t + k \leqq v$ then W_{tk} has full rank.*

PROOF. First, we claim that

$$W_{tk}W'_{tk} = \sum_{i=0}^{t} \binom{v-2t}{k-2t+i} W'_{it} W_{it} . \tag{5.1}$$

The (m, n)-entry on the left-hand side of (5.1) is the inner product of the mth and nth rows of W_{tk}; that is, it is the number of k-subsets containing simultaneously the mth and nth t-subset. This number will be $(v - (2t - \alpha))C(k - (2t - \alpha))$ if the intersection of these two t-subsets contains α elements.

To compute the corresponding entry on the right-hand side of (5.1), notice the (m, n)-entry of $W'_{it}W_{it}$ is the number of i-subsets contained simultaneously in the mth and nth t-subsets; that is αCi, since there are α elements in the intersection of these two t-subsets. So, the corresponding entry on the right-hand side is

$$\sum_{i=0}^{t} \binom{v-2t}{k-2t+i}\binom{\alpha}{i} .$$

That the expressions given for the entries on the two sides are equal is a well-known combinatorial identity. Thus, the claim is proven.

$W'_{it}W_{it}$ is nonnegative definite for $i = 0, 1, \cdots, t$. When $i = t$, $W'_{it}W_{it} = I$. Thus, $W_{tk}W'_{tk}$ is positive definite. Therefore, W_{tk} has full rank.

PROOF OF PROPOSITION 5.1. Let $r(P)$ denote the rank of the matrix P and suppose $r(P) > b^*_{\min}$; that is, suppose there exists d, a BIB design for v and k with $b^* = b^*_{\min} > r(P)$. Now since the number of columns of P_{d^*} is b^* and since $r(P_{d^*}) \leqq r(P)$, it follows that P_{d^*} is not full rank. Thus, by Theorem 4.1, there exists d_1, a BIB design for v and k such that $b_1{}^* < b^* = b^*_{\min}$, a contradiction. Therefore $b^*_{\min} \leqq r(P)$. But $r(P) = vC2$, by Lemma 5.1.

To establish the lower bound, note that in a BIB design each pair of elements of V appears at least once. Let n be the smallest number of blocks needed to cover the $vC2$ pairs of V. Observe that in order to cover all pairs, each element of V must appear in at least $\{(v-1)/(k-1)\}$ distinct blocks. So, the average number of distinct blocks in which each element appears, nk/v, must be at least $\{(v-1)/(k-1)\}$. Thus $b^*_{\min} \geqq \{(v/k)\{(v-1)/(k-1)\}\}$.

REMARK. A result analogous to $b^*_{\min} \leqq vC2$ in the context of sampling was given by Wynn (1977). Also, the argument used to establish the lower bound has been used by those investigating the problems of minimal covering. [See, for example, Kalbfleisch and Stanton (1968).]

COROLLARY 5.1. $b^*_{\min} \geqq b/\lambda$.

PROOF. In a BIB (v, b, r, k, λ), $(v/k) = (b/\lambda)(v-1)/(k-1)$. So $\{(v/k)\{(v-1)/(k-1)\}\} \geqq b/\lambda$.

PROPOSITION 5.2. *Suppose a* BIB $(v, b, r, k, \lambda \mid b^*)$ *has frequency vector* $(f_1, \cdots, f_{vCk})$. *Then*

(i) $b \geqq vf_i$, $i = 1, \cdots, vCk$; *and*
(ii) $b^* = v$ *or* $b^* \geqq v + 2$.

PROOF. (i) is a result of Mann (1969).
Let $f = \max\{f_i : i = 1, \cdots, vCk\}$. Then

$$f \geqq (\textstyle\sum_{i=1}^{vCk} f_i)/b^* = b/b^* .$$

Thus (ii) follows from (i) and Theorem 3.2 of van Lint and Ryser (1972).

6. Mathematical programming approach. Some of the results so far presented can be rephrased in the language of mathematical programming. For each fixed value of λ, Lemma 4.1 says that each feasible integer solution of the system

$$PF = \lambda 1 \quad (6.1)$$
$$F \geqq 0$$

corresponds to the frequency vector of a BIB (v, b, r, k, λ). The set of all rational feasible solutions to this system corresponds to the set of frequency vectors of BIB (v, k) designs, after appropriate scaling as in the proof of Theorem 4.1.

Now there is always at least one rational feasible solution to (6.1), namely the solution corresponding to the frequency vector of $d(v, k)$. Proposition 5.1 then follows from the well-known fact that whenever there is a feasible solution to a system, then there is a basic feasible solution.

All feasible solutions are convex combinations of the basic feasible solutions, so the classification of all BIB (v, k) reduces to finding all basic feasible solutions, to (6.1); that is, to finding all of the vertices of the polytope defined by (6.1). When applying designs with repeated blocks in practice, we are not, of course, interested in finding all solutions to (6.1). Rather, we seek a solution which excludes, or at least minimizes the occurrence of certain blocks. We may find such a design by introducing an objective function which assigns positive cost to the blocks which we wish to avoid and zero cost to the other blocks. The standard linear programming algorithms for minimizing this objective function will then produce the desired design. If we are limited to a certain number of blocks, as is generally the case in nonsampling situations, we then have the integer programming problem of minimizing the objective function over the feasible integer solutions to (6.1).

7. Characterization of the support of a BIB design. Given a set $\mathscr{S}$ of distinct blocks from $v\Sigma k$, it would be desirable to know whether there exists a BIB design with $\mathscr{S}$ as its support. This section contains several criteria for determining whether $\mathscr{S}$ is the support of a BIB design.

For each pair, α, in $v\Sigma 2$, let λ_α be the number of blocks containing α. To avoid trivialities we assume throughout this section that the $\lambda_\alpha \geqq 1$ for all α in $v\Sigma 2$; that is, $\mathscr{S}$ is a covering of $v\Sigma 2$.

LEMMA 7.1. *Suppose S_i is a block in $\mathscr{S}$ and α, β are two pairs contained in S_i with $\lambda_\alpha = 1$. If $\mathscr{S}$ is the support of a* BIB *design, then $\lambda_\beta = 1$.*

PROOF. Let $F = (f_1, \cdots, f_{vCk})$ be the frequency vector of a BIB design with support $\mathscr{S}$. Suppose $\lambda_\beta > 1$. Then there exists $S_j \neq S_i$ in $\mathscr{S}$ such that β is contained in S_j. Thus $f_j \neq 0$. Then the β-entry of the vector PF is at least as large as $f_i + f_j > f_i$, but the α-entry is f_i, contradicting Lemma 4.1.

EXAMPLE 7.1. Let $\mathscr{S}$ consist of

123	145	167	178
246	257	258	347 .
356	348	168	

Although $\mathscr{S}$ covers all pairs for $v = 8$ and $k = 3$, it cannot be the support of a BIB design. Let $\alpha = (6, 8)$ and $\beta = (1, 6)$. Then $\lambda_\alpha = 1$ and $\lambda_\beta = 2$, but both are contained in 168.

COROLLARY 7.1. *If $\mathscr{S}$ is the support of a* BIB *design, then the number of pairs that are contained exactly once in $\mathscr{S}$ is divisible by $kC2$.*

PROOF. Each block of $\mathscr{S}$ which contains a pair appearing only once in $\mathscr{S}$ contains exactly $kC2$ such pairs, by Lemma 7.1, and hence the result.

In attempting to construct a BIB design with minimum support, it would seem natural to start with a minimal covering of the pairs. The following proposition shows when this strategy will succeed.

PROPOSITION 7.1. *If $\mathscr{S}$ is a minimal covering of the pairs of elements of V, then $\mathscr{S}$ is the support of a* BIB *design if and only if $\mathscr{S}$ is itself a* BIB *design.*

PROOF. If $\mathscr{S}$ is a BIB design it is its own support. To show the converse, suppose $\mathscr{S}$ is not a BIB design. Then there exists a pair α such that $\lambda_\alpha > 1$. Now α is contained in some block S of $\mathscr{S}$. Now there is a pair β in S such that $\lambda_\beta = 1$; for, if not, we could remove the block S and $\mathscr{S}$-$\{S\}$ would still be a covering. So, by Lemma 7.1, $\mathscr{S}$ is not the support of a BIB design.

Given a set of distinct blocks, $\mathscr{S}$, we associate with each block $S \in \mathscr{S}$ a vector.

$$T(S, \mathscr{S}) = (\lambda_{\alpha_1}, \lambda_{\alpha_2}, \cdots, \lambda_{\alpha_{kC2}})$$

where α_j is the jth pair contained in S. We say that the block S is *pair balanced in* $\mathscr{S}$ if $T(S, \mathscr{S}) = (c, c, \cdots, c)$.

PROPOSITION 7.2. *If every block in the pair covering $\mathscr{S}$ is pair balanced, then $\mathscr{S}$ is the support of a* BIB *design.*

PROOF (by construction). Collect all blocks of $\mathscr{S}$ with the same T vector. This partitions $\mathscr{S}$ into classes $\mathscr{S}_1, \mathscr{S}_2, \cdots, \mathscr{S}_t$ to that every block in a given class, $\mathscr{S}_i$, has the same $T = (n_i, \cdots, n_i)$, say. To produce a BIB design with $\mathscr{S}$ as its support, take l/n_i copies of each block in $\mathscr{S}_i$, $i = 1, 2, \cdots, t$, where l is the least common multiple of n_i, $i = 1, 2, \cdots, t$.

Notice that each pair of $v\Sigma 2$ appears in exactly one class, $\mathscr{S}_i$. Moreover, each pair which appears in $\mathscr{S}_i$ appears exactly n_i times. Thus, the above procedure guarantees that every pair will appear exactly l times in the design.

EXAMPLE 7.2. Let $v = 7$ and $k = 3$. Let $\mathscr{S}$ be:

124	346	672	435
156	235	647	572.
137	457	263	

It can be checked that $\mathscr{S}$ is a covering and that each of its blocks is pair balanced in $\mathscr{S}$. Indeed, each block in the first column has $T = (1, 1, 1)$ and each of the remaining blocks has $T = (2, 2, 2)$. Therefore, by taking 2 copies of the blocks in the first column and one copy of the remaining blocks, we obtain a BIB $(7, 14, 6, 3, 2 \mid 11)$.

8. BIB designs with $v = 8$ and $k = 3$. Using the relations $rv = bk$ and $\lambda(v-1) = r(k-1)$ which hold in any BIB (v, b, r, k, λ), we can easily verify that there is no reduced BIB design based on $v = 8$ and $k = 3$. Indeed, $v = 8$ and $k = 3$ are the smallest v and k with $2 < k < v/2$ for which there is no reduced BIB. We were thus interested in finding the possible support sizes in this case.

In Section 4, we gave several examples of BIB $(8, 56, 21, 3, 6 \mid b^*)$ with $b^* < 56$. In Table 1 of this section we list some of the designs produced by a computer program implementing the algorithm described in the proof of Theorem 4.1.

TABLE 1
BIB *designs with* $v = 8$ *and* $k = 3$,
support sizes 22 to 55

b^* / b	22 / 56	23 / 56	24 / 56	25 / 56	26 / 56	27 / 56	28 / 56	29 / 56	30 / 56	31 / 56	32 / 56	33 / 56	34 / 56	35 / 56	36 / 56	37 / 56	38 / 56	39 / 56	40 / 56	41 / 56	42 / 56	43 / 56	44 / 56	45 / 56	46 / 56	47 / 56	48 / 56	49 / 56	50 / 56	51 / 112	52 / 56	53 / 112	54 / 112	55 / 112
123	—	—	—	3	1	—	—	2	—	—	3	—	—	—	2	—	3	1	3	—	1	1	2	1	1	1	1	1	1	2	1	2	2	2
124	—	—	—	—	—	2	2	—	—	3	—	—	—	—	—	—	—	1	3	1	1	1	—	1	1	1	1	1	1	3	1	2	2	2
125	1	1	1	1	—	—	—	—	4	—	1	3	—	1	—	1	1	2	—	3	2	1	1	—	2	—	1	1	1	2	1	—	1	2
126	—	—	—	—	4	2	2	3	—	2	—	2	—	—	2	—	—	—	—	1	—	1	1	3	—	2	1	1	1	2	1	3	3	2
127	2	2	2	—	1	2	2	1	—	1	—	—	3	1	1	1	1	1	—	—	1	1	1	1	1	2	2	1	1	2	1	3	2	2
128	3	3	3	2	—	—	—	—	2	—	2	1	3	4	1	4	1	1	—	1	1	1	1	—	1	—	—	1	1	2	1	2	2	3
134	—	—	—	2	—	—	—	1	1	—	2	1	1	1	1	1	1	1	—	1	1	1	1	1	1	1	1	1	1	2	1	1	1	3
135	—	—	—	1	3	4	4	2	—	3	1	—	1	1	2	1	1	1	2	1	1	1	1	1	1	1	2	1	1	3	1	3	3	3
136	5	3	5	—	2	1	1	1	2	1	—	2	1	3	1	2	1	1	1	1	1	—	1	1	1	1	—	1	1	1	1	2	2	2
137	1	3	1	—	—	1	1	—	2	2	—	2	1	1	—	1	—	1	—	2	1	2	1	1	1	1	1	1	1	2	1	2	2	2
138	—	—	—	—	—	—	—	—	1	—	—	1	2	—	—	1	—	1	—	1	1	1	—	1	1	1	1	1	1	2	1	2	2	—
145	5	5	5	—	3	—	—	2	2	—	—	3	3	3	2	3	—	2	—	1	2	1	1	1	1	1	1	1	1	4	2	2	2	1
146	1	1	1	—	—	3	3	—	2	2	—	1	2	2	—	2	—	—	2	1	—	2	—	1	1	1	2	1	1	3	1	2	2	2
147	—	—	—	3	1	—	—	1	—	—	3	—	—	—	2	—	2	1	1	1	1	—	1	1	1	1	—	1	—	—	—	2	2	2
148	—	—	—	1	2	1	1	2	1	1	1	1	—	—	1	—	3	1	—	1	1	1	3	1	1	1	1	1	2	1	1	3	3	3
156	—	—	—	4	—	—	—	2	—	1	4	—	1	—	2	—	3	1	1	—	1	—	2	1	1	1	1	1	1	2	1	3	2	2
157	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	2	—	—	1	1	1	1	1	—	1	1	1	1	2	2	2
158	—	—	—	—	—	1	2	—	—	2	—	—	1	—	—	—	—	—	1	1	—	2	—	2	—	2	1	1	1	—	—	2	2	2
167	—	—	—	1	—	—	—	—	2	—	1	1	2	1	—	2	1	2	—	2	2	2	1	—	1	—	1	1	2	2	1	1	2	2
168	—	2	—	1	—	—	—	—	—	—	1	—	—	—	1	—	1	2	2	1	2	1	1	—	2	1	1	1	—	2	1	1	1	2
178	3	1	3	2	4	3	3	4	2	3	2	3	—	2	3	1	1	1	3	1	1	—	1	2	1	1	2	1	1	5	2	2	2	2
234	4	4	4	3	2	2	2	2	3	1	2	3	1	3	1	3	2	—	—	2	—	2	2	1	—	1	1	2	1	2	1	3	3	2
235	—	—	—	—	—	2	1	—	—	2	—	—	3	—	—	—	—	2	—	—	1	1	—	2	1	2	1	—	1	1	1	1	1	2
236	—	—	—	—	—	—	—	—	3	—	1	2	1	1	—	1	1	—	1	3	1	1	1	—	1	—	1	2	1	2	1	3	3	2
237	2	2	2	—	—	—	—	—	—	—	—	1	1	2	1	2	—	1	1	1	1	—	—	—	1	—	1	1	—	3	1	2	2	2
238	—	—	—	—	3	2	3	2	—	3	—	—	—	—	2	—	—	2	1	—	2	1	1	2	2	2	1	—	2	2	1	1	1	2

245	1	1	1	1	2	—	—	2	—	—	1	—	2	1	2	1	1	—	2	—	1	1	1	1	1	1	1	1	1	2	1	2	1	3
246	1	1	1	2	1	—	—	1	—	—	1	—	1	1	1	1	1	4	—	—	3	—	1	—	3	1	1	—	1	2	1	2	2	2
247	—	—	—	—	—	2	2	—	2	2	1	1	—	1	1	1	1	1	—	1	1	1	1	2	1	1	1	1	1	2	1	1	2	3
248	—	—	—	—	1	—	—	1	1	—	1	2	2	—	1	—	1	—	1	2	—	1	1	1	—	1	1	1	1	2	1	2	2	1
256	2	2	2	2	1	2	3	1	—	2	1	—	1	3	2	3	1	1	1	—	1	1	1	1	1	1	1	1	1	3	1	2	2	2
257	2	2	2	1	3	1	1	3	1	1	2	2	—	1	2	1	1	—	1	2	—	1	2	1	—	1	1	1	2	2	1	4	4	1
258	—	—	—	1	—	1	1	—	1	1	1	1	—	—	—	—	2	1	2	1	1	2	1	1	1	1	1	2	—	2	1	3	3	2
267	—	—	—	2	—	—	—	—	2	1	2	1	2	—	—	—	2	1	3	1	1	2	1	1	1	1	—	1	1	1	1	—	—	2
268	3	3	3	—	—	2	1	1	1	1	1	1	1	1	1	1	1	—	1	1	—	—	1	1	—	1	2	1	1	2	1	2	2	2
278	—	—	—	3	2	1	1	2	1	1	1	1	—	1	1	1	1	2	1	1	2	1	1	1	2	1	1	1	1	2	1	2	2	2
345	—	—	—	—	—	—	1	—	—	1	—	—	—	—	1	—	1	—	1	1	—	1	1	1	1	1	—	1	1	—	—	2	2	1
346	—	2	—	1	1	—	—	1	—	1	2	—	—	—	1	1	1	2	1	—	2	1	1	1	1	1	1	—	1	3	1	2	2	2
347	2	—	—	—	3	—	—	2	—	—	—	—	3	—	1	—	1	2	2	—	1	1	1	1	1	1	1	—	2	3	2	2	2	1
348	—	—	2	—	—	4	3	—	2	3	—	2	1	2	1	1	—	1	2	2	2	—	—	1	2	1	2	2	—	2	1	2	2	3
356	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	1	1	1	1	1	—	1	—	1	1	1	2	1	1	1	2
357	—	—	2	5	—	—	—	—	4	—	4	3	1	2	—	2	3	2	1	2	3	2	2	1	2	1	1	2	1	2	1	2	2	2
358	6	6	4	—	3	—	—	4	2	—	1	2	—	3	3	3	1	—	1	1	—	—	1	1	—	1	1	1	1	4	2	3	3	2
367	1	1	1	—	3	5	5	4	—	4	—	—	—	1	4	1	—	—	1	—	—	—	—	3	1	3	2	1	1	2	1	2	2	2
368	—	—	—	5	—	—	—	—	1	—	3	1	3	1	—	1	3	2	1	1	1	3	2	1	1	1	1	1	1	2	1	2	2	2
378	—	—	—	1	—	—	—	—	—	—	2	—	—	—	—	—	2	—	1	1	—	1	2	—	—	—	—	1	1	—	—	2	2	3
456	—	—	—	—	—	2	1	—	3	1	1	2	1	—	—	—	2	—	1	3	—	2	1	2	—	2	1	2	1	—	1	2	3	2
457	—	—	—	—	1	4	4	2	1	4	—	1	—	1	1	1	—	1	1	1	1	1	—	1	1	1	2	1	1	3	1	2	2	3
458	—	—	—	5	—	—	—	—	—	—	4	—	—	1	—	1	2	3	1	—	2	1	2	—	2	—	1	—	1	3	1	2	2	2
467	1	1	3	3	1	—	—	1	1	—	2	3	1	2	1	1	2	—	1	2	1	—	3	—	1	—	1	2	1	2	1	3	2	2
468	3	1	1	—	3	1	2	3	—	2	—	—	1	1	3	1	—	—	1	—	—	—	—	2	—	1	—	1	1	2	1	1	1	2
478	3	5	3	—	—	—	—	—	2	—	—	1	2	2	—	3	—	1	1	1	1	3	—	1	1	2	1	1	1	2	1	2	2	1
567	4	4	2	—	2	—	—	1	—	—	—	—	1	1	1	1	—	2	1	—	1	1	—	1	1	1	1	—	—	2	1	2	2	2
568	—	—	2	—	3	2	2	2	3	2	—	3	1	2	1	2	—	1	1	2	2	1	1	1	2	1	1	1	2	2	1	2	2	2
578	—	—	—	—	—	1	1	—	—	1	—	—	4	—	2	—	1	1	—	1	1	—	1	1	1	1	1	1	1	1	1	—	—	2
678	—	—	—	—	—	1	1	—	1	1	1	1	—	1	—	1	1	1	—	1	1	1	1	1	1	1	1	1	1	2	1	4	4	2

NOTE. For each support size the number of blocks, b, is a minimum.

In Table 1 each column specifies a BIB design. The first two entries give the support size and the number of blocks in the design. The remaining entries in each column form the frequency vector of the design. Note that there is a design listed with each support size from 22 to 55. For b^* equal to 51, 53, 54, and 55 it can be shown by a counting argument that there is no BIB (8, 3) with $b < 112$; that is with $\lambda < 12$.

Acknowledgment. The reformulation in the terminology of linear programming is due to a helpful suggestion of a referee. We also thank R. M. Wilson for assistance with the proof of Lemma 5.1.

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