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# A CHARACTERIZATION OF A BIVARIATE EXPONENTIAL DISTRIBUTION[1]

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The independence of $U = \min(X, Y)$ and $V = X - Y$ or $W = |X - Y|$ is studied where $X$ and $Y$ are not assumed to be independent. The bivariate exponential distribution of Marshall and Olkin is characterized as the distribution with exponential marginals where $U$ is exponential and independent of $V$.

**1. Introduction.** For independent and identically distributed nonnegative random variables $X$ and $Y$, under appropriate continuity conditions, the independence of $U = \min(X, Y)$ and $W = |X - Y|$ characterizes the exponential distribution (see, for example, Basu (1965)). Furthermore if $X$ and $Y$ are not assumed to be identically distributed, the independence of $U$ and $V = X - Y$ imply that $X$ and $Y$ have exponential distributions, again under appropriate continuity conditions. A result of this type was proven by Ferguson (1964) who assumed that $X$ and $Y$ were absolutely continuous but not necessarily nonnegative and obtained that $X$ and $Y$ were exponential with the same location parameter. See Basu and Block (1975) for a discussion of these and other characterizations of exponential distributions.

When $X$ and $Y$ are not necessarily independent, but are assumed to have a joint density, Block and Basu (1974) have related the independence of $U$ and $V$ to a generalized lack of memory property of Marshall and Olkin (1967). In Section 2, the bivariate exponential distribution of Marshall and Olkin is characterized as the distribution with exponential marginals where $U$ is exponential and independent of $V$. This characterization is obtained via a lemma which is an improvement of the result of Block and Basu mentioned above. The implications of the independence of $U$ and $W$ when $X$ and $Y$ are not assumed to be independent are studied in Section 3.

**2. The main result.** A characterization of the bivariate exponential distribution of Marshall and Olkin (1967) will be derived in this section and is given in the following theorem.

THEOREM 2.1. *The bivariate random variable* $(X, Y)$ *has the distribution*

$$P\{X > x, Y > y\} = \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)], \quad x, y > 0 \tag{2.1}$$

*where* $\lambda_1, \lambda_2, \lambda_{12} \geqq 0$

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*if and only if* $(X, Y)$ *has exponential marginal distributions*, $\min(X, Y)$ *is exponential, and* $\min(X, Y)$ *is independent of* $X - Y$.

The proof of Theorem 2.1 makes use of the fact (see Marshall and Olkin (1967)) that the distribution given in (2.1), which we call the BVE, is the only bivariate distribution with exponential marginals such that

$$(*) \qquad \bar{F}(s_1 + t, s_2 + t) = \bar{F}(s_1, s_2)\bar{F}(t, t) \qquad s_1, s_2, t \geqq 0$$

where $\bar{F}(s, t) = P\{X > s, Y > t\}$.

Block and Basu (1974) call (*) the *loss of memory property* (LMP) and use it to derive a related absolutely continuous distribution called the ACBVE. The distributional form of the ACBVE is given by (3.3) or (4.1) of Block and Basu (1974).

The LMP has been characterized in Theorem 8.1 of Block and Basu (1974) for absolutely continuous bivariate distributions. Since the BVE is not absolutely continuous this characterization does not apply. Lemma 2.1 below is an improvement of Theorem 8.1 in which the assumption of bivariate absolute continuity is replaced by absolute continuity of the marginals only. In addition, it turns out that condition 3 of Theorem 8.1 can also be dropped. Theorem 2.1 is then an immediate consequence of the following lemma.

It should be noted that only one part of the following lemma is really required for the proof of the theorem, but because of its independent interest we include the complete lemma.

LEMMA 2.1. *Let* $(X, Y)$ *be a nonnegative bivariate random variable with absolutely continuous marginal distributions and let* $U = \min(X, Y)$ *and* $V = X - Y$. *The* LMP *holds if and only if there is a* $\theta > 0$ *such that*

(1) *$U$ and $V$ are independent*

*and*

(2) *$U$ is exponential with mean* $\theta^{-1}$.

*Furthermore if the* LMP *holds the distribution of $V$ is given by* (3) *of Theorem* 8.1 *of Block and Basu* (1974).

PROOF. Let $F_i(t)$ and $f_i(t)$ for $i = 1, 2$ be the marginal distributions and densities of $X$ and $Y$. Also let $\bar{F}_i(t) = 1 - F_i(t)$.

Assume the LMP holds. It follows that there is a $\theta > 0$ such that $P\{\min(X, Y) > s\} = \exp(-\theta s)$, $s > 0$. Furthermore it can be shown that

$$\begin{aligned} P\{X > x \mid Y = y\} &= \exp(-\theta x) f_2(y - x)[f_2(y)]^{-1} \qquad \text{if } x < y, \\ &= [\theta \bar{F}_1(x - y) - f_1(x - y)] \exp(-\theta y)[f_2(y)]^{-1} \qquad \text{if } x \geqq y, \end{aligned}$$

for all $y$ such that $f_2(y) > 0$, and

$$\begin{aligned} P\{Y > y \mid X = x\} &= \exp(-\theta y) f_1(x - y)[f_1(x)]^{-1} \qquad \text{if } y < x, \\ &= [\theta \bar{F}_2(y - x) - f_2(y - x)] \exp(-\theta x)[f_1(x)]^{-1} \qquad \text{if } y \geqq x, \end{aligned}$$

for all $x$ such that $f_2(x) > 0$.

Partitioning the event $\{\min(X, Y) \leqq s, X - Y \leqq t\}$ into three parts according to the events $\{X > Y\}$, $\{X = Y\}$ and $\{X < Y\}$ and using the preceding conditional probabilities it is not hard to show that for $s \geqq 0$

$$\begin{aligned} P\{\min(X, Y) &\leqq s, X - Y \leqq t\} \\ &= [1 - \exp(-\theta s)]\{F_1(t) + \theta^{-1}f_1(t) + 1 - \theta^{-1}[f_1(0) + f_2(0)]\} \\ &\qquad + P\{X = Y \leqq s\} \qquad \text{if} \quad t \geqq 0 \\ &= [1 - \exp(-\theta s)][1 - F_2(-t) + \theta^{-1}f_2(-t)] \qquad \text{if} \quad t < 0 . \end{aligned}$$

Letting $t \to \infty$ gives that for $s \geqq 0$

$$P\{X = Y \leqq s\} = \{\theta^{-1}[f_1(0) + f_2(0)] - 1\}[1 - \exp(-\theta s)] .$$

Substituting this into the preceding expression gives (1) and (2).

To prove the lemma in the other direction assume that (1) and (2) hold. Partitioning as in the first part of the proof gives that for $0 \leqq x \leqq y$

$$\bar{F}(x, y) = \exp(-\theta y) + \int_x^y P\{V \leqq u - y\}\theta \exp(-\theta u)\, du , \qquad x, y \geqq 0$$

so that for fixed $y > 0$ and for all $0 \leqq x \leqq y$ at which $P\{V \leqq x - y\}$ is continuous (i.e., all but at most a countable number of $x$'s)

$$\frac{\partial}{\partial x}\bar{F}(x, y) = -P\{V \leqq x - y\}\theta \exp(-\theta x) . \tag{2.2}$$

With $s_2 > 0$ and $t > 0$ fixed, (2.2) can be used to obtain

$$\frac{\partial}{\partial s_1}\bar{F}(s_1 + t, s_2 + t) = \bar{F}(t, t)\frac{\partial}{\partial s_1}\bar{F}(s_1, s_2) \qquad \text{for} \quad 0 \leqq s_1 \leqq s_2 , \quad 0 \leqq t \tag{2.3}$$

except at most a countable number of $s_1$. Integrating (2.3) gives the LMP for all $0 \leqq s_1 \leqq s_2$, $0 \leqq t$. Similar computations give the result for the remaining $s_1$, $s_2$ and $t$.

In the preceding lemma only the assumption that the marginal distribution functions are continuous (as opposed to absolutely continuous) is used in the second part of the proof. Even in the first part of the proof, the assumption of absolutely continuous marginals is not overly restrictive since it can be shown that the LMP implies that the marginals cannot be discrete.

Condition (2) of Lemma 2.1 cannot be dropped since as in the second part of the proof of Lemma 2.1 for $0 \leqq x \leqq y$

$$\bar{F}(x, y) = P\{U > y\} + \int_x^y P\{V \leqq u - y\}\, dP\{U < u\} .$$

A nonexponential choice of distribution for $U$ implies that $\bar{F}(x, x) = P\{U > x\}$ is not exponential, so that the LMP cannot hold. For example, if

$$f_U(u) = ue^{-u} , \quad u \geqq 0 , \qquad f_V(v) = \tfrac{1}{2}\exp(-|v|) ,$$

then

$$f(x, y) = \tfrac{1}{2}\min(x, y)\exp(-\max(x, y)) , \qquad x, y \geqq 0 ,$$

a distribution which does not have the LMP.

**3. The independence of** $\min(X, Y)$ **and** $|X - Y|$. Under the assumption of Lemma 2.1 it follows immediately that if $U = \min(X, Y)$ and $W = |X - Y|$ where $(X, Y)$ has the LMP, then there is a $\theta > 0$ such that

(1) $U$ and $W$ are independent,
(2) $U$ is exponential with mean $\theta^{-1}$,
(3) $P\{W \leqq w\} = \theta^{-1}\{f_1(w) + f_2(w)\} + F_1(w) + F_2(w) - 1$ for $w \geqq 0$,

where for $i = 1, 2$, $F_i(t)$ and $f_i(t)$ are respectively the marginal distributions and densities of $X$ and $Y$.

It might be conjectured that (1) and (2) characterize the LMP, but this is not the case as can be seen by the following example which was provided by T. Savits. Let $a > 0$ and $P\{Z > z\} = \exp(-\theta z)$, $z \geqq 0$. Then define

$$\begin{aligned}(X, Y) &= (Z, Z + 1) \qquad \text{if} \quad a < Z \\ &= (Z + 1, Z) \qquad \text{if} \quad 0 \leqq Z \leqq a\,.\end{aligned}$$

Clearly $U = Z$ and $W = 1$ are independent. It is not hard to show $(X, Y)$ does not satisfy the LMP.

Conditions (1) and (2) do imply that a related distribution has the LMP. This result is given in the following theorem.

THEOREM 3.1. *Let $(X, Y)$ have a bivariate distribution with absolutely continuous marginals such that*

(1) $U = \min(X, Y)$ *and* $W = (X - Y)$ *are independent*

*and*

(2) $U$ *is exponentially distributed. Then the distribution*

$$\bar{G}(x, y) = 2^{-1}P\{X > x, Y > y\} + 2^{-1}P\{X > y, Y > x\}$$

*has the* LMP.

PROOF. It is not hard to show that for $0 \leqq x \leqq y$

$$\begin{aligned}\bar{G}(x, y) &= 2P\{U > y\} + P\{U + W > y, x < U < y\} \\ &= \exp(-\theta x)\bar{G}(0, y - x)\,.\end{aligned}$$

Similarly for $0 \leqq y \leqq x$, $\bar{G}(x, y) = \exp(-\theta y)\bar{G}(x - y, 0)$. This gives that $\bar{G}(x, y)$ has the LMP.

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